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Non-uniform Small-gain Theorems for Systems with Unstable Invariant Sets

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***Abstract*— We consider the problem of small-gain analysis of asymptotic behavior in interconnected nonlinear dynamic systems. Mathematical models of these systems are allowed to be uncertain and time-varying. In contrast to standard small-gain theorems that require global asymptotic stability of each interacting component in the absence of inputs, we consider interconnections of systems that can be critically stable and have infinite input-output L_∞ gains. For this class of systems we derive small-gain conditions specifying state boundedness of the interconnection. The estimates of the domain in which the system's state remains are also provided. Conditions that follow from the main results of our paper are non-uniform in space. That is they hold generally only for a set of initial conditions in the system's state space. We show that under some mild continuity restrictions this set has a non-zero volume, hence such bounded yet potentially globally unstable motions are realizable with a non-zero probability. Proposed results can be used for the design and analysis of intermittent, itinerant and meta-stable dynamics which is the case in the domains of control of chemical kinetics, biological and complex physical systems, and non-linear optimization.**

I. INTRODUCTION

Small-Gain theorems are widely recognized as effective tools for the analysis of asymptotic behavior of the cascades and interconnections of linear and nonlinear systems [1], [2]. They are especially advantageous in those situations when mathematical models of systems are uncertain, and only estimates of the input-output properties of each component are available. The latter property together with the notions of *input-output* and *input-to-state stability* [1], [3], [4] makes the small-gain technique a promising instrument in the analysis of complex biological and physical systems, see for instance, [5], [6], [7].

Conventional small-gain results often require (global) Lyapunov asymptotic stability of unperturbed dynamics of each interacting subsystems [4]. Yet, there are physical and biological systems that fail to satisfy these requirements. Furthermore, as the methods of control expand from purely engineering applications into wider areas of science, there is a need for maintaining behavior that fail to obey the usual notion of Lyapunov stability [8]. Here are few examples of systems in which *explorative, searching* rather than Lyapunov-unstable behavior is considered useful or inherent.

Example 1. In problems of *nonlinear output regulation* Lyapunov-unstable convergence allowed to address the question of minimal information about the plant that is to be made available in order to designing an adaptive controller [9], [10] solving the adaptive output regulation problem. The proposed solution has been called the universal algorithm for adaptive control, and equations of the controlled system are as follows:

$$\begin{aligned}\dot{x} &= f(x,\lambda) + g(x,t)u, \ f:\mathbb{R}^n\times\mathbb{R}\rightarrow\mathbb{R}^n\\ u &= \beta(\lambda)x \\ \dot{\lambda} &= |h(x)|^p, \ \gamma\in\mathbb{R}_{>0}\end{aligned} \tag{1}$$

In (1) u is a control input, λ is a dynamic variable of the feedback, $\beta:\mathbb{R}\rightarrow\mathbb{R}^{m\times n}$ is a dense trajectory in $\Omega_K\subset\mathbb{R}^{m\times n}$, $h:\mathbb{R}^n\rightarrow\mathbb{R}$ is a Lipschitz function, and there exists $\beta^*\in\Omega_K$ such that the origin of

$$\dot{x} = f(x,\lambda) + g(x,t)\beta^* x \tag{2}$$

is exponentially stable.

Example 2. Systems with Lyapunov-unstable yet bounded solutions emerge in *models of decision-making and recognition* in neural systems [11], [12]. These models are networks of nonlinear oscillators [13]:

$$\begin{aligned}\dot{x}_i &= x_i\left[h_i - \left(x_i + \sum_{j\neq i}^{N}\rho_{ij}x_j\right)\right] + \eta_i(t)\\ \dot{h}_i &= -\gamma\frac{\partial U_i(h_i,I)}{\partial h_i},\end{aligned}$$

where x_i are internal states, h_i are the stimuli-dependent control parameters, $\rho_{ij}\in\mathbb{R}$ defines the strength of the competitive interaction from the state j to state i, I models external stimulation, η_i model external noise, and U_i is a "potential" determining system's response to stimulation. In general U_i depends on sate variables x_i of the system.

In these models equilibria are associated with decision stages, and trajectories are to explore the system state space along one-dimensional unstable manifolds forming heteroclinic sequences and channels [11] connecting so-called

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dissipative saddles. Saddle-node heteroclinic connections are among possible configurations in such systems:

$$\begin{aligned} &\dot{x} = f(x, \lambda, t), \ f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n \\ &\dot{\lambda} = \gamma \|x\|, \ \gamma \in \mathbb{R}_{>0}. \end{aligned} \tag{3}$$

In (3) the origin of $\dot{x} = f(x, 0, t)$ is assumed uniformly asymptotically stable, and $f(x, \lambda, t)$ be locally Lipschitz in λ uniformly in t. Controlling and determining domain of attraction for the point attractor at the origin of (3) is crucial for this concept.

