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Convergence analysis of the Max-Plus Finite Element Method for Solving Deterministic Optimal Control Problems

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***Abstract*— We consider the Max-Plus Finite Element Method for Solving Deterministic Optimal Control Problems, which is a max-plus analogue of the Petrov–Galerkin finite element method. This method, that we introduced in a previous work, relies on a max-plus variational formulation. The error in the sup-norm can be bounded from the difference between the value function and its projections on max-plus and min-plus semimodules when the max-plus analogue of the stiffness matrix is exactly known. We derive here a convergence result in arbitrary dimension for approximations of the stiffness matrix relying on the Hamiltonian, and for arbitrary discretization grid. We show that for a class of problems, the error estimate is of order $\delta+\Delta x(\delta)^{-1}$ or $\sqrt{\delta}+\Delta x(\delta)^{-1}$, depending on the choice of the approximation, where δ and Δx are, respectively, the time and space discretization steps. We give numerical examples in dimension 2.**

I. Introduction

We consider the following optimal control problem:

$$\text{maximize } \int_0^T \ell(\mathbf{x}(s), \mathbf{u}(s))\, ds + \phi(\mathbf{x}(T)) \tag{1a}$$

over the set of trajectories $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ satisfying

$$\dot{\mathbf{x}}(s) = f(\mathbf{x}(s), \mathbf{u}(s)), \quad \mathbf{x}(s) \in X, \quad \mathbf{u}(s) \in U \tag{1b}$$

for all $0 \le s \le T$ and

$$\mathbf{x}(0) = x. \tag{1c}$$

Here the *state space* X is a subset of $\mathbb{R}^n$, the set of *control values* U is a subset of $\mathbb{R}^m$, the *horizon* $T > 0$ and the *initial condition* $x \in X$ are given, and we assume that the map $\mathbf{u}(\cdot)$ is measurable and that the map $\mathbf{x}(\cdot)$ is absolutely continuous. We also assume that the *instantaneous reward* or *Lagrangian* $\ell : X \times U \to \mathbb{R}$ and the *dynamics* $f : X \times U \to \mathbb{R}^n$ are sufficiently regular maps and that the *terminal reward* ϕ is a map $X \to \mathbb{R} \cup \{-\infty\}$.

We are interested in the numerical computation of the *value function* v which associates with any $(x,t) \in X \times [0,T]$ the supremum $v(x,t)$ of $\int_0^t \ell(\mathbf{x}(s), \mathbf{u}(s))\, ds + \phi(\mathbf{x}(t))$, under the constraints (1b), for $0 \le s \le t$ and (1c). It is known that, under certain regularity assumptions, v is solution of the Hamilton–Jacobi equation

$$-\frac{\partial v}{\partial t} + H\left(x, \frac{\partial v}{\partial x}\right) = 0, \quad (x,t) \in X \times (0,T], \tag{2}$$

with initial condition $v(x,0) = \phi(x), \quad x \in X$, where $H(x,p) = \sup_{u \in U} \ell(x,u) + p \cdot f(x,u)$ is the *Hamiltonian* of the problem (see, for instance, [1], [2]).

Several techniques have been proposed in the literature to solve this problem. We mention, for example, finite difference schemes and the method of the vanishing viscosity [3], the antidiffusive schemes for advection [4], the so-called discrete dynamic programming method or semi-Lagrangian method [5], [6], [7], [8]. Recently, max-plus methods have been proposed to solve first-order Hamilton–Jacobi equations [9], [10], [11], [12].

Recall that the *max-plus semiring*, $\mathbb{R}_{\max}$, is the set $\mathbb{R} \cup \{-\infty\}$, equipped with the addition $a \oplus b = \max(a,b)$ and the multiplication $a \otimes b = a + b$. In what follows, let S^t denote the *evolution semigroup* of (2), or Lax–Oleinik semigroup, which associates with any map ϕ the function $v^t := v(\cdot, t)$, where v is the value function of the optimal control problem (1). Maslov [13] observed that the semigroup S^t is *max-plus linear*, meaning that for all maps f, g from X to $\mathbb{R}_{\max}$, and for all $\lambda \in \mathbb{R}_{\max}$, we have

$$S^t(f \oplus g) = S^t f \oplus S^t g, \quad S^t(\lambda f) = \lambda S^t f,$$

where $f \oplus g$ denotes the map $x \mapsto f(x) \oplus g(x)$, and λf denotes the map $x \mapsto \lambda \otimes f(x)$.

In [14], [15], we introduced a max-plus analogue of the finite element method, the "MFEM," to solve the deterministic optimal control problem (1). This method approximates the evolution semigroup S^t by means of a nonlinear discrete semigroup, which can be interpreted as the dynamic programming operator of a deterministic zero-sum two players game, with finite action and state spaces (unlike the method of Fleming and McEneaney which leads to a discrete optimal control problem). One interest of the MFEM is to provide, as in the case of the classical finite element method, a systematic way to compute error estimates, which can be interpreted geometrically as "projection" errors. When the value function is nonsmooth, the space of test functions must be different from the space in which the solution is represented, so that our discretization is indeed a max-plus analogue of the Petrov–Galerkin finite element method. A convenient choice of finite elements and test functions includes quadratic functions (also considered by Fleming and McEneaney [10]) and norm-like functions; see section IV.

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In the MFEM, we need to compute the value of the max-plus scalar product $\langle z \mid S^\delta w\rangle$ for each finite element w and each test function z. In some special cases, $\langle z \mid S^\delta w\rangle$ can be computed analytically. In general, we need to approximate this scalar product. Here we consider the approximation $S^\delta w(x) = w(x) + \delta H(x, \nabla w(x))$, for $x \in X$, which is also used in [9]. Our main result, Theorem 13, provides for the resulting discretization of the value function an error estimate of order $\delta + \Delta x(\delta)^{-1}$, where Δx is the "space discretization step," under classical assumptions on the control problem and the additional assumption that the value function v^t is semiconvex for all $t \in [0, T]$. This is comparable with the order obtained in the simplest discretization methods; see [16], [6], [17]. To avoid solving a difficult (nonconvex) optimization problem, we propose a further approximation of the max-plus scalar product $\langle z \mid S^\delta w\rangle$, for which we obtain an error estimate of order $\sqrt{\delta} + \Delta x(\delta)^{-1}$, which is yet comparable to the order of the existing discretization methods. Here, unlike in [14], the discretization grid need not be regular: in Theorem 13, Δx is defined for an arbitrary grid in terms of Voronoi tessellations.

