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SUFFICIENT OPTIMALITY CONDITIONS FOR DIRICHLET BOUNDARY CONTROL OF WAVE EQUATIONS

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Abstract— **We study optimal control problems for wave equations (focusing on the multidimensional wave equation) with control functions in the Dirichlet boundary conditions under pointwise control (and we admit state - by assuming weak hypotheses) constraints.**

I. INTRODUCTION

The paper is devoted to study optimal control problems for state-constrained wave equations with controls as well in state equation as in Dirichlet boundary conditions. We study problem (P) governed by multidimensional wave equation:

$$\text{minimize } J(x,u,v) = \int_{[0,T]\times\Omega} L(t,z,x(t,z),u(t,z))dtdz + \int_{\Sigma} h(t,z,v(t,z))dtdz + \int_{\Omega} l(x(T,z))\,dz$$

subject to

$$x_{tt}(t,z) - \Delta_z x(t,z) = f(t,z,x(t,z),u(t,z)) \text{ a. e. on } (0,T)\times\Omega \tag{1}$$

$$x(0,z) = \varphi(0,z),\ \ x_t(0,z) = \psi(0,z) \text{ on } \Omega \tag{2}$$

$$x(t,z) = v(t,z) \text{ on } (0,T)\times\Gamma \tag{3}$$

$$u(t,z) \in U \text{ a. e. on } (0,T)\times\Omega \tag{4}$$

$$v(t,z) \in \mathbf{V} \text{ on } (0,T)\times\Gamma \tag{5}$$

where Ω is a given bounded domain of R^n with boundary $\Gamma = \partial\Omega$ of C^2, $\Sigma = (0,T)\times\Gamma$, $U \subset R^m$ and $\mathbf{V} \subset R$ are given nonempty sets, $\mathbf{V}$ - closed; $L,\ f : [0,T]\times\bar{\Omega}\times R\times R^m \to R$, $l : R \to R$, $h : [0,T]\times\bar{\Omega}\times R \to R$ and $\varphi, \psi : R^{n+1} \to R$ are given functions, $\varphi(0,\cdot) \in L^2(\Omega)$, $\psi(0,\cdot) \in H^{-1}(\Omega)$; $x : [0,T]\times\Omega \to R$, $x \in W^{2,2}((0,T)\times\Omega)\cap C([0,T];L^2(\Omega))$ and $u : [0,T]\times\Omega \to R^m$, $v : [0,T]\times\Gamma \to R$ are Lebesgue measurable functions in suitable sets. We assume that the functions L, f, h, l are lower semicontinuous in their domains of definitions. Assuming the lower semicontinuity of theses functions only we admit that state x may satisfy some pointwise state constraints e.g. that $x(t,z) \in C$ for a.e. $(t,z) \in [0,T]\times\Omega$ with C a closed set in R. We call a trio $x(t,z)$, $u(t,z)$, $v(t,z)$ to be admissible if it satisfies (1)-(5) and $L(t,z,x(t,z),u(t,z))$, $h(t,z,v(t,z))$ are summable; then the corresponding trajectory $x(t,z)$ is said to be admissible.

There are only a few results [15], [16] on boundary control problems for the wave equation and/or for other partial differential equations of the hyperbolic type. Generally, hyperbolic equations exhibit less regularity. This is why in that paper we assume that system (1)-(5) admits at least one solution belonging to $W^{2,2}((0,T)\times\Omega)\cap C([0,T];L^2(\Omega))$.

The aim of the paper is to present sufficient optimality conditions for problem (P) in terms of dynamic programming conditions directly. In literature, there is not work which study problem (P) directly by a dynamic programming method. The only results known to the author for parabolic (also abstract case) (see e.g. [5], [6], [14] and literature therein) treat problem (P) first as an abstract problem with an abstract evolution equation (1) and then derive from abstract Hamilton-Jacobi equations suitable sufficient optimality conditions for problem (P). We would like to stress that the problem with Dirichlet boundary control is rather difficult to treat by abstract formulation of the problem as then the abstract space of function on Ω need to depend on control also on $\partial\Omega$ i.e. we need to consider the abstract space depending on control. We refer the reader to [12] and their bibliographies for more discussions on important differences between parabolic and hyperbolic systems.

We propose almost a direct method to study (P) by a dual dynamic programming approach following the method described in [17] for one dimensional case and in [8] for multidimensional case. We move all notions of a dynamic programming to a dual space (the space of multipliers) and then develop a dual dynamic approach together with a dual Hamilton-Jacobi equation and as consequence sufficient optimality conditions for (P). We also define an optimal dual feedback control and we formulate sufficient conditions for optimality in terms of it. Such approach allows us to weak significantly the assumptions on the data. An approximate minimum in terms of the dual dynamic programming is also investigated.

II. A DUAL DYNAMIC PROGRAMMING

Let $P \subset R^{n+3}$ be a set of the variables $(t,z,p) = (t,z,y^0,y)$, $(t,z) \in [0,T]\times\bar{\Omega}$, $y^0 \le 0$, $y \in R$, and let $\bar{c} = (c^0,c) \in R^2$ be fixed. The constant $\bar{c}$ is introduced because of practical purpose only, in order to make easier calculations of some relation stated below for concrete problems (see section Example). We adopt the convention that $\bar{c}p = \left(c^0y^0, cy\right)$ for $(t,z,p) \in P$. Let $\tilde{x} : P \to R$ be such a function of the variables (t,z,p) that for each admissible trajectory $x(t,z)$ there exists a function $p(t,z) = (y^0, y(t,z))$, $p \in W^{2,2}([0,T]\times\bar{\Omega})\cap C([0,T];L^2(\Omega))$, $(t,z,p(t,z)) \in P$ such that

$$x(t,z) = \tilde{x}(t,z,p(t,z)) \text{ for } (t,z) \in [0,T]\times\bar{\Omega}. \tag{6}$$

Now, let us introduce an auxiliary function $V(t,z,p) : P \to R$ being of C^2 such that the following two conditions are