Example 3. Analysis of kinetic equations:

$$\dot{x}_1 = -\lambda_1(t)x_1 + c_1(x_2, t) + u \tag{4}$$

$$\dot{x}_2 = -\lambda_2(t)x_2 + c_2(x_1, t), \tag{5}$$

where the function $\lambda_1 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$ is separated from zero, i.e. $\exists\ \lambda^* \in \mathbb{R}_{>0} : \ \lambda_1(t) \geq \lambda^*$, and $\lambda_2 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq}0$ can assume zero values over $\mathbb{R}_{\geq 0}$. The functions $c_1, c_2 : \mathbb{R} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ are globally Lipschitz in x_1, x_2, and $c_2(x_1, t)$ is non-negative (non-positive) in x_1. Variable $u \in \mathbb{R}$ constitutes an external regulatory input.

All these examples share common characterization – asymptotically stable dynamics (stable subsystem) is coupled with explorative motions in the system state space (explorative subsystem). Furthermore, right-hand sides of the equations governing these subsystems are likely to be subject of external perturbations. For instance, the $\lambda_1(t)$, $\lambda_2(t)$ in (4), (5) may vary with time, and functions $f(x, \lambda)$, $f(x, \lambda, t)$ in (1), (3) may be unknown. When precise knowledge of ordinary differential equations governing the system dynamics is not available the system can be thought of as a mere interconnection of input-output maps. Small-gain theorems [1], [2] are usually efficient in this case. These results, however, apply only under the assumption of stability of each component in the interconnection. The latter condition is violated for (1) – (5).

In the present study we aim to find a proper balance between the generality of input-output approaches [1], [2] in the analysis of convergence and the specificity of the fundamental notions of limit sets and invariance. The object of our study is a class of systems that can be decomposed into an attracting, or stable, component $\mathcal{S}_a$ and an exploratory, generally unstable, part $\mathcal{S}_w$. Typical systems of this class are nonlinear systems in cascaded form

$$\begin{aligned} \mathcal{S}_a : \ \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{z}), \\ \mathcal{S}_w : \ \dot{\mathbf{z}} = \mathbf{q}(\mathbf{z}, \mathbf{x}) \end{aligned} \tag{6}$$

where the zero solution of the $\mathbf{x}$-subsystem is asymptotically stable in the absence of input $\mathbf{z}$, and the state of the $\mathbf{z}$-subsystem are functions of $\int_{t_0}^{t} \|\mathbf{x}(\tau)\| d\tau$. Even when both subsystems in (6) are stable and the $\mathbf{x}$-subsystem does not depend on state $\mathbf{z}$, the cascade can still be unstable [14]. We show, however, that for unstable interconnections (6), under certain conditions that involve only input-to-state properties of $\mathcal{S}_a$ and $\mathcal{S}_w$, there is a set $\mathcal{V}$ in the system state space, such that trajectories starting in $\mathcal{V}$ remain bounded. The result is formally stated in Theorem 1. In case an additional measure of invariance is defined for $\mathcal{S}_a$ (in our case a steady-state characteristic), a weak, Milnor attracting set [15] emerges. Its location is completely determined by the zeros of the steady-state response of system $\mathcal{S}_a$.

Due to space limitation we concentrate on presenting the main ideas and applications of our approach rather than technical details. Proofs of the statements can be found in [16], [17].

The paper is organized as follows. Section II describes notational agreements. In Section III we specify the class of systems of our study and formally state the problem. Section IV contains main results of our paper. Namely, Theorem 1 provides a set of general sufficient conditions for non-uniform convergence, and Corollary 1 shapes these conditions into the usual small-gain formulae for a wide class of nonlinear systems. Section V provides discussion of these results and concludes the paper.

II. NOTATION

Throughout the paper we use the following notational conventions.

