Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Output Feedback H_∞ Control of Continuous-time Infinite Markovian Jump Linear Systems via LMI Methods

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Abstract— **The output feedback H_∞ control is addressed for a class of continuous-time Markov jump linear systems with the Markov process taking values in an infinite countable set $\mathcal{S}$. We consider that only an output and the jump parameters are available to the controller. Via a certain bounded real lemma, together with some extensions of Schur complements and of the projection lemma, a theorem which characterizes whether there exists a full-order solution to the disturbance attenuation problem is devised in terms of two different linear matrix inequality (LMI) feasibility problems. This result connects a certain projection approach to an LMI problem which is more amenable to computer solution, and hence for design. We conclude the paper with some algorithms for the construction of such controllers and an illustrative example.**

I. INTRODUCTION

It is a well-known fact that in order to treat adequately problems related to a wide class of dynamical systems, we do need to characterize adequately the uncertainties in the mathematical description of these systems, which can be, for instance, of an environmental and/or modeling nature. These uncertainties have many sources: noise in communications systems, atmospheric fluctuation, volatility in the economic scenario, *failures* (abrupt change in the system structure), parametric uncertainty, etc. In this paper we shall be interested in a class of linear dynamical systems which are subjected to uncertainty (change) in their structures as a consequence of abrupt phenomena. The uncertainties are characterized in the model via a Markov process. These systems are known in the literature as Markov jump linear systems (MJLS) and constitute an important class of hybrid systems (see [1]).

This class of systems (MJLS) has been the subject of intensive research over the last few decades and the associated literature is now fairly extensive. Among the great variety of open problems in this field, our interest here lies on the H_∞ control of MJLS by means of output feedback. Different from [2], [3], we assume that the underlying Markov process takes values in an infinite countable set. As pointed out in [4], the infinite setting calls for extended versions of fundamental tools from LMI theory, such as Schur complements or the projection lemma (see [5]), as well as the bounded real lemma devised in [6].

In this paper we address the output feedback H_∞ control for the case in which the state space of the Markov process is *infinite countable*, in an attempt to summarize the contributions of [4]. Among the main results, we provide a characterization of the existence of solutions to the disturbance attenuation problem in terms of two different LMI feasibility problems. Besides being necessary and sufficient, such characterization establishes the connection between a certain projection approach and an LMI problem which is more suitable for design. As a by-product, two alternative design algorithms are also provided.

The paper is organized as follows. In section II we provide the bare essentials of notation and some auxiliary results. Section III introduces the basic model together with the bounded real lemma, which will be extremely important in section IV, where the H_∞ problem is dealt with. Some tools for the design of full-order H_∞ compensators are given in section V, together with a nominal example. The paper is concluded in section VI with a highlight of the main contributions.

II. NOTATION AND AUXILIARY RESULTS

Let $\|\cdot\|$ denote the euclidean norm in the complex n-space $\mathbb{C}^n$. We define $\mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$ as the Banach space of all complex matrices $M \in \mathbb{C}^{n\times m}$, equipped with the standard induced matrix norm, also denoted by $\|\cdot\|$. Let us also define the infinite dimensional Banach space $\mathbb{H}^{m,n}_{\sup}$ of all matrices of the form $H = (H_1, H_2, \ldots)$ where $H_i \in \mathbb{M}(\mathbb{C}^m, \mathbb{C}^n)$ for every $i \in \mathcal{S} := \{1, 2, \ldots\}$, such that $\|H\|_{\sup} := \sup_{i\in\mathcal{S}} \|H_i\| < \infty$. We also write $\mathbb{H}^n_{\sup}$ in place of $\mathbb{H}^{n,n}_{\sup}$ and define $\mathbb{H}^{n*}_{\sup}$ as the subset of $\mathbb{H}^n_{\sup}$ whose elements $H = (H_1, H_2, \ldots)$ exhibit the additional property that $H_i = H_i^*$ for all $i \in \mathcal{S}$ ($H = H^*$ for short), with * denoting the conjugate transpose (we denote plain transposition by $'$). Next, we define $\tilde{\mathbb{H}}^{n+}_{\sup}$ as the set composed by all *uniformly positive* matrices $H \gg 0$, i.e., such that $H = (H_1, H_2, \ldots) \in \mathbb{H}^{n*}_{\sup}$ and $H_i \geq \varepsilon I_n$ for all $i \in \mathcal{S}$ and some $\varepsilon > 0$ independent of i (here I_n stands for the $n \times n$ identity matrix). Accordingly, we say that $L \in \tilde{\mathbb{H}}^{n-}_{\sup}$ (or is *uniformly negative*, $L \ll 0$) whenever $-L \gg 0$. For short, we write that such $H_i \gg 0$ and $L_i \ll 0$ for all $i \in \mathcal{S}$. Finally, given $R = (R_1, R_2, \ldots) \in \mathbb{H}^{n*}_{\sup}$ and $S = (S_1, S_2, \ldots) \in \mathbb{H}^{n,m}_{\sup}$ we shall write that $R_i \gg 0$ *on* $\mathcal{N}(S_i)$ whenever there exist $\varepsilon > 0$ such that $R_i \geq \varepsilon I_n$ on $\mathcal{N}(S_i)$ for all $i \in \mathcal{S}$, where $\mathcal{N}(\cdot)$ stands for the null space associated to a given complex matrix (accordingly, $\mathcal{R}(\cdot)$ is the range of complex matrices).