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satisfied:

$$V(t,z,\bar{c}p) = c^0 y^0 V_{y^0}(t,z,\bar{c}p) + cyV_y(t,z,\bar{c}p) = \bar{c}pV_p(t,z,\bar{c}p),$$

$$\text{for } (t,z) \in (0,T)\times\Omega, (t,z,\bar{c}p) \in P \tag{7}$$

$$\nabla_z V(t,z,\bar{c}p)\nu(z) = c^0 y^0 \nabla_z V_{y^0}(t,z,\bar{c}p)\nu(z),$$
$$\text{for } (t,z) \in [0,T]\times\partial\Omega, (t,z,\bar{c}p) \in P. \tag{8}$$

where $\nu(\cdot)$ is the exterior unit normal vector to $\partial\Omega$ and $\nabla V(t,z,p)$ means "∇" of the function $z \to V(t,z,p)$. The condition (7) is a generalization of transversality condition known in classical mechanics as orthogonality of momentum to the front of wave. The condition (8) is of same meaning but taken on the boundary. Similarly as in classical dynamic programming define at $(t,\tilde{p}(\cdot))$, where $\tilde{p}(\cdot) = (\tilde{y}^0, \tilde{y}(\cdot))$ is any function $\tilde{p} \in W^{2,2}(\Omega)$, $(t,z,\tilde{p}(z)) \in P$, $(t,z) \in [0,T]\times\Omega$, a dual value function S_D by the formula

$$S_D(t,\tilde{p}(\cdot)) := \inf\Big\{-c^0\tilde{y}^0 \int_{[t,T]\times\Omega} L(\tau,z,x(\tau,z),u(\tau,z))d\tau dz$$
$$-c^0\tilde{y}^0 \int_\Omega l(x(T,z))dz - c^0\tilde{y}^0 \int_{[t,T]\times\partial\Omega} h(\tau,z,v(\tau,z))d\tau dz\Big\} \tag{9}$$

where the infimum is taken over all admissible trios $x(\tau,\cdot)$, $u(\tau,\cdot)$, $v(\tau,\cdot)$, $\tau \in [t,T]$ such that

$$x(t,z) = \tilde{x}(t,z,\tilde{p}(z)) \quad \text{for} \quad z \in \Omega \tag{10}$$

$$\tilde{x}(t,z,\tilde{p}(z)) = v(t,z) \quad \text{for} \quad z \in \partial\Omega \tag{11}$$

i.e. whose trajectories start at $(t,\tilde{x}(t,\cdot,\tilde{p}(\cdot))$ and for which there exists such a function $p(\tau,z) = (\tilde{y}^0, y(\tau,z))$, $p \in W^{2,2}([t,T]\times\bar{\Omega}) \cap C([t,T];L^2(\Omega))$, $(\tau,z,\bar{c}p(\tau,z)) \in P$, that $x(\tau,z) = \tilde{x}(\tau,z,\bar{c}p(\tau,z))$ for $(\tau,z) \in (t,T)\times\bar{\Omega}$ and

$$y(t,z) = \tilde{y}(z) \quad \text{for} \quad z \in \bar{\Omega}. \tag{12}$$

Then, integrating (7) over Ω, for any function $\tilde{p}(\cdot) = (\tilde{y}^0, \tilde{y}(\cdot))$, $\tilde{p} \in W^{2,2}(\Omega)$, $(t,z,\tilde{p}(z)) \in P$, $(t,z,\bar{c}\tilde{p}(z)) \in P$, such that $x(\cdot,\cdot)$ satisfying $x(t,z) = \tilde{x}(t,z,\tilde{p}(z))$ for $z \in \overline{\Omega}$, is an admissible trajectory, we also have the equalities: $\int_\Omega V(t,z,\bar{c}\tilde{p}(z))dz + \int_{\partial\Omega} \nabla_z V(t,z,\bar{c}\tilde{p}(z))\nu(z)dz$

$= -c\int_\Omega \tilde{y}(z)x(t,z,\tilde{p}(z))dz - S_D(t,\tilde{p}(\cdot))$,
with

$$\int_\Omega c^0\tilde{y}^0 V_{y^0}(t,z,\bar{c}\tilde{p}(z))dz$$

$$+c^0\tilde{y}^0 \int_{\partial\Omega} \nabla_z V_{y^0}(t,z,\bar{c}\tilde{p}(z))\nu(z)dz = -S_D(t,\tilde{p}(\cdot)), \tag{13}$$

and assuming

$$\tilde{x}(t,z,\tilde{p}(z)) = -V_y(t,z,\bar{c}\tilde{p}(z)),$$

$$\text{for } (t,z) \in (0,T)\times\bar{\Omega}, (t,z,\bar{c}\tilde{p}(z)) \in P.$$

Denote by the symbol $\Delta_z h$ the sum of the second partial derivatives of the function $h : P \longrightarrow R$ with respect to the variable z_i, $i = 1,\ldots,n$, i. e.

$$\Delta_z h(t,z,p) := \sum_{i=1}^n \frac{\partial^2 h}{\partial z_i^2}(t,z,p). \tag{14}$$

It turns out that the function $V(t,z,p)$ being defined by (II), (13) satisfies the second order partial differential system

$$V_{tt}(t,z,\bar{c}p) - \Delta_z V(t,z,\bar{c}p) + H(t,z,-V_y(t,z,\bar{c}p),\bar{c}p) = 0, \tag{15}$$

$$(t,z) \in (0,T)\times\Omega, (t,z,\bar{c}p) \in P,$$

$$\nabla_z V(t,z,\bar{c}p)\nu(z) + H_\Sigma(t,z,\bar{c}p) = 0,$$

$$(t,z) \in (0,T)\times\partial\Omega, (t,z,\bar{c}p) \in P$$

where

$$H(t,z,x,\bar{c}p) = c^0 y^0 L(t,z,x,u(t,z,p)) \tag{16}$$

$$+cyf(t,z,x,u(t,z,p)), \quad H_\Sigma(t,z,\bar{c}p) = c^0 y^0 h(t,z,v(t,z,p))$$

and $u(t,z,p)$, $v(t,z,p)$ are optimal dual feedback controls, respectively on $(0,T)\times\Omega$ and $(0,T)\times\partial\Omega$, and the dual second order partial differential system of multidimensional dynamic programming (DSPDEMDP)

$$\sup\big\{V_{tt}(t,z,\bar{c}p) - \Delta_z V(t,z,\bar{c}p) + c^0 y^0 L(t,z,-V_y(t,z,\bar{c}p),u)$$
$$+cyf(t,z,-V_y(t,z,\bar{c}p),u) : u \in U\big\} = 0, \tag{17}$$

$$(t,z) \in (0,T)\times\Omega, (t,z,\bar{c}p) \in P$$

$$\sup\{\nabla_z V(t,z,\bar{c}p)\nu(z) + c^0 y^0 h(t,z,v) : v \in \mathbf{V}\} = 0,$$

$$(t,z) \in (0,T)\times\partial\Omega, (t,z,\bar{c}p) \in P. \tag{18}$$

Let us note that the function $\tilde{x}(t,z,p)$ introduced at the beginning of this section a little bit artificially in fact it is defined by $-V_y(t,z,p)$, where V is a solution to (17), i.e. knowing the set P and V_y we are able to know the set $\dot{X}$ where our original problem we need to consider.