Improved (quadratic) errors estimates are obtained in [18] under strongly regularity asumptions.

The present paper is an abridged version of [15] where proofs can be found.

II. Preliminaries on residuation and projections over semimodules

In this section we recall some classical residuation results (see, for example, [19], [20], [21], [22]) and their application to linear maps on idempotent semimodules (see [23], [24]). We also review some results of [25], [24] concerning projectors over semimodules. Other results on projectors over semimodules appeared in [26], [27].

A. Residuation, semimodules, and linear maps

If $(S, \leq)$ and $(T, \leq)$ are (partially) ordered sets, we say that a map $f : S \to T$ is *monotone* if $s \leq s' \implies f(s) \leq f(s')$. We say that f is *residuated* if there exists a map $f^\sharp : T \to S$ such that

$$f(s) \leq t \iff s \leq f^\sharp(t).$$

Then

$$f^\sharp(t) = \max\{s \in S \mid f(s) \leq t\} \quad \forall t \in T.$$

We shall consider situations where S (or T) is equipped with an idempotent monoid law $\oplus$ (*idempotent* means that $a \oplus a = a$). Then the *natural order* on S is defined by $a \leq b \iff a \oplus b = b$. The supremum law for the natural order, which is denoted by $\bigvee$, coincides with $\oplus$, and the infimum law for the natural order, when it exists, will be denoted by $\bigwedge$. We say that S is *complete* as a naturally ordered set if any subset of S has a least upper bound for the natural order.

If $\mathcal{K}$ is an idempotent semiring, i.e., a semiring whose addition is idempotent, we say that the semiring $\mathcal{K}$ is *complete* if it is complete as a naturally ordered set and if the left and right multiplications $\mathcal{K} \to \mathcal{K}$, $x \mapsto ax$, and $x \mapsto xa$ are residuated. The max-plus semiring, $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$, defined in the introduction, is an idempotent semiring. It is not complete, but it can be embedded into the complete idempotent semiring $\overline{\mathbb{R}}_{\max}$ obtained by adjoining $+\infty$ to $\mathbb{R}_{\max}$, with the convention that $-\infty$ is absorbing for the multiplication.

Semimodules over semirings are defined like modules over rings, mutatis mutandis; see [23], [24]. When $\mathcal{K}$ is a complete idempotent semiring, we say that a (right) $\mathcal{K}$-semimodule $\mathcal{X}$ is *complete* if it is complete as an idempotent monoid and if, for all $u \in \mathcal{X}$ and $\lambda \in \mathcal{K}$, the right and left multiplications, $R^{\mathcal{X}}_\lambda : \mathcal{X} \to \mathcal{X}$, $v \mapsto v\lambda$ and $L^{\mathcal{X}}_u : \mathcal{K} \to \mathcal{X}$, $\mu \mapsto u\mu$, are residuated (for the natural order). In a complete semimodule $\mathcal{X}$, we define, for all $u, v \in \mathcal{X}$,

$$u \backslash v \overset{\text{def}}{=} (L^{\mathcal{X}}_u)^\sharp(v) = \max\{\lambda \in \mathcal{K} \mid u\lambda \leq v\}.$$

The semimodule of functions $\mathcal{K}^X$ is defined naturally. In particular, $\overline{\mathbb{R}}^X_{\max}$ is the set of functions from X to $\mathbb{R} \cup \{\pm\infty\}$, equipped with the pointwise supremum and with the action $(\lambda, u) \mapsto \lambda + u$, where $(\lambda + u)(x) = \lambda + u(x)$, for all $x \in X$, again with the convention that $(-\infty) + \infty = -\infty$.

If $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{K}$-semimodules, we say that a map $A : \mathcal{X} \to \mathcal{Y}$ is *linear* if for all $u, v \in \mathcal{X}$ and $\lambda, \mu \in \mathcal{K}$, $A(u\lambda \oplus v\mu) = A(u)\lambda \oplus A(v)\mu$. Then, as in classical algebra, we use the notation Au instead of $A(u)$. When A is residuated and $v \in \mathcal{Y}$, we use the notation $A \backslash v$ or $A^\sharp v$ instead of $A^\sharp(v)$.

If X and Y are two sets, $\mathcal{K}$ is a complete idempotent semiring, and $a \in \mathcal{K}^{X \times Y}$, we construct the linear operator A from $\mathcal{K}^Y$ to $\mathcal{K}^X$ which associates with any $u \in \mathcal{K}^Y$ the function $Au \in \mathcal{K}^X$ such that $Au(x) = \bigvee_{y \in Y} a(x, y)u(y)$. We say that A is the *kernel operator* with *kernel* or *matrix* a. We shall often use the same notation A for the operator and the kernel (so $A(x, y) = a(x, y)$). In particular, when $\mathcal{K} = \overline{\mathbb{R}}_{\max}$, we have

$$Au(x) = \sup_{y \in Y}(A(x, y) + u(y)). \tag{3}$$

As is well known (see, for instance, [22]), the kernel operator A is residuated, and

$$(A \backslash v)(y) = \bigwedge_{x \in X} A(x, y) \backslash v(x).$$

When $\mathcal{K} = \overline{\mathbb{R}}_{\max}$, $A \backslash v$ is given by

$$(A \backslash v)(y) = \inf_{x \in X}(-A(x, y) + v(x)) = [-A^*(-v)](y), \tag{4}$$

where A^* denotes the *transposed operator* $\mathcal{K}^X \to \mathcal{K}^Y$, which is associated with the kernel $A^*(y, x) = A(x, y)$. (In (4), we use the convention that $+\infty$ is absorbing for addition.)

