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Feedback Solutions of Optimal Control Problems with DAE Constraints*

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Abstract

An optimal feedback control has been obtained for linear-quadratic optimal control problems with constraints described by differential-algebraic equations (DAEs). For that purpose, a new implicit Riccati equation (Riccati differential-algebraic system) is provided, and its solvability is investigated. It is shown that one can do without the strong consistency conditions as used in several previous papers. Furthermore, the solvability of the resulting closed loop system is considered and the relations between Riccati equations and Hamiltonian systems are elucidated.

1. INTRODUCTION

Feedback solutions via Riccati differential equations are a known and proven tool for solving linear-quadratic optimal control problems. If an explicit ordinary differential equation (ODE) in the state equation is replaced by a differential-algebraic equation (DAE)

$$Ex' = Cx + Du, \tag{1}$$

with E being a singular constant square matrix, several different generalizations of the Riccati-ansatz are possible. For this, quite a lot of references are available (in particular for the case of constant coefficients); however, we can mention here only part of them. We refer to [1]-[3] for further sources.

In (1) and in following equalities, the argument t is dropped, and the given relations are meant pointwise for all $t \in [0,T]$. The superscript * will denote the transpose.

In [1] it was first noted that the standard Riccati equation modification

$$E^*Y'E = -E^*YC - C^*YE + \\ +(S+E^*YD)R^{-1}(S^*+D^*YE) - W, \tag{2}$$

which is considered to be *obvious*, leads to unacceptable solvability conditions. Consequently, more specific Riccati approaches that skillfully make use of the inherent structures find favor with [1]. Starting from a singular value decomposition $UEV = \operatorname{diag}(\Sigma, 0)$ and certain rank conditions, lower dimensional Riccati equations of the form $\Sigma Y'\Sigma = \dots$ are introduced. From the point of view of DAE theory the rank conditions used in [1] imply that the related Hamilton–Lagrange system is a regular DAE with tractability index one (cf. [4]).

Kurina has investigated the implicit Riccati equation of the form

$$E^*Y' = -Y^*C - C^*Y + (S+Y^*D)R^{-1}(S^*+D^*Y) - W \tag{3}$$

in a more general Hilbert space setting. See, for example, [5]. References, concerning the study of this equation are contained in [6]. Like (2), (3) is also primarily a matrix-DAE; however, (3) has much better solvability properties than (2).

Kunkel and Mehrmann [3] consider the Riccati DAE

$$(E^*YE)' = -E^*YC - C^*YE + \\ +(S+E^*YD)R^{-1}(S^*+D^*YE) - W, \tag{4}$$

which generalizes (2) to allow for time-dependent coefficients E. However, this equation is as unsuitable as its time-invariant version (2), and the authors have to admit that, *unfortunately, this approach can only be used in very special cases since, for $E(t)$ singular, the solutions of* (4) *and the Euler–Lagrange equation are not related via $u = -R^{-1}(S+D^*YE)x$, as in the case of nonsingular $E(t)$.*

If, in (1), there is no constant matrix E in front of the derivative but a time-dependent matrix, it makes sense to change to a DAE with a *properly formulated leading term*

$$A(Bx)' = Cx + Du, \tag{5}$$

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with well-matched A and B (cf. [7]). The corresponding initial condition is

$$A(0)B(0)x(0) = z_0 \tag{6}$$

with $z_0 \in \operatorname{im}(A(0)B(0))$. For arguments that state the leading term in this way we refer to [7], [8]. Notice that, in particular, such equations arise in circuit simulation via modified nodal analysis (see, e.g., [9], [10]).

Under the assumption that B is continuously differentiable, the terminal problem for the special implicit Riccati equation is proved to be relevant in [11] (in a more general Hilbert space setting).

The generalization of (2), adopting the bad solvability properties of (2), is investigated in [12] for a DAE with a properly formulated leading term when B is continuous.