- Symbol $\mathbb{R}$ denotes the field of real numbers, symbol $\mathbb{R}_+$ stands for the following subset of $\mathbb{R}$: $\mathbb{R}_+ = \{x \in \mathbb{R} |\ x \geq 0\}$; $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers and its extension to the negative domain respectively.
- Symbol $\mathcal{C}^k$ denotes the space of functions that are at least k times differentiable.
- $\mathcal{K}$ denotes the class of all strictly increasing continuous functions $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\kappa(0) = 0$. If, in addition, $\lim_{s\rightarrow\infty} \kappa(s) = \infty$ we say that $\kappa \in \mathcal{K}_\infty$.
- Symbol $\mathcal{KL}$ denotes the class of functions $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\beta(\cdot, s) \in \mathcal{K}$ for each $s \in \mathbb{R}_+$, and $\beta(r, \cdot)$ is monotonically decreasing to zero for each $r \in \mathbb{R}_+$.
- Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{x}$ can be partitioned into two vectors $\mathbf{x}_1 \in \mathbb{R}^q$, $\mathbf{x}_1 = (x_{11}, \ldots, x_{1q})^T$, $\mathbf{x}_2 \in \mathbb{R}^p$, $\mathbf{x}_2 = (x_{21}, \ldots, x_{2p})^T$ with $q + p = n$, then $\oplus$ denotes their concatenation: $\mathbf{x} = \mathbf{x}_1 \oplus \mathbf{x}_2$.
- The symbol $\|\mathbf{x}\|$ denotes the Euclidian norm in $\mathbf{x} \in \mathbb{R}^n$.
- By $L^n_\infty[t_0, T]$ we denote the space of all functions $\mathbf{f} : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ such that $\|\mathbf{f}\|_{\infty,[t_0,T]} = \sup\{\|\mathbf{f}(t)\|, t \in [t_0, T]\} < \infty$, and $\|\mathbf{f}\|_{\infty,[t_0,T]}$ stands for the $L^n_\infty[t_0, T]$ norm of $\mathbf{f}(t)$.
- Let $\mathcal{A}$ be a set in $\mathbb{R}^n$ and $\|\cdot\|$ be the usual Euclidean norm in $\mathbb{R}^n$. By the symbol $\|\cdot\|_\mathcal{A}$ we denote the following induced norm:

$$\|\mathbf{x}\|_\mathcal{A} = \inf_{\mathbf{q}\in\mathcal{A}} \{\|\mathbf{x} - \mathbf{q}\|\}$$

III. PROBLEM FORMULATION

Similar to [16], we consider a system that can be decomposed into two interconnected subsystems, $\mathcal{S}_a$ and $\mathcal{S}_w$:

$$\begin{aligned} \mathcal{S}_a : (u_a, \mathbf{x}_0) \mapsto \mathbf{x}(t) \\ \mathcal{S}_w : (u_w, \mathbf{z}_0) \mapsto \mathbf{z}(t) \end{aligned} \tag{7}$$

where $u_a \in \mathcal{U}_a \subseteq L_\infty[t_0, \infty]$, $u_w \in \mathcal{U}_w \subseteq L_\infty[t_0, \infty]$ are the spaces of inputs to $\mathcal{S}_a$ and $\mathcal{S}_w$, respectively $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{z}_0 \in$

$\mathbb{R}^m$ represent initial conditions, and $\mathbf{x}(t) \in \mathcal{X} \subseteq L^n_\infty[t_0, \infty]$, $\mathbf{z}(t) \in \mathcal{Z} \subseteq L^m_\infty[t_0, \infty]$ are the system states.

System $\mathcal{S}_a$ represents the contracting dynamics. More precisely, we require that $\mathcal{S}_a$ is input-to-state stable [18] with respect to a compact set $\mathcal{A}$:

Assumption 1 (Globally stable dynamics):

$$\mathcal{S}_a: \quad \|\mathbf{x}(t)\|_{\mathcal{A}} \leq \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, t - t_0) + c\|u_a(t)\|_{\infty,[t_0,t]}, \ \forall\, t_0 \in \mathbb{R}_+,\ t \geq t_0 \tag{8}$$

where the function $\beta(\cdot, \cdot) \in \mathcal{KL}$, and $c > 0$ is some positive constant.

In what follows we will assume that the function $\beta(\cdot, \cdot)$ and constant c are known or can be estimated a-priori. Clearly, Assumption 1 holds for (4). In particular, when $\mathcal{A} = 0$ the function $\beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, t - t_0)$ is defined as $\beta(|x_1(t_0)|, t - t_0) = e^{-\lambda^*(t-t_0)}|x_1(t_0)|$, and coefficient $c = C_1/\lambda^*$ where C_1 is the Lipschitz constant of $c_1(x_2, t)$ with respect to x_2.

The system $\mathcal{S}_w$ stands for a critically stable, explorative, wandering subsystem (compartment). We will restrict our attention to those systems $\mathcal{S}_w$ that satisfy the following constraints:

Assumption 2 (Critically stable, wandering dynamics): The system $\mathcal{S}_w$ is forward-complete:

$$u_w(t) \in \mathcal{U}_w \Rightarrow \mathbf{z}(t) \in \mathcal{Z},\ \forall\ t \geq t_0,\ t_0 \in \mathbb{R}_+$$

and there exists an "output" function $h : \mathbb{R}^m \to \mathbb{R}$, and two "bounding" functions $\gamma_0 \in \mathcal{K}_{\infty,e}$, $\gamma \in \mathcal{K}_{\infty,e}$ such that the following integral inequality holds:

$$\mathcal{S}_w: \int_{t_0}^{t} \gamma_1(u_w(\tau))d\tau \leq h(\mathbf{z}(t_0)) - h(\mathbf{z}(t)) \leq \int_{t_0}^{t} \gamma_0(u_w(\tau))d\tau,\ \forall\ t \geq t_0,\ t_0 \in \mathbb{R}_+ \tag{9}$$