B. Projectors on semimodules

Let $\mathcal{K}$ be a complete idempotent semiring and $\mathcal{V}$ denote a *complete subsemimodule* of a complete semimodule $\mathcal{X}$, i.e., a subset of $\mathcal{X}$ that is stable by arbitrary sups and by the action of scalars. We call *canonical projector* on $\mathcal{V}$ the map

$$P_{\mathcal{V}} : \mathcal{X} \to \mathcal{X}, \quad u \mapsto P_{\mathcal{V}}(u) = \max\{v \in \mathcal{V} \mid v \leq u\}. \tag{5}$$

Let W denote a *generating family* of a complete subsemimodule $\mathcal{V}$, which means that any element $v \in \mathcal{V}$ can be written as $v = \bigvee\{w\lambda_w \mid w \in W\}$ for some $\lambda_w \in \mathcal{K}$. It is known that

$$P_{\mathcal{V}}(u) = \bigvee_{w\in W} w(w\backslash u)$$

(see, for instance, [24]). If $B : \mathcal{U} \to \mathcal{X}$ is a residuated linear operator, then when $\mathcal{U}$ and $\mathcal{X}$ are complete semimodules over $\mathcal{K}$, the image $\operatorname{im} B$ of B is a complete subsemimodule of $\mathcal{X}$, and

$$P_{\operatorname{im} B} = B \circ B^\sharp. \tag{6}$$

The max-plus finite element methods relies on the notion of projection on an image, parallel to a kernel, which was introduced by Cohen, Gaubert, and Quadrat in [25]. The following theorem, of which Corollary 2 below is an immediate corollary, is a variation on the results of [25, Section 6].

Theorem 1 (projection on an image parallel to a kernel): Let $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$ be complete semimodules over $\mathcal{K}$. Let $B : \mathcal{U} \to \mathcal{X}$ and $C : \mathcal{X} \to \mathcal{Y}$ be two residuated linear operators over $\mathcal{K}$. Let $\Pi_B^C = B \circ (C \circ B)^\sharp \circ C$. We have $\Pi_B^C = \Pi_B \circ \Pi^C$, where $\Pi_B = B \circ B^\sharp$ and $\Pi^C = C^\sharp \circ C$. Moreover, Π_B^C is a projector, meaning that $(\Pi_B^C)^2 = \Pi_B^C$, and for all $x \in \mathcal{X}$

$$\Pi_B^C(x) = \max\{y \in \operatorname{im} B \mid Cy \le Cx\}.$$

When $\mathcal{K} = \overline{\mathbb{R}}_{\max}$, and $C : \overline{\mathbb{R}}_{\max}^X \to \overline{\mathbb{R}}_{\max}^Y$ is a kernel operator, $\Pi^C = C^\sharp \circ C$ has an interpretation similar to (6):

$$\Pi^C(v) = C^\sharp \circ C(v) = -P_{\operatorname{im} C^*}(-v) = P^{-\operatorname{im} C^*}(v),$$

where $-\operatorname{im} C^*$ is thought of as a $\overline{\mathbb{R}}_{\min}$-subsemimodule of $\overline{\mathbb{R}}_{\min}^X$ and $P^{\mathcal{V}}$ denotes the projector on a $\overline{\mathbb{R}}_{\min}$-semimodule $\mathcal{V}$, so that

$$P^{-\operatorname{im} C^*}(v) = \min\{w \in -\operatorname{im} C^* \mid w \ge v\},$$

where $\le$ denotes here the usual order on $\overline{\mathbb{R}}^X$. When $B : \overline{\mathbb{R}}_{\max}^U \to \overline{\mathbb{R}}_{\max}^X$ is also a kernel operator, we have

$$\Pi_B^C = P_{\operatorname{im} B} \circ P^{-\operatorname{im} C^*}.$$

III. The max-plus finite element method

A. Max-plus variational formulation

Let $\mathcal{V}$ be a complete semimodule of functions from X to $\overline{\mathbb{R}}_{\max}$. Let $S^t : \mathcal{V} \to \mathcal{V}$ and v^t be defined as in the introduction. Using the semigroup property $S^{t+t'} = S^t \circ S^{t'}$, for $t, t' > 0$, we get

$$v^{t+\delta} = S^\delta v^t, \quad t = 0, \delta, \dots, T-\delta, \tag{7}$$

with $v^0 = \phi$ and $\delta = \frac{T}{N}$ for some positive integer N. Let $\mathcal{W} \subset \mathcal{V}$ be a complete $\overline{\mathbb{R}}_{\max}$-semimodule of functions from X to $\overline{\mathbb{R}}_{\max}$ such that for all $t \ge 0$, $v^t \in \mathcal{W}$. We choose a "dual" semimodule $\mathcal{Z}$ of "test functions" from X to $\overline{\mathbb{R}}_{\max}$. Recall that the max-plus *scalar product* is defined by

$$\langle u \mid v\rangle = \sup_{x\in X} u(x) + v(x)$$

for all functions $u, v : X \to \overline{\mathbb{R}}_{\max}$. We replace (7) by

$$\langle z \mid v^{t+\delta}\rangle = \langle z \mid S^\delta v^t\rangle \quad \forall z \in \mathcal{Z} \tag{8}$$

for $t = 0, \delta, \dots, T-\delta$, with $v^\delta, \dots, v^T \in \mathcal{W}$. Equation (8) can be seen as the analogue of a *variational* or *weak formulation*. Kolokoltsov and Maslov used this formulation in [28] and [29, Section 3.2] to define a notion of generalized solution of Hamilton–Jacobi equations. We use it in the next section to construct an approximation algorithm for the value function, which is obtained by taking for $\mathcal{W}$ and $\mathcal{Z}$ finitely generated semimodules.

B. Ideal max-plus finite element method

We consider a semimodule $\mathcal{W}_h \subset \mathcal{W}$ generated by the family $\{w_i\}_{1\le i\le p}$. We call *finite elements* the functions w_i. We approximate v^t by $v_h^t \in \mathcal{W}_h$, that is,

$$v^t \simeq v_h^t = \bigvee_{1\le i\le p} w_i\lambda_i^t,$$

where $\lambda_i^t \in \overline{\mathbb{R}}_{\max}$. We also consider a semimodule $\mathcal{Z}_h \subset \mathcal{Z}$ with generating family $\{z_j\}_{1\le j\le q}$. The functions $z_1, \dots, z_q$ will act as test functions. We replace (8) by

$$\langle z \mid v_h^{t+\delta}\rangle = \langle z \mid S^\delta v_h^t\rangle \quad \forall z \in \mathcal{Z}_h \tag{9}$$

for $t = 0, \delta, \dots, T-\delta$, with $v_h^\delta, \dots, v_h^T \in \mathcal{W}_h$. The function v_h^0 is a given approximation of ϕ. Since $\mathcal{Z}_h$ is generated by $z_1, \dots, z_q$, (9) is equivalent to

$$\langle z_j \mid v_h^{t+\delta}\rangle = \langle z_j \mid S^\delta v_h^t\rangle \quad \forall 1 \le j \le q \tag{10}$$

for $t = 0, \delta, \dots, T-\delta$, with $v_h^t \in \mathcal{W}_h$, $t = 0, \delta, \dots, T$.