In this paper we work with the Riccati DAE

$$B^*(A^*YB^-)'B = -Y^*C - C^*Y +$$

$$+(S+Y^*D)R^{-1}(S^*+D^*Y) - W \tag{7}$$

and the terminal value condition

$$A(T)^*Y(T)B(T)^- = B(T)^{-*}VB(T)^-, \tag{8}$$

where B is assumed to be just continuous. Here, the solutions meet the symmetry condition $A^*YB^- = B^{-*}Y^*A$ (B^- is a special, generalized inverse).

The difficulties with (4) were illustrated in [3] by means of a small academic problem. This special problem has been analyzed in details in [6] to show that things work well when using more appropriate Riccati DAE (7).

The detailed proofs of the statements from this paper are given in [6].

2. OPTIMAL FEEDBACK CONTROL

We deal with the quadratic cost functional

$$J(u,x) := \frac{1}{2}\langle x(T), Vx(T)\rangle + \frac{1}{2}\int_0^T \{\langle x(t), W(t)x(t)\rangle +$$

$$+2\langle x(t), S(t)u(t)\rangle + \langle u(t), R(t)u(t)\rangle\}dt \tag{9}$$

to be minimized on pairs $(u,x) \in \mathscr{C} \times \mathscr{C}^1_B$ satisfying the IVP (5), (6).

The coefficients in (9), (5) are matrices $W(t) \in L(\mathbb{R}^m,\mathbb{R}^m)$, $R(t) \in L(\mathbb{R}^l,\mathbb{R}^l)$, $S(t) \in L(\mathbb{R}^l,\mathbb{R}^m)$, $A(t) \in L(\mathbb{R}^n,\mathbb{R}^k)$, $B(t) \in L(\mathbb{R}^m,\mathbb{R}^n)$, $C(t) \in L(\mathbb{R}^m,\mathbb{R}^k)$, $D(t) \in L(\mathbb{R}^l,\mathbb{R}^k)$, $t \in [0,T]$, which depend continuously on t, and $V \in L(\mathbb{R}^m,\mathbb{R}^m)$.

The coefficients determining the cost (9) satisfy the following standard assumptions: $W(t), R(t)$, and V are symmetric, $R(t)$ is positive definite, and $\begin{bmatrix} W(t) & S(t) \\ S(t)^* & R(t)\end{bmatrix}$ is positive semidefinite, $t \in [0,T]$.

We use the symbols $\mathscr{C}$ and $\mathscr{C}^1$ for continuous and continuously differentiable function spaces respectively and

$$\mathscr{C}^1_B := \{x \in \mathscr{C} : Bx \in \mathscr{C}^1\},\ \mathscr{C}^1_{A^*} := \{\psi \in \mathscr{C} : A^*\psi \in \mathscr{C}^1\}.$$

The value z_0 is given. The leading term of the DAE (5) is assumed to be properly stated in the sense that the decomposition

$$\ker A(t) \oplus \operatorname{im} B(t) = \mathbb{R}^n,\ t \in [0,T], \tag{10}$$

holds true, and both subspaces forming this direct sum have constant dimensions and are spanned by continuously differentiable on $[0,T]$ functions (cf. [7]).

A pair $(u,x) \in \mathscr{C} \times \mathscr{C}^1_B$ satisfying the IVP (5), (6) is said to be *admissible*.

Let $K(t) \in L(\mathbb{R}^n,\mathbb{R}^n)$ denote the projector that realizes decomposition (10), $\ker K(t) = \ker A(t)$, $\operatorname{im} K(t) = \operatorname{im} B(t)$, $t \in [0,T]$.

In addition to $K(t)$ we introduce $Q(t) \in L(\mathbb{R}^m,\mathbb{R}^m)$, $Q_*(t) \in L(\mathbb{R}^k,\mathbb{R}^k)$, which are the orthoprojectors onto $\ker(A(t)B(t))$ and $\ker(B(t)^*A(t)^*)$, respectively; furthermore, $P(t) := I - Q(t), P_*(t) := I - Q_*(t)$, $t \in [0,T]$. The projector functions Q, P, Q_*, and P_* are continuous.

It is natural to assume that $V = VP(T)$ (see, e.g., [13]).