For $H \in \mathbb{H}^{p,m}_{\sup}$ and $L \in \mathbb{H}^{n,p}_{\sup}$ we have, in a natural way, that $\|HL\|_{\sup} \leq \|H\|_{\sup}\|L\|_{\sup}$ and thus $HL := (H_1L_1, H_2L_2, \ldots) \in \mathbb{H}^{n,m}_{\sup}$. Moreover, given $F \in \mathbb{H}^{\ell,m}_{\sup}$ we have that $[H\ F] := ([H_1\ F_1], [H_2\ F_2], \ldots) \in \mathbb{H}^{(p+\ell),m}_{\sup}$ (the

This work was partially supported by the Brazilian National Research Council-CNPq, under the Grants No. 301740/2007-0, 470527/2007-2 and 141363/2007-0, and by FAPERJ under the Grant No. E-26/100.579/2007.

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analogous holding for vertical or diagonal block concatenation). We denote $0_{\ell\times m}$ by the zero matrix in either $\mathbb{C}^{\ell,m}$ or $\mathbb{H}^{m,\ell}_{\sup}$, the same holding for the identity matrices $I_\ell \in \mathbb{C}^{\ell\times\ell}$, $I_\ell \in \tilde{\mathbb{H}}^{\ell+}_{\sup}$. Whenever the size of any of those matrices have no importance or may be easily deduced by the context, it will be omitted. In addition, we define $\mathrm{Her}(H) := H + H^*$, $\mathfrak{C}(H, L) := L^*HL$, and sometimes represent off-diagonal blocks of a given self-adjoint matrix (that is, a matrix in a set such as $\mathbb{H}^{n*}_{\sup}$) by $*$, while entries with absolutely no importance are denoted by $\star$. Furthermore, the Kronecker product of complex matrices is denoted by $\otimes$, in the usual way (see [7]).

Concerning the random objects, fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a right-continuous filtration $\mathcal{F}_t \subset \mathcal{F}$ on $t \in \mathbb{R}_+ := [0, \infty)$. In addition, let $E(\cdot)$ denote the usual mathematical expectation and define L^n_2 as the space of all second order random variables $(\Omega, \mathcal{F}) \mapsto \mathbb{C}^n$. We also define the Lebesgue space $L^n_2(\mathbb{R}_+)$ of all stochastic processes $y = \{(y(t), \mathcal{F}_t);\, t \in \mathbb{R}_+,\, y(\cdot) \in \mathbb{C}^n\}$ such that $\|y\|_{\mathbb{R}_+} := (\int_0^\infty E[\|y(t)\|^2]dt)^{1/2}$ is finite.

A. Some auxiliary results

The following theorem extends the well-known result on Schur complements to our setting.

Theorem 1: (uniform Schur complements). Given $U = (U_1, U_2, \dots) \in \mathbb{H}^{p*}_{\sup}$, $V = (V_1, V_2, \dots) \in \mathbb{H}^{q,p}_{\sup}$ and $W = (W_1, W_2, \dots) \in \mathbb{H}^{q*}_{\sup}$, the following are equivalent:

(i) $\left[\begin{smallmatrix} U & V \\ V^* & W\end{smallmatrix}\right] \gg 0$;
(ii) $U \gg 0$ and $W - V^*U^{-1}V \gg 0$;
(iii) $W \gg 0$ and $U - VW^{-1}V^* \gg 0$.

Moreover, the same holds replacing $\gg$ by $\ll$.

Proof: See [4, Theorem 2.2]. ■

A consequence of the preceding result is as follows.