Since (10) need not have a solution, we look for its maximal subsolution, i.e., the maximal solution $v_h^{t+\delta} \in \mathcal{W}_h$ of

$$\langle z_j \mid v_h^{t+\delta}\rangle \le \langle z_j \mid S^\delta v_h^t\rangle \quad \forall 1 \le j \le q. \tag{11a}$$

We also take for the approximate value function v_h^0 at time 0 the maximal solution $v_h^0 \in \mathcal{W}_h$ of

$$v_h^0 \le v^0. \tag{11b}$$

Let us denote by W_h the max-plus linear operator from $\overline{\mathbb{R}}_{\max}^p$ to $\mathcal{W}$ with matrix $W_h = \operatorname{col}(w_i)_{1\le i\le p}$ and by Z_h^* the max-plus linear operator from $\mathcal{W}$ to $\overline{\mathbb{R}}_{\max}^q$ whose transposed matrix is $Z_h = \operatorname{col}(z_j)_{1\le j\le q}$. This means that $W_h\lambda = \bigvee_{1\le i\le p} w_i\lambda_i$ for all $\lambda = (\lambda_i)_{i=1,\dots,p} \in \overline{\mathbb{R}}_{\max}^p$, and $(Z_h^* v)_j = \langle z_j \mid v\rangle$ for all $v \in \mathcal{W}$ and $j = 1, \dots, q$. Applying Theorem 1 to $B = W_h$ and $C = Z_h^*$ and noting that $\mathcal{W}_h = \operatorname{im} W_h$, we get the following corollary.

Corollary 2: The maximal solution $v_h^{t+\delta} \in \mathcal{W}_h$ of (11a) is given by $v_h^{t+\delta} = S_h^\delta(v_h^t)$, where

$$S_h^\delta = \Pi_{W_h}^{Z_h^*} \circ S^\delta.$$

Note that $\Pi_{W_h}^{Z_h^*} = P_{\mathcal{W}_h} \circ P^{-\mathcal{Z}_h}$. The following proposition provides a recursive equation verified by the vector of coordinates of v_h^t.

Proposition 3: Let $v_h^t \in \mathcal{W}_h$ be the maximal solution of (11) for $t = 0, \delta, \dots, T$. Then, for every $t = 0, \delta, \dots, T$, there exists a maximal $\lambda^t \in \mathbb{R}^p_{\max}$ such that $v_h^t = W_h \lambda^t$, $t = 0, \delta, \ldots, T$, which can be determined recursively from

$$\lambda^{t+\delta} = (Z_h^* W_h) \backslash (Z_h^* S^\delta W_h \lambda^t) \tag{12a}$$

for $t = 0, \ldots, T - \delta$ with the initial condition

$$\lambda^0 = W_h \backslash \phi. \tag{12b}$$

For $1 \le i \le p$ and $1 \le j \le q$, we define

$$(M_h)_{ji} = \langle z_j \mid w_i \rangle, \tag{13}$$

$$(K_h)_{ji} = \langle z_j \mid S^\delta w_i \rangle, \tag{14}$$

$$= \langle (S^*)^\delta z_j \mid w_i \rangle, \tag{15}$$

where S^* is the *transposed semigroup* of S, which is the evolution semigroup associated with the optimal control problem (1) in which the sign of the dynamics is changed. The matrices M_h and K_h, which represent, respectively, the max-plus linear operators $Z_h^* W_h$ and $Z_h^* S^\delta W_h$, may be thought of as the max-plus analogues of the mass and stiffness matrices, respectively.

The ideal max-plus finite element method (Algorithm 1) is the algorithm derived from Proposition 3, assuming that the "mass" and "stiffness" matrices M_h and K_h are computed by oracles.

Algorithm 1 IDEAL MAX-PLUS FINITE ELEMENT METHOD

1: Choose the finite elements $(w_i)_{1\le i\le p}$ and $(z_j)_{1\le j\le q}$. Choose the time discretization step $\delta = \frac{T}{N}$.
2: Compute the matrix M_h and the matrix K_h defined in (13), (14), or (15).
3: Compute $\lambda^0 = W_h \backslash \phi$ and $v_h^0 = W_h \lambda^0$.
4: For $t = \delta, 2\delta, \ldots, T$, compute $\lambda^t = M_h \backslash (K_h \lambda^{t-\delta})$ and $v_h^t = W_h \lambda^t$.

For the convenience of the reader, we rewrite the elements of Algorithm 1 with the usual notation:

$$(M_h)_{ji} = \sup_{x\in X} \big(z_j(x) + w_i(x)\big), \tag{16}$$

$$(K_h)_{ji} = \sup_{x\in X} \big(z_j(x) + S^\delta w_i(x)\big), \tag{17}$$

$$= \sup_{x\in X} \big(w_i(x) + (S^*)^\delta z_j(x)\big).$$

Equation (12a) may be written explicitly, using (3) and (4), for $1 \le i \le p$, as

$$\lambda_i^{t+\delta} = \min_{1\le j\le q} \Big(-(M_h)_{ji} + \max_{1\le k\le p} \big((K_h)_{jk} + \lambda_k^t\big)\Big).$$

Finally, we have, for all $x \in X$ and $t = 0, \delta, \ldots, T - \delta$,

$$v_h^{t+\delta} = \sup_{1\le i\le p} \Big(w_i(x) + \lambda_i^{t+\delta}\Big).$$

C. Effective max-plus finite element method

In the ideal max-plus finite element method, we assume that the matrices M_h and K_h are exactly known. We shall see in section IV that for natural choices of finite elements and test functions, computing every entry of the matrix M_h is equivalent to solving a maximization problem in which the objective function is concave and the feasible set is convex. This problem can be approached by standard optimization methods. When the domain X has a "simple" shape, for instance when X is a hypercube, the entries of the matrix M_h can even be computed analytically. Hence, the assumption that M_h is accurately known is not a restrictive one. The same is not true for K_h. Indeed, evaluating every scalar product $\langle z \mid S^\delta w\rangle$ leads to a new optimal control problem since

$$\langle z \mid S^\delta w\rangle = \max z(\mathbf{x}(0)) + \int_0^\delta \ell(\mathbf{x}(s), \mathbf{u}(s)) ds + w(\mathbf{x}(\delta)),$$

where the maximum is taken over the set of trajectories $\big(\mathbf{x}(\cdot), \mathbf{u}(\cdot)\big)$ satisfying (1b). This problem is simpler to approximate than problem (1), because the horizon δ is small, and the functions z and w have a regularizing effect.