In case system $\mathcal{S}_w$ is specified in terms of vector-fields

$$\dot{\mathbf{z}} = \mathbf{f}_z(\mathbf{z}, u_w),\ \mathbf{f}_z(\cdot, \cdot) \in \mathcal{C}^1, \tag{10}$$

Assumption 2 can be viewed, for example, as postulating the existence of a function $h : \mathbb{R}^m \to \mathbb{R}_+$ of which the evolution in time is a mere integration of the input $u_w(t)$. In general, for $u_w :\ u_w(t) \geq 0\ \ \forall\ t \in \mathbb{R}_+$, inequality (9) implies *monotonicity* of function $h(\mathbf{z}(t))$ in t. Regarding the function $\gamma_0(\cdot)$ in (9), we assume that for any $M \in \mathbb{R}_+$ there exists a function $\gamma_{0,1} : \mathbb{R}_+ \to \mathbb{R}_+$ and a *non-decreasing* function $\gamma_{0,2} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\gamma_0(a \cdot b) \leq \gamma_{0,1}(a) \cdot \gamma_{0,2}(b), \forall\ a, b \in [0, M]. \tag{11}$$

Requirement (11) is a technical assumption which will be used in the formulation and proof of the main results of the paper. Yet, it is not too restrictive; it holds, for instance, for a wide class of locally Lipschitz functions $\gamma_0(\cdot) :\ \gamma_0(a\cdot b) \leq L_0(M) \cdot (a \cdot b)$, $L_0(M) \in \mathbb{R}_+$. Another example for which the assumption holds is the class of polynomial functions $\gamma_0(\cdot) :\ \gamma_0(a \cdot b) = (a \cdot b)^p = a^p \cdot b^p$, $p > 0$. No further restrictions will be imposed a-priori on $\mathcal{S}_a$, $\mathcal{S}_w$.

Now consider the interconnection of (8), (9) with coupling $u_a(t) = h(\mathbf{z}(t))$, and $u_s(t) = \|\mathbf{x}(t)\|_{\mathcal{A}}$. Equations for the combined system can be written as:

$$\begin{aligned} &\|\mathbf{x}(t)\|_{\mathcal{A}} \leq \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, t - t_0) + c\|h(\mathbf{z}(t))\|_{\infty,[t_0,t]} \\ &\int_{t_0}^{t} \gamma_1(\|\mathbf{x}(\tau)\|_{\mathcal{A}})d\tau \leq h(\mathbf{z}(t_0)) - h(\mathbf{z}(t)) \\ &\qquad\qquad \leq \int_{t_0}^{t} \gamma_0(\|\mathbf{x}(\tau)\|_{\mathcal{A}})d\tau. \end{aligned} \tag{12}$$

A diagram illustrating the general structure of the entire system (12) is given in figure 1.

In what follows we aim to derive simple small-gain conditions for interconnection (12) that can be used to determine state boundedness of the system. Given that conventional notion of the input-output gain hardly applies to subsystem $\mathcal{S}_w$, we do not wish to present these conditions in the standard form, e.g. that *the loop gain is less than unit* [1]. We rather search for conditions that can be formulated as follows:

$$c \cdot G(\beta, \gamma_0, \gamma_1) < 1, \tag{13}$$

where $G(\cdot)$ is a functional $\beta(\cdot)$, $\gamma_0(\cdot)$ and $\gamma_1(\cdot)$ in (12). Despite that the "gains" in (13) refer to the different spaces, equation (13) has familiar small-gain form. Small-gain like conditions (13) follow as a corollary (Corollary 1) from a more general statement (Theorem 1). Detailed formulations of these results are provided in the next section.

IV. MAIN RESULTS

Before we formulate the main results of this section let us first comment briefly on the machinery of our analysis. First of all we introduce three sequences

$$\mathcal{S} = \{\sigma_i\}_{i=0}^{\infty}, \quad \Xi = \{\xi_i\}_{i=0}^{\infty}, \quad \mathcal{T} = \{\tau_i\}_{i=0}^{\infty}$$

The first sequence, $\mathcal{S}$, partitions the interval $[0, h(\mathbf{z}_0)]$, $h(\mathbf{z}_0) > 0$ into the union of shrinking subintervals H_i:

$$[0, h(\mathbf{z}_0)] = \cup_{i=0}^{\infty} H_i,\ H_i = [\sigma_{i+1}h(\mathbf{z}_0), \sigma_i h(\mathbf{z}_0)] \tag{14}$$

We define this property in the form of Property 1

Property 1 (Partition of $\mathbf{z}_0$): The sequence $\mathcal{S}$ is strictly monotone and converging

$$\{\sigma_n\}_{n=0}^{\infty} : \lim_{n\to\infty} \sigma_n = 0,\ \sigma_0 = 1 \tag{15}$$

Sequences Ξ and $\mathcal{T}$ will specify the desired rates $\xi_i \in \Xi$ of the contracting dynamics (8) in terms of function $\beta(\cdot, \cdot)$ and $\tau_i \in \mathcal{T}$. Let us, therefore, impose the following constraint on the choice of Ξ, $\mathcal{T}$.