III. A VERIFICATION THEOREM

The most important conclusion of a dynamic programming is a verification theorem. We present it in a dual form accordingly to our dual dynamic programming approach described in the previous section.

Theorem Let $\bar{x}(t,z)$, $\bar{u}(t,z)$, $(t,z) \in (0,T)\times\bar{\Omega}$, $\bar{v}(t,z)$, $(t,z) \in (0,T)\times\partial\Omega$, be an admissible trio. Assume that there exist $\bar{c} = (c^0,c) \in R^2$ and a C^2 solution $V(t,z,p)$ of DSPDEMDP (17) on P such that (7), (8) hold. Let further $\overline{p}(t,z) = (\overline{y}^0, \overline{y}(t,z))$, $\overline{p} \in W^{2,2}([0,T]\times\Omega) \cap C([0,T];L^2(\Omega))$, $\bar{p} \in L^2([0,T]\times\partial\Omega)$, $(t,z,\overline{cp}(t,z)) \in P$, be such a function that $\overline{x}(t,z) = -V_y(t,z,\overline{cp}(t,z))$ for $(t,z) \in (0,T)\times\bar{\Omega}$. Suppose that $V(t,z,p)$ satisfies the boundary condition for $(T,z,\bar{c}p) \in P$,

$$c^0\overline{y}^0 \int_\Omega (d/dt)V_{y^0}(T,z,\bar{c}p)dz = c^0\overline{y}^0 \int_\Omega l(-V_y(T,z,\bar{c}p))dz. \tag{19}$$

Moreover, assume that

$$\begin{aligned} V_{tt}(t,z,\overline{cp}(t,z)) - \Delta_z V(t,z,\overline{cp}(t,z)) \\ +c^0\overline{y}^0 L(t,z,-V_y(t,z,\overline{cp}(t,z)),\overline{u}(t,z)) \\ +c\overline{y}(t,z) f(t,z,-V_y(t,z,\overline{cp}(t,z)),\overline{u}(t,z)) = 0, \\ \text{for } (t,z)\in(0,T)\times\Omega, (\nabla_z)V(t,z,\overline{cp}(t,z))\nu(z) \\ +c^0y^0 h(t,z,\bar{v}(t,z)) = 0, \text{ for } (t,z)\in(0,T)\times\partial\Omega. \end{aligned} \tag{20}$$

Then $\bar{x}(t,z)$, $\bar{u}(t,z)$, $(t,z)\in(0,T)\times\Omega$, $\bar{v}(t,z)$, $(t,z)\in(0,T)\times\partial\Omega$, is an optimal trio relative to all admissible trios $x(t,z)$, $u(t,z)$, $(t,z)\in(0,T)\times\Omega$, $v(t,z)$, $(t,z)\in(0,T)\times\partial\Omega$, for which there exists such a function $p(t,z)=(\overline{y}^0,y(t,z))$, $p\in W^{2,2}([0,T]\times\Omega)\cap C([0,T];L^2(\Omega))$, $p\in L^2([0,T]\times\partial\Omega)$, $(t,z,\bar{c}p(t,z))\in P$, that $x(t,z)=-V_y(t,z,\bar{c}p(t,z))$ for $(t,z)\in(0,T)\times\Omega$, $v(t,z)=-V_y(t,z,\bar{c}p(t,z))$ for $(t,z)\in(0,T)\times\partial\Omega$ and

$$y(0,z)=\overline{y}(0,z) \quad \text{for} \quad z\in\Omega. \tag{21}$$

Proof: Let $x(t,z)$, $u(t,z)$, $(t,z)\in(0,T)\times\Omega$, $v(t,z)$, $(t,z)\in(0,T)\times\partial\Omega$, be an admissible trio for which there exists such a function $p(t,z)=(\overline{y}^0,y(t,z))$, $p\in W^{2,2}([0,T]\times\Omega)\cap C([0,T];L^2(\Omega))$, $p\in L^2([0,T]\times\partial\Omega)$, $(t,z,\bar{c}p(t,z))\in P$, that $x(t,z)=-V_y(t,z,\bar{c}p(t,z))$ for $(t,z)\in(0,T)\times\Omega$, $v(t,z)=-V_y(t,z,\bar{c}p(t,z))$ for $(t,z)\in(0,T)\times\partial\Omega$ and (21) is satisfied. From transversality condition (7), (8), we obtain that for $(t,z)\in(0,T)\times\Omega$,

$$\begin{aligned} &V_{tt}(t,z,\bar{c}p(t,z)) - \Delta_z V(t,z,\bar{c}p(t,z)) = \\ &c^0\overline{y}^0\left[\left(d^2/dt^2\right)V_{y^0}(t,z,\bar{c}p(t,z)) - (\Delta_z)V_{y^0}(t,z,\bar{c}p(t,z))\right] \\ &+cy(t,z)\left[\left(d^2/dt^2\right)V_y(t,z,\bar{c}p(t,z)) - (\Delta_z)V_y(t,z,\bar{c}p(t,z))\right], \end{aligned} \tag{22}$$

(since V is of C^2, $V_y(t,z,\bar{c}p(t,z))=-x(t,z)$ and $x\in W^{2,2}([0,T]\times\Omega)$ therefore, by (7), the above derivatives make sense) and for $(t,z)\in(0,T)\times\partial\Omega$,

$$\begin{aligned} (\nabla_z)V(t,z,\bar{c}p(t,z))\nu(z) \\ = c^0\overline{y}^0(\nabla_z)V_{y^0}(t,z,\bar{c}p(t,z))\nu(z). \end{aligned} \tag{23}$$