Having the projectors K, P, and P_*, we introduce the generalized inverses B^- of B and A^{*-} of A^* by

$$\begin{aligned} B^-BB^- &= B^-, & BB^-B &= B, \\ BB^- &= K, & B^-B &= P \end{aligned}$$

and by similar relations for A^{*-}. Notice that B^- and A^{*-} are uniquely determined by these relations and continuous on $[0,T]$.

Next we consider the terminal value problem (7), (8). Equation (7) generalizes the (well-known for $A = I$, $B = I$) Riccati differential equation and may be understood as a Riccati DAE.

Lemma 1 *If $Y : [0,T] \to L(\mathbb{R}^m,\mathbb{R}^k)$ is continuous with a continuously differentiable part A^*YB^-, and if it satisfies the terminal value problem* (7), (8), *then the symmetry relation*

$$A^*YB^- = B^{-*}Y^*A$$

becomes true.

Remark 1 If Y solves (7), (8) and if, additionally, the condition $A^*YQ = 0$ is given, then it follows that

$$B^*A^*Y = Y^*AB \tag{11}$$

must hold. Conversely, relation (11) implies $A^*YQ = 0$.

Theorem 1 *Let Y be a solution of the terminal value problem $(7),(8)$, and let the condition $A^*YQ = 0$ be fulfilled. Let $x_* \in \mathscr{C}_B^1$ be a solution of the IVP*

$$A(Bx)' = Cx - DR^{-1}(S^* + D^*Y)x, \quad A(0)B(0)x(0) = z_0, \tag{12}$$

and let

$$u_* := -R^{-1}(S^* + D^*Y)x_*.$$

Then (u_*, x_*) is an optimal pair, i.e. it holds for each admissible pair (u,x) that

$$J(u,x) \geq J(u_*,x_*) = \frac{1}{2}\langle z_0, A(0)^{*-}B(0)^{-*}Y(0)^* z_0\rangle.$$

The linear-quadratic optimal control problem (9), (5), (6) is closely related to the boundary value problem (BVP)

$$\begin{bmatrix} A & 0 \\ 0 & -B^* \end{bmatrix}\left(\begin{bmatrix} B & 0 \\ 0 & A^* \end{bmatrix}\begin{bmatrix} x \\ \psi \end{bmatrix}\right)' =$$

$$= \begin{bmatrix} C - DR^{-1}S^* & -DR^{-1}D^* \\ W - SR^{-1}S^* & C^* - SR^{-1}D^* \end{bmatrix}\begin{bmatrix} x \\ \psi \end{bmatrix}, \tag{13}$$

$$A(0)B(0)x(0) = z_0, \tag{14}$$

$$B(T)^*A(T)^*\psi(T) = Vx(T). \tag{15}$$

If this BVP has a solution pair x_*, ψ_*, then $u_* := -R^{-1}(S^*x_* + D^*\psi_*)$ is an optimal control. This can be realized by slightly modifying Lemma 2.2 in [13]. Conversely, if u_*, x_* is an optimal pair, and if the composed matrix function $[AB - CQ, D]$ has on $[0,T]$ full row rank, then there exists an adjoint function ψ_* such that x_*, ψ_* solve the BVP (13)–(15) (see [14]). If A and B are nonsingular, then the full rank condition is always given. For singular A and B, if the full rank condition fails to be valid, then it may happen (see [14]) that there is an optimal pair u_*, x_*, but an adjoint function to solve the BVP does not exist. Assuming the rank condition to be satisfied, we can use the BVP (13)–(15) as a sufficient and necessary optimality condition.

In case of $A = B = I$, system (13) is nothing else than the Hamiltonian ODE associated with the standard linear-quadratic optimal control problem. For singular A and B, (13) is a DAE with a properly stated leading term. We adopt the notion *Hamiltonian system* for this DAE. This is justified, since under certain conditions the inherent dynamic part in (13) actually shows a Hamiltonian flow (see [4]).

While, e.g., in [1], [12] the Riccati-type DAEs are constructed to solve the Hamiltonian system, here a direct optimality proof is applied to Theorem 1 and, at the same time, our new Riccati DAE system is justified.