Property 2 (Rate of contraction, Part 1): For the given sequences Ξ, $\mathcal{T}$ and function $\beta(\cdot, \cdot) \in \mathcal{KL}$ in (8) the following inequality holds:

$$\beta(\cdot, T_i) \leq \xi_i \beta(\cdot, 0),\ \forall\ T_i \geq \tau_i \tag{16}$$

Property 2 states that for the given, yet arbitrary, factor ξ_i and time instant t_0, the amount of time τ_i is needed for the state $\mathbf{x}$ in order to reach the domain:

$$\|\mathbf{x}\|_{\mathcal{A}} \leq \xi_i \beta(\|\mathbf{x}(t_0)\|_{\mathcal{A}}, 0)$$

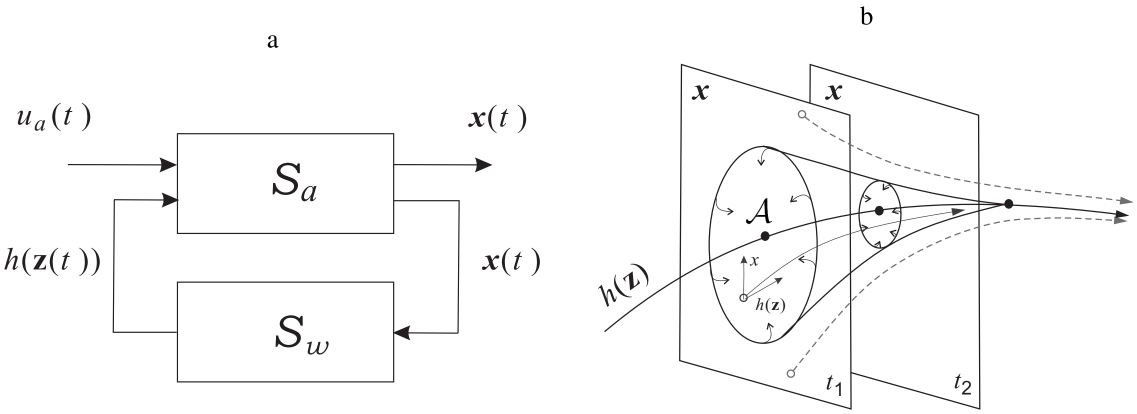


Fig. 1. **a**. The class of interconnected systems S_a and S_w. System $\mathcal{S}_a$, the "contracting system", has an attracting invariant set $\mathcal{A}$ in its state space, system $\mathcal{S}_w$ does not necessarily have an attracting set. It represents the "wandering" dynamics. A typical example of such behavior is the dynamics of the flow in a neighborhood of a saddle point in three-dimensional space (diagram **b**).

In order to specify the desired convergence rates ξ_i, it will be necessary to define another measure in addition to (16). This is a measure of the propagation of initial conditions $\mathbf{x}_0$ and input $h(\mathbf{z}_0)$ to the state $\mathbf{x}(t)$ of the contracting dynamics (8) when the system travels in $h(\mathbf{z}(t)) \in [0, h(\mathbf{z}_0)]$. For this reason we introduce two systems of functions, Φ and Υ:

$$\Phi: \begin{array}{ll} \phi_j(s) & = \phi_{j-1} \circ \rho_{\phi,j}(\xi_{i-j} \cdot \beta(s,0)), \ j = 1, \ldots, i \\ \phi_0(s) & = \beta(s,0) \end{array} \tag{17}$$

$$\Upsilon: \begin{array}{ll} \upsilon_j(s) & = \phi_{j-1} \circ \rho_{\upsilon,j}(s), \ j = 1, \ldots, i \\ \upsilon_0(s) & = \beta(s,0) \end{array} \tag{18}$$

where the functions $\rho_{\phi,j}, \ \rho_{\upsilon,j} \in \mathcal{K}$ satisfy the following inequality

$$\phi_{j-1}(a+b) \leq \phi_{j-1} \circ \rho_{\phi,j}(a) + \phi_{j-1} \circ \rho_{\upsilon,j}(b) \tag{19}$$

Notice that in case $\beta(\cdot, 0) \in \mathcal{K}_\infty$ the functions $\rho_{\phi,j}(\cdot)$, $\rho_{\upsilon,j}(\cdot)$ will always exist [2]. The properties of sequence Ξ which ensure the desired propagation rate of the influence of initial condition $\mathbf{x}_0$ and input $h(\mathbf{z}_0)$ to the state $\mathbf{x}(t)$ are specified in Property 3.