Since $x(t,z)=-V_y(t,z,\bar{c}p(t,z))$, for $(t,z)\in(0,T)\times\bar{\Omega}$, (1) shows that for $(t,z)\in(0,T)\times\Omega$,

$$\begin{aligned} \left(d^2/dt^2\right)V_y(t,z,\bar{c}p(t,z)) - (\Delta_z)V_y(t,z,\bar{c}p(t,z)) = \\ -f(t,z,-V_y(t,z,\bar{c}p(t,z)),u(t,z)) \end{aligned} \tag{24}$$

and boundary control (3) shows that for $(t,z)\in(0,T)\times\partial\Omega$,

$$-V_y(t,z,\bar{c}p(t,z)) = v(t,z).$$

We conclude from (22)-(24) that for $(t,z)\in(0,T)\times\Omega$,

$$\begin{aligned} c^0\overline{y}^0[\left(d^2/dt^2\right)V_{y^0}(t,z,\bar{c}p(t,z)) - (\Delta_z)V_{y^0}(t,z,\bar{c}p(t,z)) \\ +L(t,z,-V_y(t,z,\bar{c}p(t,z)),u(t,z))] = V_{tt}(t,z,p(t,z)) \\ -\Delta_z V(t,z,p(t,z)) + c^0\overline{y}^0 L(t,z,-V_y(t,z,\bar{c}p(t,z)),u(t,z)) \\ +cy(t,z)f(t,z,-V_y(t,z,\bar{c}p(t,z)),u(t,z)) \end{aligned} \tag{25}$$

and for $(t,z)\in(0,T)\times\partial\Omega$,

$$\begin{aligned} c^0\overline{y}^0(\nabla_z)V_{y^0}(t,z,\bar{c}p(t,z))\nu(z) + c^0\overline{y}^0 h(t,z,v(t,z)) \\ = (\nabla_z)V(t,z,\bar{c}p(t,z))\nu(z) + c^0\overline{y}^0 h(t,z,v(t,z)), \end{aligned} \tag{26}$$

Hence, by (17) and (25), we infer that

$$\begin{aligned} c^0\overline{y}^0[\left(d^2/dt^2\right)V_{y^0}(t,z,\bar{c}p(t,z)) - (\Delta_z)V_{y^0}(t,z,\bar{c}p(t,z)) \\ +L(t,z,-V_y(t,z,\bar{c}p(t,z)),u(t,z))] \le 0 \quad \text{for } (t,z)\in(0,T)\times\Omega \end{aligned} \tag{27}$$

and for $(t,z)\in(0,T)\times\partial\Omega$

$$c^0\overline{y}^0(\nabla_z)V_{y^0}(t,z,\bar{c}p(t,z))\nu(z) + c^0\overline{y}^0 h(t,z,v(t,z)) \le 0 \tag{28}$$

and finally, after integrating (27) that

$$\begin{aligned} c^0\overline{y}^0\int_{[0,T]\times\Omega}\Big[\left(d^2/dt^2\right)V_{y^0}(t,z,\bar{c}p(t,z)) \\ -(div\nabla_z)V_{y^0}(t,z,\bar{c}p(t,z))dtdz \\ \le -c^0\overline{y}^0\int_{[0,T]\times\Omega}L(t,z,x(t,z),u(t,z))dtdz. \end{aligned} \tag{29}$$

Similarly in the set $(0,T)\times\partial\Omega$ we have

$$\begin{aligned} c^0\overline{y}^0\int_{[0,T]\times\partial\Omega}(\nabla_z)V_{y^0}(t,z,\bar{c}p(t,z))\nu(z)dtdz \\ \le -c^0\overline{y}^0\int^0_{[0,T]\times\partial\Omega}h(t,z,v(t,z))dtdz. \end{aligned} \tag{30}$$

Thus from (29), (30), (19), (21), and the Green formula it follows that

$$\begin{aligned} c^0\overline{y}^0\int_\Omega[l(-V_y(T,z,\bar{c}p(T,z))) \\ -(d/dt)V_{y^0}(0,z,c^0\overline{y}^0,c\overline{y}(0,z))dz \\ -c^0\overline{y}^0\int_{[0,T]}\left(\int_{\partial\Omega}(\nabla_z)V_{y^0}(t,z,c^0\overline{y}^0,cy(t,z))\nu(z)dz\right)dt \\ \le -c^0\overline{y}^0\int_{[0,T]\times\Omega}L(t,z,x(t,z),u(\tau,z))dtdz. \end{aligned} \tag{31}$$

So by (31) and (30) we get

$$\begin{aligned} -c^0\overline{y}^0\int_\Omega(d/dt)V_{y^0}(0,z,c^0\overline{y}^0,c\overline{y}(0,z))dz \\ \le -c^0\overline{y}^0\int_{[0,T]\times\Omega}L(t,z,x(t,z),u(t,z))dtdz \\ -c^0\overline{y}^0\int_\Omega l(x(T,z))dz - c^0\overline{y}^0\int^0_{[0,T]\times\partial\Omega}h(t,z,v(t,z))dtdz. \end{aligned} \tag{32}$$

In the same manner applying (20) and (25) we have for $(t,z)\in(0,T)\times\Omega$

$$\begin{aligned} c^0\overline{y}^0[\left(d^2/dt^2\right)V_{y^0}(t,z,\overline{cp}(t,z)) - (\Delta_z)V_{y^0}(t,z,\overline{cp}(t,z)) \\ +L(t,z,-V_y(t,z,\overline{cp}(t,z)),\bar{u}(t,z))] = 0, \end{aligned}$$

and for $(t,z)\in(0,T)\times\partial\Omega$,

$$c^0\overline{y}^0(\nabla_z)V_{y^0}(t,z,\overline{cp}(t,z))\nu(z) + c^0\overline{y}^0 h(t,z,\bar{v}(t,z)) = 0.$$