We call "the effective max-plus finite element method" the method obtained by replacing in Algorithm 1 the matrix K_h by an approximation.

We first discuss the approximation of $S^\delta w$ for every finite element w. The Hamilton–Jacobi equation (2) suggests approximating $S^\delta w$ by the function $[S^\delta w]_H$ such that

$$[S^\delta w]_H(x) = w(x) + \delta H(x, \nabla w(x)) \quad \forall x \in X. \tag{18}$$

Let $[S^\delta W_h]_H$ denote the max-plus linear operator from $\overline{\mathbb{R}}^p_{\max}$ to $\mathcal{W}$ with matrix $[S^\delta W_h]_H = \mathrm{col}([S^\delta w_i]_H)_{1\le i\le p}$, which means that

$$[S^\delta W_h]_H \lambda = \bigvee_{1\le i\le p} [S^\delta w_i]_H \lambda_i$$

for all $\lambda = (\lambda_i)_{1\le i\le p} \in \mathbb{R}^p_{\max}$. The above approximation of $S^\delta w$ yields an approximation of the matrix K_h by the matrix $K_{H,h} := Z_h^*[S^\delta W_h]_H$, whose entries are given, for $1 \le i \le p$ and $1 \le j \le q$, by

$$(K_{H,h})_{ji} = \sup_{x\in X} \Big(z_j(x) + w_i(x) + \delta H(x, \nabla w_i(x))\Big). \tag{19}$$

Let A_{ji} denote the set where the optimum of the function $x \mapsto z_j(x) + w_i(x)$ is attained. Computing $(K_{H,h})_{ji}$ in (19) requires solving an optimization problem, which is nothing but a perturbation of the optimization problem associated with the computation of $(M_h)_{ji}$. We may exploit this observation by approximating $K_{H,h}$ by the matrix $\tilde{K}_{H,h}$ with entries

$$(\tilde{K}_{H,h})_{ji} = \sup_{x\in A_{ji}} \Big(z_j(x) + w_i(x)\Big) + \sup_{x\in A_{ji}} H(x, \nabla w_i(x))$$

$$= \langle z_j \mid w_i\rangle + \delta \sup_{x\in A_{ji}} H(x, \nabla w_i(x)) \tag{20}$$

for $1 \le i \le p$ and $1 \le j \le q$. When A_{ji} has only one element, (20) yields a convenient approximation of K_h.

IV. Error analysis

A. General error estimates

In what follows we denote by $\|v\|_\infty = \sup_{i\in I} |v(i)| \in \mathbb{R}\cup\{+\infty\}$ the sup-norm of any function $v : I \to \mathbb{R}$. We also use the same notation $\|v\|_\infty = \sup_{i\in I}|v_i|$ for a vector $v = (v_i)_{i\in I}$. For any two sets I and J, a map $\Phi : \mathbb{R}^I \to \mathbb{R}^J$ is said to be monotone and homogeneous if it is monotone for the natural order and if for all $u \in \mathbb{R}^I$ and $\lambda \in \mathbb{R}$, $\Phi(u + \lambda) = \Phi(u) + \lambda$ with $(u + \lambda)(i) = u(i) + \lambda$. Monotone homogeneous maps are nonexpansive for the sup-norm: $\|\Phi(u)-\Phi(v)\|_\infty \le \|u-v\|_\infty$; see [30]. In particular, max-plus or min-plus linear operators are nonexpansive for the sup-norm. We denote $\bar\tau_\delta = \{0, \delta, \dots, T\}$, $\tau_\delta^+ = \bar\tau_\delta\backslash\{0\}$, and $\tau_\delta^- = \bar\tau_\delta\backslash\{T\}$.

The following lemma shows that the error of the ideal max-plus finite element method is controlled by the projection errors $\|\Pi_{W_h}^{Z_h^*}(v^t) - v^t\|_\infty$.

Lemma 4: For $t \in \bar\tau_\delta$, let v^t be the value function at time t and v_h^t be its approximation given by the ideal max-plus finite element method. We have

$$\|v_h^T - v^T\|_\infty \le \|\Pi_{W_h}(v^0) - v^0\|_\infty + \sum_{t\in\tau_\delta^+} \|\Pi_{W_h}^{Z_h^*}(v^t) - v^t\|_\infty.$$

Corollary 5: For $t \in \bar\tau_\delta$, let v^t be the value function at time t and v_h^t be its approximation given by the effective max-plus finite element method, implemented with an approximation $\tilde K_h$ of K_h. We have

$$\begin{aligned}\|v_h^T - v^T\|_\infty \le \left(1+\frac{T}{\delta}\right)\Big(&\sup_{t\in\bar\tau_\delta}(\|\Pi^{Z_h^*}v^t - v^t\|_\infty \\ &+\|\Pi_{W_h}v^t - v^t\|_\infty) + \max_{1\le i\le p}\|[S^\delta w_i]_H - S^\delta w_i\|_\infty \\ &+\|\tilde K_h - K_{H,h}\|_\infty\Big).\end{aligned}$$

B. Projection errors

Recall that a function f is *c-semiconvex* if $f(x) + \frac{c}{2}\|x\|_2^2$, where $\|\cdot\|_2$ is the standard euclidean norm of $\mathbb{R}^n$, is convex. A function f is *c-semiconcave* if $-f$ is c-semiconvex. Spaces of semiconvex functions were intensively used in the max-plus based approximation method of Fleming and McEneaney [10].

We shall use the following finite elements.

Definition 6 (P_1 finite elements): We call the P_1 *finite element* or *Lipschitz finite element* centered at point $\hat x \in X$, with constant $a > 0$, the function $w(x) = -a\|x - \hat x\|_1$, where $\|x\|_1 = \sum_{i=1}^n |x_i|$ is the l^1-norm of $\mathbb{R}^n$.

The family of Lipschitz finite elements of constant a generates, in the max-plus sense, the semimodule of Lipschitz continuous functions from X to $\bar{\mathbb{R}}$ of Lipschitz constant a with respect to $\|\cdot\|_1$.