Property 3 (Rate of contraction, Part 2): The sequences

$$\sigma_n^{-1} \cdot \phi_n(\|\mathbf{x}_0\|_{\mathcal{A}}), \quad \sigma_n^{-1} \cdot \left(\sum_{i=0}^{n} \upsilon_i(c|h(\mathbf{z}_0)|\sigma_{n-i}) \right),$$

$n = 0, \ldots, \infty$, are bounded from above, e.g. there exist functions $B_1(\|\mathbf{x}_0\|)$, $B_2(|h(\mathbf{z}_0)|, c)$ such that

$$\sigma_n^{-1} \cdot \phi_n(\|\mathbf{x}_0\|_{\mathcal{A}}) \leq B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) \tag{20}$$

$$\sigma_n^{-1} \cdot \left(\sum_{i=0}^{n} \upsilon_i(c|h(\mathbf{z}_0)|\sigma_{n-i}) \right) \leq B_2(|h(\mathbf{z}_0)|, c) \tag{21}$$

for all $n = 0, 1, \ldots, \infty$
For a large class of functions $\beta(s, 0)$, for instance those that are Lipschitz in s, these conditions reduce to more transparent ones which can always be satisfied by an appropriate choice of sequences Ξ and $\mathcal{S}$. This case is considered in detail as a corollary of the main theorem.

The main differences between the standard and the presently proposed approaches for the analysis of asymptotic behavior of dynamical systems are illustrated with figure 2.

In order to prove the emergence of the trapping region we consider the following collection of volumes induced by the sequence $\mathcal{S}_i$ and the corresponding partition (14) of the interval $[0, h(\mathbf{z}_0)]$:

$$\Omega_i = \{\mathbf{x} \in \mathcal{X}, \mathbf{z} \in \mathcal{Z} | \ h(\mathbf{z}(t)) \in H_i\} \tag{22}$$

For the given initial conditions $\mathbf{x}_0 \in \mathcal{X}$, $\mathbf{z}_0 \in \mathcal{Z}$ two alternative possibilities exist. First, there exists an i such that the trajectory $\mathbf{x}(t, \mathbf{x}_0) \oplus \mathbf{z}(t, \mathbf{z}_0)$ enters Ω_i and stays there forever. Hence for $t \to \infty$ the state will converge into

$$\begin{aligned} \Omega_a = & \{\mathbf{x} \in \mathcal{X}, \ \mathbf{z} \in \mathcal{Z} | \\ & \|\mathbf{x}\|_{\mathcal{A}} \leq c \cdot h(\mathbf{z}_0), \ \mathbf{z}: \ h(\mathbf{z}) \in [0, h(\mathbf{z}_0)]\} \end{aligned} \tag{23}$$

The second alternative is that for each $i = 0, 1, \ldots$ the trajectory $\mathbf{x}(t, \mathbf{x}_0) \oplus \mathbf{z}(t, \mathbf{z}_0)$ enters Ω_i and leaves sometimes later. Let t_i be the time instances when it hits the hyper-surfaces $h(\mathbf{z}(t)) = h(\mathbf{z}_0)\sigma_i$. Then the state of the coupled system stays in $\cup_{i=0}^{\infty} \ \Omega_i$ only if the sequence $\{t_i\}_{i=0}^{\infty}$ diverges. Theorem 1 provides sufficient conditions specifying the latter case in terms of the properties of sequences $\mathcal{S}$, Ξ, $\mathcal{T}$ and function $\gamma_0(\cdot)$ in (12). For a large class of interconnections (12) it is possible to formulate these conditions in terms of the input-output properties of systems $\mathcal{S}_a$ and $\mathcal{S}_w$ explicitly, i.e. in terms of functions $\beta(\cdot, \cdot)$, $\gamma_0(\cdot)$, and the values of c. This is presented an immediate corollary of Theorem 1.

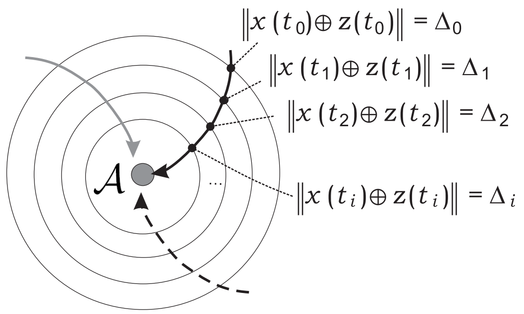
Theorem 1 (Non-uniform Small-gain Theorem 1): Let systems $\mathcal{S}_a$, $\mathcal{S}_w$ be given and satisfy Assumptions 1, 2. Consider their interconnection (12) and suppose there exist sequences $\mathcal{S}$, Ξ, and $\mathcal{T}$ satisfying Properties 1–3. In addition, suppose that the following conditions hold:

1) There exists a positive number $\Delta_0 > 0$ such that

$$\frac{1}{\tau_i} \frac{(\sigma_i - \sigma_{i+1})}{\gamma_{0,1}(\sigma_i)} \geq \Delta_0 \ \forall \ i = 0, 1, \ldots, \infty \tag{24}$$

<table>
<tr><th>Standard</th><th>Proposed</th></tr>
<tr><td>1) Domain of attraction is a neighborhood

2) Implies Lyapunov stability


</td><td>1) Domain of attraction is a set of positive measure (not necessarily a neighborhood)

2) Allows to analyze convergence in Lyapunov-unstable systems

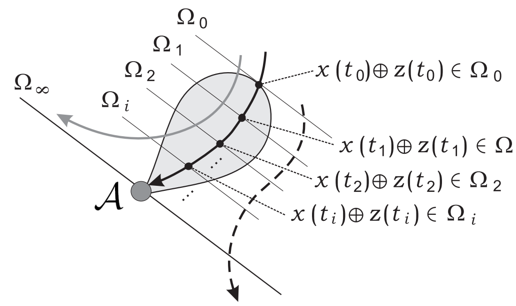

</td></tr>
<tr><td>Given: a sequence of diverging time instances t_i</td><td>Given: a sequence of sets Ω_i whose distance Δ_i to $\mathcal{A}$ is converging to zero</td></tr>
<tr><td>Prove: convergence of norms $\|\mathbf{x}(t_i) \oplus \mathbf{z}(t_i)\| = \Delta_i$ to zero</td><td>Prove: divergence of $\{t_i\}$, where $t_i: \ \mathbf{x}(t_i) \oplus \mathbf{z}(t_i) \in \Omega_i$</td></tr>
</table>

Fig. 2. Key differences between the conventional concept of convergence (left panel) and the concept of weak, non-uniform, convergence (right panel). In the uniform case, trajectories which start in a neighborhood of $\mathcal{A}$ remain in a neighborhood of $\mathcal{A}$ (solid and dashed lines). In the non-uniform case, only a fraction of the initial conditions in a neighborhood of $\mathcal{A}$ will produce trajectories which remain in a neighborhood of $\mathcal{A}$ (solid black line). In the most general case a necessary condition for this to happen is that the sequence $\{t_i\}$ diverges. In our current problem statement divergence of $\{t_i\}$ implies boundedness of $\|\mathbf{x}(t)\|_{\mathcal{A}}$. To show state boundedness and convergence of $\mathbf{x}(t)$ to $\mathcal{A}$ an additional information on the system dynamics will be required.

2) The set Ω_γ of all points $\mathbf{x}_0$, $\mathbf{z}_0$ satisfying the inequality

$$\gamma_{0,2}(B_1(\|\mathbf{x}_0\|_{\mathcal{A}}) + B_2(|h(\mathbf{z}_0)|, c)+c|h(\mathbf{z}_0)|) \leq h(\mathbf{z}_0)\Delta_0 \tag{25}$$

is not empty.

3) Partial sums of elements from $\mathcal{T}$ diverge:

$$\sum_{i=0}^{\infty} \tau_i = \infty \tag{26}$$

Then for all $\mathbf{x}_0$, $\mathbf{z}_0 \in \Omega_\gamma$ the state $\mathbf{x}(t, \mathbf{z}_0) \oplus \mathbf{z}(t, \mathbf{z}_0)$ of system (12) converges into the set specified by (23).
(See [16] for details of the proof).

Remark 1: Conditions 1), 3) of the theorem can be easily checked for the given sequences $\mathcal{S}$, $\mathcal{T}$. Verifying condition 2), however, might be a nontrivial operation. Therefore, a simpler statement that does not involve explicit verification of condition 2) of Theorem 1 is desirable.

In what follows we will show that this goal can be achieved in case additional information about the function $\beta(\cdot, \cdot)$ is available. This information is the knowledge of functions $\beta_x(\cdot)$, $\beta_t(\cdot)$ in the following factorization:

$$\beta(\|\mathbf{x}\|_{\mathcal{A}}, t) \leq \beta_x(\|\mathbf{x}\|_{\mathcal{A}}) \cdot \beta_t(t), \tag{27}$$

where $\beta_x(\cdot) \in \mathcal{K}$ and $\beta_t(\cdot) \in \mathcal{C}^0$ is strictly decreasing[1] with

$$\lim_{t \to \infty} \beta_t(t) = 0 \tag{28}$$

It is shown in [19] (Lemma 8) that factorization (27) is always achievable for any $\mathcal{KL}$ function. In case the function $\beta_x(\cdot)$ in the factorization (27) is Lipschitz the conditions of Theorem 1 reduce to a single and easily verifiable inequality.