Remark 2 In [13] we dealt with linear-quadratic optimal control problems in a more general Hilbert space setting, where R is not necessarily invertible and the side conditions are given as $(Bx)' = Cx + Du$, $B(0)x(0) = z_0$. Sufficient solvability conditions are derived by investigating the structure as well as the inherent flow of a linear (abstract) descriptor system associated with a sufficient extremal condition.

3. SOLVABILITY OF THE RICCATI DAE SYSTEM

In this section we consider solutions of the problem (7), (8), which satisfy the condition

$$P_*YQ = 0. \tag{16}$$

Each solution Y that must be continuous with a continuously differentiable part A^*YB^- can be decomposed as

$$\begin{aligned} Y &= P_*YP + Q_*YP + Q_*YQ \\ &= A^{*-}A^*YB^-B + Q_*YP + Q_*YQ. \end{aligned}$$

We are going to show that the components

$$U := A^*YB^- \in \mathscr{C}^1, \;\; \mathscr{V} := Q_*YP, \;\; Z := Q_*YQ = YQ \in \mathscr{C}$$

satisfy a standard Riccati differential equation, a linear equation, and an algebraic Riccati equation, respectively.

Multiplying (7) by Q from the left and right, then by Q from the left and P from the right, and also by B^{-*} from the left and B^- from the right, we obtain the system

$$0 = -(YQ)^*CQ - QC^*YQ +$$
$$+(QS + (YQ)^*D)R^{-1}(S^*Q + D^*YQ) - QWQ, \tag{17}$$
$$0 = -(YQ)^*CP - QC^*YP +$$
$$+(QS + (YQ)^*D)R^{-1}(S^*P + D^*YP) - QWP, \tag{18}$$
$$K^*(A^*YB^-)'K = -(YB^-)^*CB^- - B^{-*}C^*YB^- +$$
$$+ (B^{-*}S + (YB^-)^*D)R^{-1}(S^*B^- + D^*YB^-) - B^{-*}WB^-. \tag{19}$$

Since multiplication of (7) by P from the left and Q from the right yields (18) once more, we know (7) to be equivalent to (17)–(19). Obviously, the component $Z = Q_*YQ = YQ$ satisfies (cf. (17)) the algebraic Riccati equation

$$0 = -Z^*Q_*CQ - QC^*Q_*Z +$$
$$+(QS + Z^*Q_*D)R^{-1}(S^*Q + D^*Q_*Z) - QWQ \tag{20}$$

and the trivial conditions $P_*Z = 0$, $ZP = 0$.

Next, from (18) we obtain a linear relation for the components Z, U, and $\mathscr{V}$, namely,

$$MQ_*\mathscr{V} + MP_*A^{*-}UB = -Z^*Q_*CP +$$
$$+(QS + Z^*Q_*D)R^{-1}S^*P - QWP, \qquad (21)$$

where

$$M := QC^* - (QS + Z^*Q_*D)R^{-1}D^*, \; M = QM.$$

Notice that, if the conditions

$$\mathrm{im}MQ_* = \mathrm{im}Q, \; \ker M \cap \mathrm{im}Q_* = 0 \qquad (22)$$

are fulfilled, we also have $\ker MQ_* = \ker Q_*$; further

$$(MQ_*)^+ MQ_* = Q_*, \; MQ_*(MQ_*)^+ = Q,$$

and equation (21) determines $\mathscr{V}$ uniquely, depending on Z and U. Let us then write

$$\mathscr{V} = C_1 + C_2A^{*-}UB, \qquad (23)$$

with

$$C_1 := (MQ_*)^+\{-Z^*Q_*CP + (QS + Z^*Q_*D)R^{-1}S^*P - QWP\},$$
$$C_2 := -(MQ_*)^+ MP_*.$$

Notice that $(MQ_*)^+$ is continuous. It holds that $C_1 = Q_*C_1 = C_1P$, $C_2 = Q_*C_2 = C_2P_*$.