Further we have

$$\begin{aligned} -c^0\overline{y}^0\int_\Omega(d/dt)V_{y^0}(0,z,c^0\overline{y}^0,c\overline{y}(0,z))dz \\ = -c^0\overline{y}^0\int_{[0,T]\times\Omega}L(t,z,\bar{x}(t,z),\bar{u}(t,z))dtdz \\ -c^0\overline{y}^0\int_\Omega l(\bar{x}(T,z))dz - c^0\overline{y}^0\int_{[0,T]\times\partial\Omega}h(t,z,\bar{v}(t,z))dtdz. \end{aligned} \tag{33}$$

Combining (32) with (33) gives

$$-c^0\overline{y}^0\int_{[0,T]\times\Omega}L(t,z,\overline{x}(t,z),\overline{u}(t,z))dtdz-c^0\overline{y}^0\int_\Omega l(\overline{x}(T,z))dz$$
$$-c^0\overline{y}^0\int^0_{[0,T]\times\partial\Omega}h(t,z,\overline{v}(t,z))dtdz\le$$
$$-c^0\overline{y}^0\int_{[0,T]\times\Omega}L(t,z,x(t,z),u(t,z))dtdz$$
$$-c^0\overline{y}^0\int_\Omega l(x(T,z))dz-c^0\overline{y}^0\int^0_{[0,T]\times\partial\Omega}h(t,z,v(t,z))dtdz, \quad (34)$$

which completes the proof. ■

Remark The requirement in the theorem that $V(t,z,p)$ is a C^2 solution of DSPDEMDP (17) on P such that (7), (8) and (19) hold look very complicated and difficult to be satisfied. However if we rewrite it to more nice form it is not different much from known PDE systems. Thus let us assume that $l\equiv 0$, put $w=V_y$ and rewrite (7) to the form $V_{y^0}=\frac{1}{c^0y^0}V-\frac{y}{c^0y^0}w$ (for $y^0<0$) then we get that V must satisfy in P the following system of equations

$$V_{y^0}=\frac{1}{c^0y^0}V-\frac{cy}{c^0y^0}w, \quad (35)$$
$$V_y=w,$$
$$V_{tt}-\Delta_zV+H(t,z,-w,\bar{c}p)=0$$

with initial (end) condition

$$V_{y^0}(T,z,\bar{c}p)=0$$

and Neumann boundary condition

$$\nabla_zV(t,z,\bar{c}p)\nu(z)+H_\Sigma(t,z,\bar{c}p)=0,$$
$$(t,z)\in(0,T)\times\partial\Omega,(t,z,\bar{c}p)\in P, \quad (36)$$

where H and H_Σ are defined in the former section. If it happens that both H and H_Σ are smooth enough functions and $n\ge 4$ (dimension of Ω) then the existence of continuous and then smooth solutions for (35)-(36) can be obtained by standard fixed point method (compare [16] and the smooth case for $\Omega=R^n$ [18] , see also [13])

IV. AN OPTIMAL DUAL FEEDBACK CONTROL

It often occurs that for engineers and in practice a feedback control is more important than a value function. It turns out that dual dynamic programming approach allows also to investigate a kind of feedback control which we name a dual feedback control. Surprisingly it can have a better properties than classical one - now our state equation depends only on parameter and not additionally on state in feedback function, which made the state equation difficult in order to solve.

Definition A pair of functions $u=\tilde{u}(t,z,p)$ from P of the points $(t,z,p)=(t,z,y^0,y)$, $(t,z)\in(0,T)\times\Omega$, $y^0\le 0$, $y\in R$, into U and $\tilde{v}(t,z,p)$ from a subset P of those points $(t,z,p)=(t,z,y^0,y)$, $(t,z)\in(0,T)\times\partial\Omega$, $(t,z,p)\in P$, into $\mathbf{V}$ is called a dual feedback control, if there is any solution $\tilde{x}(t,z,p)$, P, of the partial differential equation

$$\tilde{x}_{tt}(t,z,p)-\Delta_z\tilde{x}(t,z,p)=f(t,z,\tilde{x}(t,z,p),\tilde{u}(t,z,p)) \quad (37)$$

satisfying boundary condition

$$\tilde{x}(t,z,p)=\tilde{v}(t,z,p)\quad\text{on}\quad(0,T)\times\Gamma,(t,z,p)\in P$$

such that for each admissible trajectory $x(t,z)$, $(t,z)\in[0,T]\times\Omega$, there exists such a function $p(t,z)=(y^0,y(t,z))$, $p\in W^{2,2}([0,T]\times\Omega)\cap C([0,T];L^2(\Omega))$, $p\in L^2([0,T]\times\partial\Omega)$, $(t,z,p(t,z))\in P$, that (6) holds.

Definition A dual feedback control $(\overline{u}(t,z,p)$, $\overline{v}(t,z,p))$ is called an optimal dual feedback control, if there exist a function $\overline{x}(t,z,p)$, $(t,z,p)\in P$, corresponding to $\overline{u}(t,z,p)$, $\overline{v}(t,z,p))$ as in Definition IV, and a function $\overline{p}(t,z)=(\overline{y}^0,\overline{y}(t,z))$, $\overline{p}\in W^{2,2}([0,T]\times\Omega)\cap C([0,T];L^2(\Omega))$, $\overline{p}\in L^2([0,T]\times\partial\Omega)$, $(t,z,\overline{p}(t,z))\in P$, $(t,z,\overline{cp}(t,z))\in P$ with $\overline{c}=(c^0,c)$, such that dual value function S_D (see (9)) is defined at $(t,\overline{p}(t,\cdot))$ by $\overline{u}(\tau,z,p)$, $\overline{v}(\tau,z,p))$ and corresponding to them $\overline{x}(\tau,z,p)$, $(\tau,z,p)\in P$, $\tau\in[t,T]$, i.e.

$$S_D(t,\overline{p}(t,\cdot))$$
$$=-c^0\overline{y}^0\int_{[t,T]\times\Omega}L(\tau,z,\overline{x}(\tau,z,\overline{p}(\tau,z)),\overline{u}(\tau,z,\overline{p}(\tau,z)))d\tau dz$$
$$-c^0\overline{y}^0\int_\Omega l\left(\overline{x}(T,z,\overline{p}(T,z))\right)dz$$
$$-c^0y^0\int_{[t,T]\times\partial\Omega}h(\tau,z,\overline{v}(\tau,z,\overline{p}(\tau,z)))d\tau dz$$

and moreover there is $V(t,z,p)$ satisfying (7) and 8 for which V_{y^0} satisfies the equality

$$\int_\Omega c^0y^0V_{y^0}(t,z,\overline{cp}(t,z))dz$$
$$+c^0\overline{y}^0\int_{\partial\Omega}(\nabla_z)V_{y^0}(t,z,\overline{cp}(t,z))\nu(z)dz=-S_D\left(t,\overline{p}(t,\cdot)\right)$$

and V_y satisfies

$$V_y(t,z,\overline{c}p)=-\overline{x}(t,z,p)$$

The next theorem is nothing more than the above verification theorem formulated in terms of a dual feedback control.