Definition 7 (P_2 finite elements): We call the P_2 *finite element* or *quadratic finite element* centered at point $\hat x \in X$, with Hessian $c > 0$, the function $w(x) = -\frac{c}{2}\|x - \hat x\|_2^2$.

When $X = \mathbb{R}^n$, the family of quadratic finite elements with Hessian c generates, in the max-plus sense, the semimodule of l.s.c. c-semiconvex functions with values in $\bar{\mathbb{R}}$.

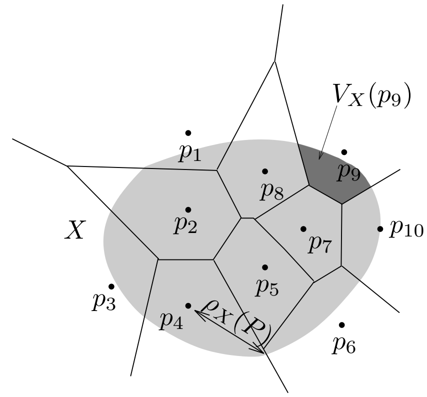

Fig. 1. Voronoi tessellation.

When C is a nonempty convex subset of $\mathbb{R}^n$ and $c > 0$, a function is said to be *c-strongly convex* on C if and only if $f - \frac{1}{2}c\|\cdot\|_2^2$ is convex on C. A function f is *c-strongly concave* on C if $-f$ is c-strongly convex on C.

Let P be a finite subset of $\mathbb{R}^n$. The *Voronoi cell* of a point $p \in P$ is defined by

$$V(p) = \{x \in \mathbb{R}^n \mid \|x - p\|_2 \le \|x - q\|_2 \;\forall q \in P\}.$$

The family $\{V(p)\}_{p\in P}$ constitutes a subdivision of $\mathbb{R}^n$, which is called a Voronoi tessellation (see [31] for an introduction to Voronoi tessellations). We define *the restriction of* $V(p)$ *to* X to be $V_X(p) = V(p)\cap X$. We define $\rho_X(P)$ to be the *maximal radius* of the restriction to X of the Voronoi cells of the points of P:

$$\rho_X(P) := \sup_{p\in P}\sup_{x\in V_X(p)} \|x - p\|_2.$$

Observe that

$$\rho_X(P) := \sup_{x\in X}\inf_{p\in P}\|x - p\|_2.$$

The previous definitions are illustrated in Figure 1. The set X is in light gray, $P = \{p_1, \dots, p_{10}\}$, $V_X(P_9)$ is in dark gray, and $\rho_X(P)$ is indicated by a bidirectional arrow.

The next two lemmas bound the projection error in term of the radius of Voronoi cells.

Lemma 8 (primal projection error): Let X be a compact convex subset of $\mathbb{R}^n$. Let $v : X \to \mathbb{R}$ be a c-semiconvex and Lipschitz continuous function with Lipschitz constant L_v with respect to the euclidean norm. Let $v_c(x) = v(x) + \frac{c}{2}\|x\|_2^2$. Let $\hat X = X + \mathrm{B}_2(0, \frac{L_v}{c})$, let $\hat X_h$ be a finite subset of $\mathbb{R}^n$, and let $\mathcal{W}_h$ denote the complete subsemimodule of $\overline{\mathbb{R}}_{\max}^X$ generated by the family $(w_{\hat x_h})_{\hat x_h\in\hat X_h}$, where $w_{\hat x_h}(x) = -\frac{c}{2}\|x - \hat x_h\|_2^2$. Then

$$\|v - P_{\mathcal{W}_h}v\|_\infty \le c\,\mathrm{diam}\,X\rho_{\hat X}(\hat X_h).$$

Lemma 9 (dual projection error): Let X be a bounded subset of $\mathbb{R}^n$ and X_h a finite subset of $\mathbb{R}^n$. Let $v : X \to \mathbb{R}$ be a given Lipschitz continuous function with Lipschitz constant L_v with respect to the euclidean norm. Let $\mathcal{Z}_h$ denote the complete semimodule of $\overline{\mathbb{R}}_{\max}^X$ generated by the P_1 finite elements $(z_{x_h})_{x_h \in X_h}$ centered at the points of X_h with constant $a \geq L_v$. Then

$$\|v - P^{-\mathcal{Z}_h} v\|_\infty \leq n(a + L_v)\rho_X(X_h).$$

C. The approximation errors

To state an error estimate, we make the following standard assumptions (see [2], for instance):

– (H1) $f : X \times U \to \mathbb{R}^n$ is bounded and Lipschitz continuous with respect to x, meaning that there exist $L_f > 0$ and $M_f > 0$ such that

$$\|f(x,u) - f(y,u)\|_2 \leq L_f\|x - y\|_2 \quad \forall x, y \in X, u \in U,$$

$$\|f(x,u)\|_2 \leq M_f \qquad \forall x \in X, u \in U.$$

– (H2) $\ell : X \times U \to \mathbb{R}$ is bounded and Lipschitz continuous with respect to x, meaning that there exist $L_\ell > 0$ and $M_\ell > 0$ such that

$$|\ell(x,u) - \ell(y,u)| \leq L_\ell\|x - y\|_2 \quad \forall x, y \in X, u \in U,$$

$$|\ell(x,u)| \leq M_\ell \qquad \forall x \in X, u \in U.$$

We shall also need the further assumptions:

– (H3) The domain X is invariant by the dynamics: for all $\mathbf{u} : [0,T] \to U$ and for all $x \in X$, the solution $\mathbf{x}_{\mathbf{u},x}$ of $\dot{\mathbf{x}}_{\mathbf{u},x}(s) = f(\mathbf{x}_{\mathbf{u},x}(s), u(s))$, $s \geq 0$, and $\mathbf{x}_{\mathbf{u},x}(0) = x$ satisfies $\mathbf{x}_{\mathbf{u},x}(s) \in X$ for all $s \geq 0$.

– (H4) The domain X is invariant by the discretized dynamics in time $\delta > 0$: for all $u \in U$ and for all $x \in X$, $x + \delta f(x,u) \in X$.

In the main results, the domain will be also assumed to be convex. Then assumption (H4) implies (H3).