Finally, we turn to (19). Since K is continuously differentiable and $UK = U$, $K^*U = U$ hold true, we may write

$$K^*(A^*YB^-)'K = K^*U'K = U' - K^{*\prime}U - UK'.$$

Recall that U is symmetric due to Lemma 1. Using (23) we derive $YP = C_1 + C_3A^{*-}UB, \quad C_3 := C_2 + P_*$.

Thus we obtain, from (19), the following differential equation for U:

$$U' = -U\widetilde{C} - \widetilde{C}^*U + U\widetilde{D}R^{-1}\widetilde{D}^*U - \widetilde{W}, \qquad (24)$$

where

$$\widetilde{C}^* := -K^{*\prime} + B^{-*}C^*C_3A^{*-} - B^{-*}(S + C_1^*D)R^{-1}D^*C_3A^{*-},$$
$$\widetilde{D}^* := D^*C_3A^{*-},$$
$$\widetilde{W} := B^{-*}\{PWP + PC_1^*CP + PC^*C_1P -$$
$$-P(S + C_1^*D)R^{-1}(S^* + D^*C_1)P\}B^- = \widetilde{W}^*.$$

Lemma 2 *Let condition* (22) *be given, and additionally,*

$$\mathrm{im}Z = \mathrm{im}Q_*, \quad \ker Z = \ker Q. \qquad (25)$$

Then, (24) *represents a standard Riccati differential equation with a symmetric, positive semidefinite coefficient* $\widetilde{W}$.

The following assertion reflects what we have derived.

Theorem 2 *If Y is a solution of the Riccati-type terminal value problem* (7), (8), (16), *and if the conditions* (22) *and* (25) *are fulfilled, then the component $Z = Q_*YQ$ is a solution of the algebraic Riccati equation* (20), *$U = A^*YB^-$ is a solution of the standard Riccati differential equation* (24), *and $\mathscr{V} = Q_*YP$ satisfies* (21).

Conversely, considering now the following decoupled system for the unknown functions $Z, U, \mathscr{V}$ to be given (cf. (20), (16), (24), (8), (21)) as

$$0 = -Z^*Q_*CQ - QC^*Q_*Z +$$
$$+(QS + Z^*Q_*D)R^{-1}(S^*Q + D^*Q_*Z) - QWQ, \qquad (26)$$
$$P_*Z = 0, \qquad (27)$$
$$ZP = 0, \qquad (28)$$
$$U' = -U^*\widetilde{C} - \widetilde{C}^*U + U^*\widetilde{D}R^{-1}\widetilde{D}^*U - \widetilde{W}, \qquad (29)$$
$$U(T) = B(T)^{-*}VB(T)^-, \qquad (30)$$
$$MQ_*\mathscr{V} = -MP_*A^{*-}UB - QWP - Z^*Q_*CP +$$
$$+(QS + Z^*Q_*D)R^{-1}S^*P, \qquad (31)$$

we may try to compose a solution Y of the original Riccati system (7), (8), (16) from the solutions $Z, U, \mathscr{V}$. Let us recall that the coefficients $\widetilde{C}, \widetilde{D}, \widetilde{W}$, and M as defined above depend on Z.

If Z is a solution of the algebraic equation (26), then $Z + P_*\widetilde{Z}$, where $\widetilde{Z}$ is an arbitrary $k \times m$ matrix function, is also a solution of (26). By means of (27), the arbitrary solution part belonging to $\mathrm{im}P_*$ is fixed as zero.

By multiplication of (26) from both sides by Q we realize that, if Z solves (26), then ZQ does also. By means of condition (28) we pick up solutions with $Z = ZQ$. From (27), (28) we have $Z = Q_*ZQ$.

Obviously, (26) itself is symmetric, but Z is not so necessarily. Notice that Z has k rows and m columns. If $m = k$ and $Q_* = Q$ (i.e., $\ker AB = \ker(AB)^*$), then Z can be expected to be symmetric.

What we need is a continuous solution Z that satisfies the conditions (25) and

$$\mathrm{im}MQ_* = \mathrm{im}Q, \quad \ker MQ_* = \ker Q_*, \qquad (32)$$

with $M = QC^* - (QS + Z^*Q_*D)R^{-1}D^*$.