Theorem Let $(\overline{u}(t,z,p)$, $\overline{v}(t,z,p))$ be a dual feedback control in P. Suppose that there exist $\overline{c}=(c^0,c)\in R^2$ and a C^2 solution $V(t,z,p)$ of (17) on P such that (7) and (19) hold. Let $\overline{p}(t,z)=(\overline{y}^0,\overline{y}(t,z))$, $\overline{p}\in W^{2,2}([0,T]\times\Omega)\cap C([0,T];L^2(\Omega))$, $\overline{p}\in L^2([0,T]\times\partial\Omega)$, $(t,z,\overline{p}(t,z))\in P$, $(t,z,\overline{cp}(t,z))\in P$, be such a function that $\overline{x}(t,z)=\overline{x}(t,z,\overline{p}(t,z))$, $\overline{u}(t,z)=\overline{u}(t,z,\overline{p}(t,z))$,$(t,z)\in(0,T)\times\Omega$, $\overline{v}(t,z)=\overline{v}(t,z,\overline{p}(t,z))$, $(t,z)\in(0,T)\times\partial\Omega$, is an admissible trio, where $\overline{x}(t,z,p)$, $(t,z,p)\in P$, is corresponding to $\overline{u}(t,z,p)$ and $\overline{v}(t,z,p)$ as in Definition IV. Assume further that V_y and V_{y^0} satisfy:

$$V_y(t,z,\overline{c}p)=-\overline{x}(t,z,p)$$
$$(t,z)\in[0,T]\times\Omega,(t,z,p)\in P,\ (t,z,\overline{c}p)\in P,$$

$$c^0\overline{y}^0\int_\Omega V_{y^0}(t,z,\overline{cp}(t,z))dz$$
$$+c^0\overline{y}^0\int_{[0,T]}\left(\int_{\partial\Omega}(\nabla_z)V_{y^0}(t,z,c^0\overline{y}^0,c\overline{y}(t,z))\nu(z)dz\right)dt$$
$$=-c^0\overline{y}^0\int_{[0,T]\times\Omega}L(t,z,\overline{x}(t,z,\overline{p}(t,z)),\overline{u}(t,z,\overline{p}(t,z)))dtdz$$
$$-c^0\overline{y}^0\int_\Omega l\left(\overline{x}(T,z,\overline{p}(T,z))\right)dz$$
$$-c^0\overline{y}^0\int_{[t,T]\times\partial\Omega}h(\tau,s,\overline{v}(\tau,z,\overline{p}(\tau,z)))d\tau ds.$$

Then $(\bar{u}(t,z,p),\ \bar{v}(t,z,p))$ is an optimal dual feedback control.

V. EXAMPLE

The example is linear but with control on the boundary so it cannot be treated by any classical dynamic programming approach. Moreover it is an applied example.

We shall consider a simply case of structural acoustic system. Let $\Omega = \{(z_1,z_2) : -1 \le z_1 \le 1,\ -1 \le z_2 \le 1\}$ be the domain occupied by an acoustic medium (air). The boundary S of the domain consists of two parts S_1 and S_2. The part S_1 corresponds to a thin wall (a shell) and S_2 corresponds to a hard wall. An external acoustic eld, through structural acoustic coupling, leads to high interior sound pressure levels in Ω . Piezoelectric elements (patches) used to active control in order to reduce the sound pressure levels in Ω. Coupled problems for structural acoustic systems were studied in a series of works, see [1]–[4] , [10], [11] and references therein. The acoustic dynamics is described by the equation

$$x_{tt}(t,z) - c_0^2 \Delta_z x(t,z) = 0 \text{ in } Q = (0,1)\times\Omega. \tag{33}$$

Here c_0^2 is a positive constant and the pressure function q in the acoustic medium is defined by $q(t,z) = \rho_0 x_t(t,z)$ where ρ_0 is the density of the acoustic medium in the ground state, ρ_0 a positive constant. Let us put:

$$\begin{aligned}
&L(t,z,x,u) := 0\\
&f(t,z,x,u) := 0\\
&h(t,z,v) := \Big(v - \cos(t-\tfrac{\pi}{2})\sin(\tfrac{1}{2c_0}z_1)\sin(\tfrac{1}{2c_0}z_2)\Big)^2\\
&l(x(\cdot)) = \rho_0 \textstyle\int_\Omega x(1,z)\,dz,\ \varphi(0,z) = 0
\end{aligned}$$

where $(t,z,x,u) \in [0,1]\times\Omega\times R\times R$, $n = 2$, $Y := \{(y^0,y) \in R^2 : y^0 \le 0,\ \ y \in R\}$, $\bar{c} := (c^0,c) \in R^2$, $\bar{c}p \in Y$, so $c^0 > 0$, $c \in R$, $U = R$, $\mathbf{V} = R$. We have chosen the simplest case of h in order one does not concentrate on technicalities which are not related to the described dual method itself.