Lemma 10 (Approximation of $S^\delta w$): Let X be a convex subset of $\mathbb{R}^n$. We make assumptions (H1), (H2), (H3), and (H4). Let $w : x \to \mathbb{R}$ be such that w is $\mathcal{C}^1$ on a neighborhood of X, Lipschitz continuous with Lipschitz constant L_w with respect to the euclidean norm, c_1-semiconvex, and c_2-semiconcave. Then there exists $K_1 > 0$ such that $\|[S^\delta w]_H - S^\delta w\|_\infty \leq K_1\delta^2$, for $\delta > 0$, where $[S^\delta w]_H$ is given by (18).

Proposition 11: Let X be a compact convex subset of $\mathbb{R}^n$. We consider an u.s.c. and strongly concave function $\varphi : X \to \mathbb{R}$ with modulus $c > 0$ and a Lipschitz continuous function $\psi : X \to \mathbb{R}$ with Lipschitz constant L_ψ with respect to the euclidean norm. Then the maximum of φ on X is attained at a unique point $x_0 \in X$, i.e., $\arg\max_X \varphi = \{x_0\}$ and

$$\left|\sup_{x \in X}\big(\varphi(x) + \delta\psi(x)\big) - \big(\varphi(x_0) + \delta\psi(x_0)\big)\right| \leq L_\psi\delta\sqrt{\frac{2\delta M}{c}},$$

where $M = \sup_{x\in X}\psi(x) - \inf_{x\in X}\psi(x)$.

Remark 12 (Approximation of K_h by $\tilde{K}_H$): To have an error estimate of the approximation of the matrix $K_{H,h}$ by the matrix $\tilde{K}_{H,h}$, we apply Corollary 11 in the case where

$$\varphi(x) = w_i(x) + z_j(x) \quad \text{and} \quad \psi(x) = H(x, \nabla w_i(x))$$

for a suitable choice of the finite elements w_i and test functions z_j. Using assumptions (H1) and (H2), we have that, for all $x \in X$, $|\psi(x)| \leq M_f\|\nabla w\|_\infty + M_\ell$, where $\|\nabla w\|_\infty = \|\|\nabla w\|_2\|_\infty$ and $\nabla w = (\nabla w_i)_{1 \leq i \leq p}$. We deduce that

$$\sup\psi - \inf\psi \leq 2\big(M_f\|\nabla w\|_\infty + M_\ell\big).$$

Moreover, $H(\cdot, p)$ and $H(x, \cdot)$ are Lipschitz continuous with Lipschitz constants $L_f\|p\|_2 + L_\ell$ and M_f, respectively. Hence, ψ is Lipschitz continuous with Lipschitz constant

$$L_\psi = L_f\|\nabla w\|_\infty + L_\ell + M_f\|D^2 w_i\|_\infty.$$

D. Final estimation of the error of the MFEM

We now state our main convergence result, which holds for quadratic finite elements and Lipschitz test functions.

Theorem 13: Let X be a compact convex subset of $\mathbb{R}^n$ with nonempty interior and $\hat{X} = X + \mathrm{B}_2(0, \frac{L}{c})$, where $L > 0$, $c > 0$. Choose any finite sets of discretization points $X_h \subset \mathbb{R}^n$ and $\hat{X}_h \subset \mathbb{R}^n$. Let

$$\Delta x = \max(\rho_X(X_h), \rho_{\hat{X}}(\hat{X}_h)).$$

We make assumptions (H1), (H2), (H3), and (H4) and assume that the value function at time t, v^t, is c-semiconvex and Lipschitz continuous with constant L with respect to the euclidean norm for all $t \geq 0$. Let us choose quadratic finite elements $w_{\hat{x}_h}$ of Hessian c, centered at the points $\hat{x}_h$ of $\hat{X}_h$. Let us choose, as test functions, the Lipschitz finite elements z_{x_h} with constant $a \geq L$, centered at the points x_h of X_h. For $t = 0, \delta, \dots, T$, let v_h^t be the approximation of v^t given by the max-plus finite element method implemented with the approximation $K_{H,h}$ of K_h given by (19). Then there exists a constant $C_1 > 0$ such that

$$\|v_h^T - v^T\|_\infty \leq C_1\left(\delta + \frac{\Delta x}{\delta}\right).$$

When the approximation $K_{H,h}$ is replaced by $\tilde{K}_{H,h}$, given by (20), this inequality becomes

$$\|v_h^T - v^T\|_\infty \leq C_2\left(\sqrt{\delta} + \frac{\Delta x}{\delta}\right)$$

for some constant $C_2 > 0$.

V. Numerical results

We implemented the MFEM described in section III using the max-plus toolbox of Scilab [32] (in dimension 1) and specific programs written in C (in dimension 2). We used the approximation $\tilde{K}_{H,h}$ of the matrix K_h. The matrix M_h can always be computed analytically. We present here numerical experiments for optimal control problems in dimension 2. Dimension 1 examples were shown in [14], [15]. In all the examples below, the Hamiltonian H, and thus the stiffness matrix $\tilde{K}_{H,h}$, have been computed analytically. We avoided storing the (full) matrices M_h and $\tilde{K}_{H,h}$ when the number of discretization points was large.

Example 14 (linear quadratic problem): We consider the case where $U = \mathbb{R}^2$, $X = \mathbb{R}^2$, $\phi \equiv 0$,

$$\ell(x,u) = -\frac{x_1^2 + x_2^2}{2} - \frac{u_1^2 + u_2^2}{2}, \quad \text{and} \quad f(x,u) = u.$$

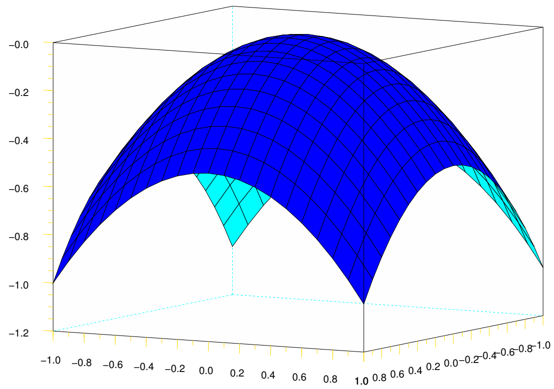

Fig. 2. Max-plus approximation of a linear quadratic control problem (Example 14).