These requirements ensure that the coefficients $\widetilde{C}$, $\widetilde{D}$, and $\widetilde{W}$ in (29) are well defined and continuous. Additionally, $\widetilde{W}$ is symmetric and positive semidefinite. It turns out that (29) is a standard Riccati differential equation, and the solution U of the terminal value problem (29), (30) is symmetric, $U = U^*$.

Lemma 3 *We are given a continuous solution Z of* (26)–(28) *such that the conditions* (25), (32) *are fulfilled. Then, for the unique solution U of the resulting standard Riccati differential equation* (29), *which satisfies the terminal condition* (30), *the relations*

$$U = U^*, \quad U = UK, \quad U = K^*UK$$

hold true.

Having the matrix functions U and Z, we compose

$$\mathscr{V} := (MQ_*)^+\{-MP_*A^{*-}UB - QWP - Z^*Q_*CP+$$
$$+(QS + Z^*Q_*D)R^{-1}S^*P\}$$

to satisfy (31) and, finally,

$$Y := A^{*-}UB + Z + \mathscr{V}.$$

Under the assumptions of Lemma 3, both $\mathscr{V}$ and Y are continuous. It holds that

$$A^*YB^- = K^*UK = U, \quad Q_*YP = Q_*\mathscr{V}P = \mathscr{V},$$
$$Q_*YQ = Q_*ZQ = Z.$$

The component A^*YB^- of Y is continuously differentiable and symmetric. Straightforward calculations in the direction opposite to that which we realized to provide system (26)–(31) will show Y to be a solution of our system (7), (8), (16). By this, the following assertion providing the solution Y for Theorem 1 is proved.

Theorem 3 *Let the algebraic Riccati system* (26)–(28) *has a continuous solution Z that satisfies the conditions* (25) *and* (32). *Then, the original Riccati DAE system* (7), (8), (16) *has a continuous solution Y whose component A^*YB^- is continuously differentiable and symmetric. Additionally, it holds that $A^*YQ = 0$.*

Remark 3 For special solvability assertions concerning algebraic Riccati equations as well as standard Riccati differential equations, we refer to [15].

Remark 4 Some observations concerning the numerical treatment of the terminal value problem (7), (8), (16) are given in [6], Remark 3.7.

4. SOLVABILITY OF THE CLOSED LOOP PROBLEM

To confirm the existence of an optimal control u_* with the minimal cost $J(u_*,x_*)$ from Theorem 1, in addition to the existence of a Riccati DAE solution Y, one necessarily needs to confirm the existence of a solution of the resulting IVP (12).

Clearly, if A and B are nonsingular, then the IVP (12) has always an uniquely determined solution for each arbitrary z_0. In the case of singular A and B the situation is different, and so for time-invariant descriptor systems (cf. e.g., [1]) one takes care to obtain a closed loop system that has no so-called *impulsive behavior* for any $z_0 \in im(A(0)B(0))$. Within the scope of DAE theory, this means that one should have closed loop systems (12) that are regular with tractability index one.

The tractability index generalizes the Kronecker index of matrix pencils to time-varying DAEs. The basic tools in this concept are special decoupling projectors computed from the coefficients of the given DAE and certain characteristic subspaces. The tractability index for DAEs with a properly stated leading term is defined as in [7], [8]. A brief description is given in [4], [6].

Theorem 4 *Let the conditions of Theorem* 3 *be given, $m = k$, and Y be a solution of the Riccati DAE system* (7), (8), (16). *Then the DAE* (12) *is regular with tractability index one, and there is exactly one solution $x_* \in \mathscr{C}_B^1$ of the IVP* (12).

Theorem 5 *Let the conditions of Theorem* 3 *be given, $m > k$, and Y be a solution of the Riccati DAE system* (7), (8), (16). *Then there are solutions $x_* \in \mathscr{C}_B^1$ of the IVP* (12).