The Hamiltonian $H : [0,1]\times\Omega\times R\times Y \to R$

$$H(t,z,x,\bar{c}p) = 0.$$

Therefore the first equation of DPDEMDP (17) has the form

$$V_{tt}(t,z,\bar{c}p) - c_0^2\Delta_z V(t,z,\bar{c}p) = 0 \text{ in } Q = (0,1)\times\Omega\times R.$$

The solutions to (33), depending on $\{a_j\}$, have the form

$$\begin{aligned}
x(t,z) = \sum_{j\ge 2} a_j \cos(jt - \frac{\pi}{2})\sin(\frac{1}{2c_0}jz_1)\sin(\frac{1}{2c_0}jz_2)\\
+\frac{1}{2}\cos(t-\frac{\pi}{2})\sin(\frac{1}{2c_0}z_1)\sin(\frac{1}{2c_0}z_2).
\end{aligned}$$

We take into account only those solutions which belong to $W^{2,2}((0,1)\times\Omega)\cap C([0,1];L^2(\Omega))$. By controls on the boundary we take just the functions

$$v = x\big|_{(0,1)\times\partial\Omega}\ .$$

Our aim is to minimize the pressure level in Ω and the cost of activations of controls on the boundary i.e. we minimize the functional

$$J(x,v) = \rho_0\int_\Omega x(1,z)\,dz + \int_{(0,1)\times\partial\Omega} h(t,z,v(t,z))dtdz.$$

For solution to (**??**) we take

$$\begin{aligned}
V(t,z,\bar{c}p) = c^0y^0\sin(t-\frac{\pi}{2})\sin(\frac{1}{2c_0}z_1)\sin(\frac{1}{2c_0}z_2) + c^0y^0cyt\\
+(cy)^2 + tcy - cy\cos(t-\frac{\pi}{2})\sin(\frac{1}{2c_0}z_1)\sin(\frac{1}{2c_0}z_2).
\end{aligned}$$

If we take $c^0 = 1$, $c = 1/2$, $\bar{y}^0 = -1$ then we easily check that V from (**??**) satisfies transversality conditions (7), (8) and boundary condition (19). Let us observe that

$$-V_y(t,z,\bar{c}p(t,z)) = x(t,z)$$

for $\bar{y}^0 = -1$,

$$y(t,z) = -2\sum_{j\ge 2} a_j\cos(jt-\frac{\pi}{2})\sin(\frac{1}{2c_0}jz_1)\sin(\frac{1}{2c_0}jz_2).$$

Let us take for $\nu(z)$ the exterior unit normal vector to $\partial\Omega$ the vector

$$\nu(z) = \begin{cases} (1,0) \text{ for } z = (-1,z_2),\ -1\le z_2\le 1,\\ (1,0) \text{ for } z = (1,z_2),\ -1\le z_2\le 1,\\ (0,1) \text{ for } z = (z_1,-1),\ -1\le z_1\le 1,\\ (0,1) \text{ for } z = (z_1,1),\ -1\le z_1\le 1, \end{cases}$$

Then, taking into account (**??**) we see that $(\nabla_z)V(t,z,\overline{cp}(t,z))\nu(z) = 0$, for $(t,z) \in (0,1)\times\partial\Omega$. Therefore $(\nabla_z)V(t,z,\overline{cp}(t,z))\nu(z) + c^0y^0h(t,z,\bar{v}(t,z)) = 0$, for $(t,z)\in(0,1)\times\partial\Omega$ is realized for

$$\bar{v}(t,z) = \cos(t-\frac{\pi}{2})\sin(\frac{1}{2c_0}z_1)\sin(\frac{1}{2c_0}z_2).$$

Hence

$$\bar{x}(t,z) = \cos(t-\frac{\pi}{2})\sin(\frac{1}{2c_0}z_1)\sin(\frac{1}{2c_0}z_2).$$

VI. AN ε-OPTIMIZATION

If we want to solve concrete problem (1)-(4) for particular data then usually we are not able to solve it exactly especially the problem we consider is nonlinear. Therefore each possibility to approximate our optimal problem (1)-(4) may turn out very useful. Below we find a certain type of a such approximation.

Definition Let $\varepsilon > 0$ and $c^0 > 0$ be fixed. A function $S_{\varepsilon D}(t,p(t,\cdot))$ is called an ε -dual value function, if

$$\begin{aligned}
S_D(t,p(t,\cdot)) \le S_{\varepsilon D}(t,p(t,\cdot))\\
\le S_D(t,p(t,\cdot)) - 2\varepsilon c^0\overline{y}^0_\varepsilon T\,vol(\Omega)
\end{aligned}$$

for any fixed $\overline{y}^0_\varepsilon < 0$.

Definition Let $\varepsilon > 0$ and $\overline{c} = (c^0,c)\in R^2$, $c^0 > 0$ be fixed and let $\widetilde{V}(t,z,p)$ be a given C^2 function such that (7) and (19) hold. Let $\overline{x}_\varepsilon(t,z)$, $\overline{u}_\varepsilon(t,z)$, $t\in(0,T)\times\Omega$, $\overline{v}_\varepsilon(t,z)$, $t\in(0,T)\times\partial\Omega$, be an admissible trio and let $\overline{p}_\varepsilon(t,z) = (\overline{y}^0_\varepsilon,\overline{y}_\varepsilon(t,z))$, $\overline{p}_\varepsilon \in W^{2,2}([0,T]\times\Omega)\cap C([0,T];L^2(\Omega))$, $\overline{p}_\varepsilon \in$

$L^2([0,T]\times\partial\Omega)$, $(t,z,\overline{cp}_\varepsilon(t,z))\in P$, be such a function that $\bar{x}_\varepsilon(t,z)=-\widetilde{V}_y(t,z,\overline{cp}_\varepsilon(t,z))$ for $(t,z)\in(0,T)\times\Omega$. The trio $\bar{x}_\varepsilon(t,z)$, $\bar{u}_\varepsilon(t,z)$, $(t,z)\in(0,T)\times\Omega$, $\bar{v}_\varepsilon(t,z)$, $t\in(0,T)\times\partial\Omega$, is called an ε-optimal trio if,

$$\begin{aligned}&-c^0\bar{y}^0_\varepsilon\int_{[0,T]\times\Omega}L(t,z,\bar{x}_\varepsilon(t,z),\bar{u}_\varepsilon(t,z))dtdz\\ &-c^0\bar{y}^0_\varepsilon\int_\Omega l(\bar{x}_\varepsilon(T,z)dz-c^0\bar{y}^0_\varepsilon\int_{[0,T]\times\partial\Omega}h(t,s,\bar{v}_\varepsilon(t,z))dtds\\ &\le -c^0\bar{y}^0_\varepsilon\int_{[0,T]\times\Omega}L(t,z,x(t,z),u(t,z))dtdz\\ &-c^0\bar{y}^0_\varepsilon\int_\Omega l(x(T,z)dz-c^0\bar{y}^0_\varepsilon\int_{[0,T]\times\partial\Omega}h(t,s,v(t,z))dtds\\ &-\varepsilon c^0\bar{y}^0_\varepsilon T\,vol\,(\Omega).\end{aligned}$$