Without loss of generality, we assume that the state $\mathbf{x}(t)$ of system $\mathcal{S}_a$ satisfies the following equation

$$\|\mathbf{x}(t)\|_{\mathcal{A}} \leq \|\mathbf{x}(t_0)\|_{\mathcal{A}} \cdot \beta_t(t - t_0) + c \cdot \|h(\mathbf{z}(\tau, \mathbf{z}_0))\|_{\infty, [t_0, t]}, \tag{29}$$

where $\beta_t(0)$ is greater or equal to one. Given that $\beta_t(t)$ is strictly decreasing and continuous, there is a (continuous) mapping $\beta_t^{-1} : [0, \beta_t(0)] \mapsto \mathbb{R}_+$:

$$\beta_t^{-1} \circ \beta_t(t) = t, \ \forall \ t > 0 \tag{30}$$

The small-gain criterion for interconnection (12) in which the dynamics of $\mathcal{S}_a$ is governed by (29) is provided below:

Corollary 1 (Non-Uniform Small-gain Theorem 2): Consider interconnection (12) where the system $\mathcal{S}_a$ satisfies inequality (29) and the function $\gamma_0(\cdot)$ is Lipschitz:

[1] If $\beta_t(\cdot)$ is not strictly monotone, it can always be majorized by a strictly decreasing function

$|\gamma_0(s)| \leq D_{\gamma,0} \cdot |s|$. Then there exists a set Ω_γ of initial conditions corresponding to the trajectories converging to Ω_a if the following condition is satisfied

$$D_{\gamma,0} \cdot c \cdot \mathcal{G} < 1, \tag{31}$$

where

$$\mathcal{G} = \beta_t^{-1}\left(\frac{d}{\kappa}\right)\frac{k}{k-1}\left(\beta_t(0)\left(1+\frac{\kappa}{1-d}\right)+1\right)$$

for some $d \in (0,1)$, $\kappa \in (1,\infty)$. In particular, Ω_γ contains the following domain

$$\|\mathbf{x}(t_0)\|_{\mathcal{A}} \leq \frac{h(\mathbf{z}(t_0))}{\beta_t(0)}\left[\frac{1}{D_{\gamma,0}}\left(\beta_t^{-1}\left(\frac{d}{\kappa}\right)\right)^{-1}\frac{k-1}{k} - c\left(\beta_t(0)\left(1+\frac{\kappa}{1-d}\right)+1\right)\right].$$

In case the function $h(\mathbf{z})$ in (12) is continuous, the volume of the set Ω_γ is nonzero in $\mathbb{R}^n \oplus \mathbb{R}^m$.
Proof of the corollary is provided in [16].

V. DISCUSSION AND CONCLUSION

Let us now briefly outline domains of potential applications of the presented analysis framework (non-uniform small-gain theorems). First of all, it is worth mentioning that the results, as well as other technical statements from [16], apply to problems covered in Examples 1, and 2 in Section I. In addition, the novel framework has recently been shown successful for *solving problems of state and parameter reconstruction for systems in non-canonical adaptive observer form*. In [20], [21] we showed that trading Lyapunov stability for convergence enables solving problems of state and parameter reconstruction for these important classes of systems. Dynamics of such observers satisfy

$$\begin{aligned} \dot{x} &= A(t)x + b(t,x,\lambda') - b(t,x,\lambda), \ \lambda' \in \mathbb{R} \\ \dot{\lambda} &= \gamma\|c^T x\|, \ \gamma \in \mathbb{R}_{>0}, \ c \in \mathbb{R}^n \end{aligned} \tag{32}$$

with $\dot{x} = A(t)x$ being uniformly asymptotically stable and $b(t,x,\lambda)$ being globally Lipschitz in λ respectively. Equations (32) can be directly translated into (12) to which Theorem 1 and Corollary 1 apply. Examples of how these results can be used to tackle the problem of adaptive regulation for systems with general yet Lipschitz nonlinear parameterization are discussed in [16].

More recent developments demonstrate that the non-uniform small-gain approaches can serve as the machinery for solving long-standing ill-posed problems such as proving adaptive capabilities of recurrent neural networks with fixed weights in the problems of adaptive classification of uncertain temporal signals [22].

Presented framework of analysis is not limited to systems in Examples 1, and 2. It can be extended to more general settings, as in Example 3, where interconnection of an input-to-state stable system with a system of which the dynamics is critically stable or in a vicinity of the critical regime. Details of such generalization are available in [17].

An interesting question however remains: whether a similar technique can in principle be derived to deal with interconnections of systems with finite escape time, such as $\dot{x}_1 = -x_1 + x_1^2$, $\dot{\lambda} = -\gamma x_1^2$. Finding answer to this and other related questions is the topic for future studies.

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