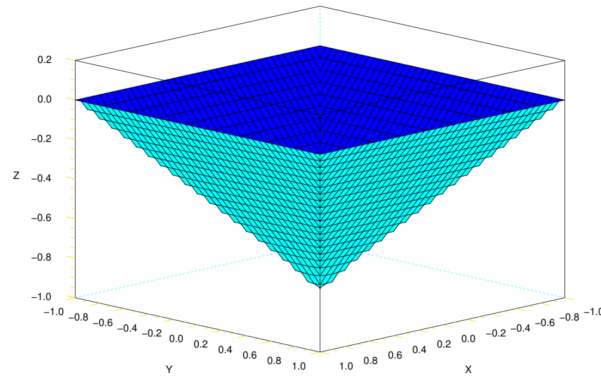

Fig. 3. Max-plus approximation of the distance problem (Example 15).

For $x \in X$, the value functions at time t is

$$v(x,t) = -\frac{1}{2}\tanh(t)(x_1^2 + x_2^2).$$

The domain X is unbounded; therefore ℓ and f do not satisfy assumptions (H1) and (H2). We will restrict the domain to the set $[-5;5]^2$. We choose quadratic finite elements w_i and z_j of Hessian c centered at the points of the regular grid $\big((\mathbb{Z}\Delta x) \cap [-6,6]\big)^2$. We represent in Figure 2 the solution given by our algorithm in the case where $T = 5$, $\delta = 0.5$, $\Delta x = 0.1$, and $c = 1$. The L_∞-error is $9 \cdot 10^{-5}$.

Example 15 (distance problem): We consider the case where $T = 1$, $\phi \equiv 0$, $X = [-1,1]^2$, $U = [-1,1]^2$, $\ell(x,u) = -1$ if $x \in \mathrm{int}X$, and $\ell(x,u) = 0$ if $x \in \partial X$, $f(x,u) = 7u$ if $x \in \mathrm{int}X$, and $f(x,u) = 0$ if $x \in \partial X$. For $x \in X$, the value function at time t is

$$v(x,t) = \max\big(-t, \max(|x_1|,|x_2|) - 1\big).$$

We choose quadratic finite elements w_i of Hessian c centered at the points of the regular grid $\big((\mathbb{Z}\Delta x) \cap [-3,3]\big)^2$ and Lipschitz finite elements z_j with constant a centered at the points of the regular grid $\big((\mathbb{Z}\Delta x) \cap [-1,1]\big)^2$. We represent in Figure 3 the solution given by our algorithm in the case where $T = 1$, $\delta = 0.05$, $\Delta x = 0.025$, $a = 3$, and $c = 1$. The L_∞-error is of order 0.05.

Example 16 (rotating problem): We consider here the Mayer problem where $T = 1$, $X = \mathrm{B}_2(0,1)$, $U = \{0\}$, $\phi(x) = -\frac{1}{2}x_1^2 - \frac{3}{2}x_2^2$, $\ell(x,u) = 0$, and $f(x,u) = (-x_2, x_1)$. For $x \in X$, the value function at time t is

$$v(x,t) = -\frac{1}{2}(-x_2\sin(t)+x_1\cos(t))^2 - \frac{3}{2}(x_2\cos(t)+x_1\sin(t))^2.$$

We choose quadratic finite elements w_i and z_j of Hessians c_w and c_z, respectively, centered at the points of the regular grid $\big((\mathbb{Z}\Delta x) \cap [-2,2]\big)^2$. We represent in Figure 4 the solution given by our algorithm in the case where $\delta = \Delta x = 0.05$, $c_w = 4$, and $c_z = 3$. The L_∞-error is 0.046.

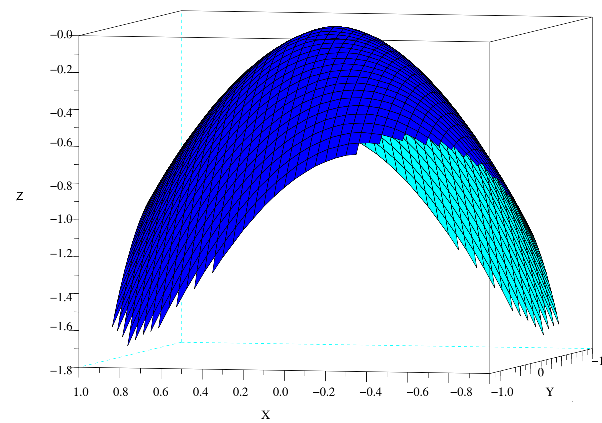

Fig. 4. Max-plus approximation of the rotating problem (Example 16).

Example 17: We consider the case where $U = \mathbb{R}$, $X = \mathbb{R}^2$, $\phi(x) = -x_1^2 - 2x_2^2$,

$$\ell(x,u) = -x_1^2 - \frac{u^2}{2}, \quad \text{and} \quad f(x,u) = (x_2, u)^T.$$

We choose quadratic finite elements w_i and z_j of Hessian c_w and c_z, respectively, centered at the points of the grids $\big((\mathbb{Z}\Delta x) \cap [-2,2]\big)^2$ and $\big((\mathbb{Z}\Delta x) \cap [-11,11]\big)^2$, respectively. We represent in Figure 5 the solution given by our algorithm in the case where $T = 1$, $\delta = 0.05$, $\Delta x = 0.025$, $c_w = 10$, and $c_z = 1$. The L_∞-error is 0.11. (We compared the max-plus approximation with the solution of the problem given by the Riccati equation.)

We have tested our method on examples that fulfill the assumptions of Theorem 13 (see Example 16) but also on problems that do not fulfill these assumptions. The method is efficient even in the second case. The only difficulty comes from the full character of the matrices M_h and K_h, which limits the number of discretization points. To treat higher dimensional examples, we need higher-order approximations (when the value function is regular enough). This is the object of a subsequent work.

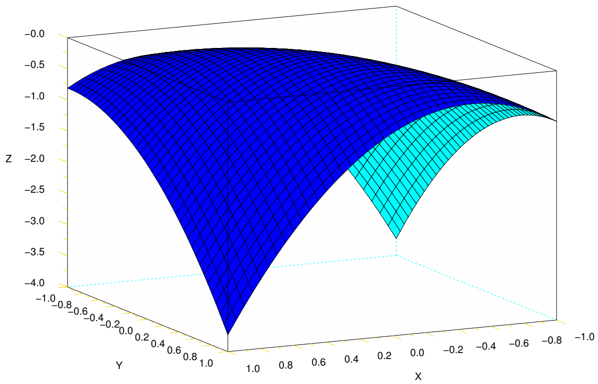

Fig. 5. Max-plus approximation of the solution of the control problem of Example 17.

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