for all admissible trios $x(t,z)$, $u(t,z)$, $t\in(0,T)\times\Omega$, $v(t,z)$, $t\in(0,T)\times\partial\Omega$, for which there exists such a function $p(t,z)=(\bar{y}^0_\varepsilon,y(t,z))$, $p\in W^{2,2}([0,T]\times\Omega)\cap C([0,T];L^2(\Omega))$, $p\in L^2([0,T]\times\partial\Omega)$, $(t,z,\bar{c}p(t,z))\in P$, that

$$x(t,z)=-\widetilde{V}_y(t,z,\bar{c}p(t,z))\ \text{for}\ (t,z)\in(0,T)\times\Omega$$

and

$$y(0,z)=\bar{y}_\varepsilon(0,z)\quad\text{for}\quad z\in\Omega$$

Similarly as The Verification Theorem one can prove the following ε-verification theorem

Theorem Let $\bar{x}_\varepsilon(t,z)$, $\bar{u}_\varepsilon(t,z)$, $(t,z)\in(0,T)\times\Omega$, $\bar{v}_\varepsilon(t,z)$, $t\in(0,T)\times\partial\Omega$, be an admissible trio. Assume that there exist $\varepsilon>0$, $\bar{c}=(c^0,c)\in R^2$, $c^0>0$, and a C^2 function $\widetilde{V}(t,z,p)$ such that for $(t,z,\bar{c}p)\in P$:

$$\begin{aligned}&\sup\Big\{\widetilde{V}_{tt}(t,z,\bar{c}p)-\Delta_z\widetilde{V}(t,z,\bar{c}p)+c^0y^0L(t,z,-\widetilde{V}_y(t,z,\bar{c}p),u)\\ &+cyf(t,z,-\widetilde{V}_y(t,z,\bar{c}p),u):u\in U\Big\}\le-\varepsilon c^0\bar{y}^0_\varepsilon,\\ &\sup\{\nabla_z\tilde{V}(t,z,\bar{c}p)\nu(z)+c^0y^0h(t,z,v):v\in\mathbf{V}\}\le-\varepsilon c^0\bar{y}^0_\varepsilon,\\ &(t,z)\in(0,T)\times\partial\Omega,(t,z,\bar{c}p)\in P\end{aligned}$$

$$\begin{aligned}&\widetilde{V}(t,z,\bar{c}p)=\bar{c}p\widetilde{V}_p(t,z,\bar{c}p),(t,z)\in(0,T)\times\Omega,(t,z,\bar{c}p)\in P,\\ &\nabla_zV(t,z,\bar{c}p)\nu(z)=c^0y^0\nabla_zV_{y^0}(t,z,\bar{c}p)\nu(z),\\ &\text{for}\ (t,z)\in(0,T)\times\partial\Omega,(t,z,\bar{c}p)\in P.\end{aligned}$$

Let further $\bar{p}_\varepsilon(t,z)=(\bar{y}^0_\varepsilon,\bar{y}_\varepsilon(t,z))$, $\bar{p}_\varepsilon\in W^{2,2}([0,T]\times\Omega)\cap C([0,T];L^2(\Omega))$, $\bar{p}_\varepsilon\in L^2([0,T]\times\partial\Omega)$, $(t,z,\overline{cp}_\varepsilon(t,z))\in P$, be such a function that $\bar{x}_\varepsilon(t,z)=-\widetilde{V}_y(t,z,\overline{cp}_\varepsilon(t))$ for $(t,z)\in(0,T)\times\Omega$. Suppose that $\widetilde{V}(t,z,p)$ satisfies the boundary condition for $(T,z,\bar{c}p)\in P$,

$$c^0\bar{y}^0_\varepsilon\int_\Omega(d/dt)\tilde{V}_{y^0}(T,z,\bar{c}p)dz=c^0\bar{y}^0_\varepsilon\int_\Omega l\left(-\tilde{V}_y(T,z,\bar{c}p)\right)dz.$$

Moreover, suppose that for almost all $(t,z)\in(0,T)\times\Omega$,

$$\begin{aligned}&\widetilde{V}_{tt}(t,z,\overline{cp}_\varepsilon(t,z))-\Delta_z\widetilde{V}(t,z,\overline{cp}_\varepsilon(t,z))\\ &+c^0\bar{y}^0_\varepsilon L(t,z,-\widetilde{V}_y(t,z,\overline{cp}_\varepsilon(t,z)),\bar{u}_\varepsilon(t,z))\\ &+c\bar{y}_\varepsilon(t,z)f(t,z,-\widetilde{V}_y(t,z,\overline{cp}_\varepsilon(t,z)),\bar{u}_\varepsilon(t,z))\ge0,\end{aligned}$$

$$(\nabla_z)\tilde{V}(t,z,\overline{cp}_\varepsilon(t,z))\nu(z)+c^0\bar{y}^0_\varepsilon h(t,z,\bar{v}_\varepsilon(t,z))\ge0,$$

for almost all $(t,z)\in(0,T)\times\partial\Omega$. Then $\bar{x}_\varepsilon(t,z)$, $\bar{u}_\varepsilon(t,z)$, $(t,z)\in(0,T)\times\Omega$, $\bar{v}_\varepsilon(t,z)$, $t\in(0,T)\times\partial\Omega$, is an ε-optimal trio relative to all admissible trios $x(t,z)$, $u(t,z)$, $(t,z)\in(0,T)\times\Omega$, $v(t,z)$, $t\in(0,T)\times\partial\Omega$, for which there exists such a function $p(t,z)=(\bar{y}^0_\varepsilon,y(t,z))$, $p\in W^{2,2}([0,T]\times\Omega)\cap C([0,T];L^2(\Omega))$, $p\in L^2([0,T]\times\partial\Omega)$, $(t,z,\bar{c}p(t,z))\in P$, that

$$x(t,z)=-\widetilde{V}_y(t,z,\bar{c}p(t,z))\ \text{for}\ (t,z)\in(0,T)\times\Omega$$

is satisfied.

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