5. RICCATI EQUATIONS AND HAMILTONIAN SYSTEMS

Theorem 6 *Given a solution Y of* (7), (8) *with $A^*YQ = 0$, if the continuous matrix function $X : [0,T] \to L(\mathbb{R}^p,\mathbb{R}^m)$, with a continuously differentiable part BX, satisfies the equation*

$$A(BX)' = (C - DR^{-1}S^* - DR^{-1}D^*Y)X,$$

then the pair $X, \Psi := YX$ forms a solution of the Hamiltonian system

$$A(BX)' = (C - DR^{-1}S^*)X - DR^{-1}D^*\Psi, \qquad (33)$$

$$-B^*(A^*\Psi)' = (W - SR^{-1}S^*)X + (C^* - SR^{-1}D^*)\Psi. \quad (34)$$

Ψ is continuous with $A^*\Psi$ being continuously differentiable.

The above pair X, Ψ combines p columns of solutions of the differential-algebraic Hamiltonian system (13). If one tries to solve the system (33), (34), one is confronted by the index of the DAE (13). Equation (13) has a properly stated leading term since (5) has one. Equation (13) is a square system having $m + k$ equations and $m + k$ unknown functions.

Theorem 7 *If A and B remain nonsingular,* (13) *represents an implicit regular ODE (regular DAE with tractability index zero). Otherwise, for the DAE* (13) *to be regular with tractability index one, it is necessary and sufficient that the following two conditions are satisfied:*

$$[AB - CQ,\, D] \quad \text{has full row rank } k,$$

$$\mathrm{im}[Q(C^* - SR^{-1}D^*)Q_*,\, Q(W - SR^{-1}S^*)Q] = \mathrm{im}Q.$$

Remark 5 In [1], descriptor systems (1) in an *SVD coordinate system* play a special role, and, in particular, the invertibility of a certain matrix $\bar{R}$ (cf. [1]) is a basic property assumed to be given in all four versions of the Riccati differential equations studied in [1, section IV]. From the viewpoint of DAE theory, for those very special systems of the form

$$\begin{bmatrix} A & 0 \\ 0 & -B^* \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} B & 0 & 0 \\ 0 & A^* & 0 \end{bmatrix} \begin{bmatrix} x \\ \psi \\ u \end{bmatrix} \right)' = \begin{bmatrix} C & 0 & D \\ W & C^* & S \\ S^* & D^* & R \end{bmatrix} \begin{bmatrix} x \\ \psi \\ u \end{bmatrix},$$

the invertibility of $\bar{R}$ exactly means regularity with tractability index one (cf. [4]).

Theorem 8 *Let* $X(t) \in L(\mathbb{R}^m, \mathbb{R}^m), \Psi(t) \in L(\mathbb{R}^m, \mathbb{R}^k)$ *be continuous on [0,T], and such that their m columns belong to* $\mathscr{C}_B^1$ *and* $\mathscr{C}_{A^*}^1$*, respectively, and*

$$A(BX)' = (C - DR^{-1}S^*)X - DR^{-1}D^*\Psi,$$

$$-B^*(A^*\Psi)' = (W - SR^{-1}S^*)X + (C^* - SR^{-1}D^*)\Psi$$

is satisfied. Let X be nonsingular and let $X^{-1}B^-$ *belong to* $\mathscr{C}^1$*. Let* $Y := \Psi X^{-1}$ *be such that*

$$P_* Y Q = 0, \quad A^* Y B^- = B^{-*} Y^* A.$$

Then, Y is continuous with a continuously differentiable part A^*YB^- *and satisfies the Riccati DAE system* (7), (16).

If X, Ψ in Theorem 8 are chosen to meet the terminal conditions $B(T)^*A(T)^*\Psi(T) = V$, $A(T)B(T)X(T) = A(T)B(T)$, then it follows that the terminal condition (8) is satisfied.

6. CONCLUSION

We have presented optimal feedback controls of linear-quadratic optimal control problems with constraints described by general linear DAEs with variable coefficients by suitably formulating a Riccati DAE system, similarly to the classical example in which the constraints are described by explicit ODEs. Compared to earlier papers and some less suitable Riccati DAEs, we could do without several restrictive assumptions.

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