Corollary 1: Suppose $\Psi = (\Psi_1, \Psi_2, \dots) \in \mathbb{H}^{p,\ell}_{\sup}$ is such that $\dim(\mathcal{N}(\Psi_i)) \geq 1$ for any $i \in \mathcal{S}$. Then the following are equivalent, for any $U = (U_1, U_2, \dots) \in \mathbb{H}^{p*}_{\sup}$, $V = (V_1, V_2, \dots) \in \mathbb{H}^{q,p}_{\sup}$ and $W = (W_1, W_2, \dots) \in \mathbb{H}^{q*}_{\sup}$:

(i) $\left[\begin{smallmatrix} U & V \\ V^* & W\end{smallmatrix}\right] \gg 0$ on $\mathcal{N}([\Psi \quad 0])$;
(ii) $U - VW^{-1}V^* \gg 0$ on $\mathcal{N}(\Psi)$ and $W \gg 0$.

Proof: See [4, Corollary 2.4]. ■

The following lemma is an extension from the one depicted in [8] and is of major importance in what follows.

Lemma 1: (uniform projection lemma). Suppose $N = (N_1, N_2, \dots) \in \mathbb{H}^{p,q}_{\sup}$, $M = (M_1, M_2, \dots) \in \mathbb{H}^{p,r}_{\sup}$, and $H = (H_1, H_2, \dots) \in \mathbb{H}^{p*}_{\sup}$. Then the LMI

$$H + N^*X^*M + M^*XN \gg 0 \tag{1}$$

has a solution $X \in \mathbb{H}^{q,r}_{\sup}$ if and only if H is uniformly positive on $\mathcal{N}(N) \cup \mathcal{N}(M)$.

Proof: See [4, Lemma 2.5]. ■

III. Model Setting and Preliminaries

Consider in $(\Omega, \mathcal{F}, \mathbb{P})$ a homogeneous Markov process $\theta = \{(\theta_t, \mathcal{F}_t), t \in \mathbb{R}_+\}$, with right-continuous sample paths and countably infinite state space $\mathcal{S} = \{1, 2, \dots\}$, such that

$$\mathbb{P}(\theta_{t+dt} = j|\theta_t = i) = \begin{cases} \lambda_{ij}dt + o(dt), & i \neq j, \\ 1 + \lambda_{ii}dt + o(dt), & i = j, \end{cases} \tag{2}$$

where $0 \leq \lambda_{ij}$ for $i \neq j$, and $0 \leq \lambda_i := -\lambda_{ii} = \sum_{j\in\mathcal{S}\setminus\{i\}} \lambda_{ij} \leq \varrho$ for some $\varrho < \infty$ and all $i \in \mathcal{S}$. We assume that $\theta_0 : (\Omega, \mathcal{F}) \to \mathcal{S}$ is a random variable with distribution ν_0, and that θ_t is available for every $t \in \mathbb{R}_+$ (this last assumption reflects the fact that the control laws we shall seek are θ_t-dependent).

In order to introduce the bounded real lemma (JBRL), it suffices to consider now a simple version of the MJLS which will be stated in the next section (see (7)).

$$\Sigma : \begin{cases} \dot{\hat{x}}(t) &= \hat{A}_{\theta_t}\hat{x}(t) + \hat{B}_{\theta_t}v(t), \quad \hat{x}(0) = \hat{x}_0 \\ z(t) &= \hat{C}_{\theta_t}\hat{x}(t) + \hat{D}_{\theta_t}v(t), \quad t \in \mathbb{R}_+, \end{cases} \tag{3}$$

for $\hat{x}_0 \in L^{\hat{n}}_2$, where $\hat{A} = (\hat{A}_1, \hat{A}_2, \dots) \in \mathbb{H}^{\hat{n}}_{\sup}$ in the same way as $\hat{B} \in \mathbb{H}^{n_v,\hat{n}}_{\sup}$, $\hat{C} \in \mathbb{H}^{\hat{n},n_z}_{\sup}$ and $\hat{D} \in \mathbb{H}^{n_v,n_z}_{\sup}$. In the sequel we shall point out some fundamental facts regarding this model, including a modified version of the JBRL [6].

Denoting by $\hat{x}(\cdot, \hat{x}_0, \theta_0, v)$ the *state response* of system Σ when subjected to arbitrary initial conditions $(\hat{x}_0, \theta_0)$ and input $v \in L^{n_v}_2(\mathbb{R}_+)$, we begin with the following definition.

Definition 1: For an initial condition $\hat{x}(0) = 0$ and arbitrary θ_0 we define the *zero-state response* of (3) as $\hat{x}_{zs}(\cdot) = \hat{x}(\cdot, 0, \theta_0, v)$. On the other hand, for arbitrary initial conditions but an identically zero input, we have the *zero-input response*, $\hat{x}_{zi}(\cdot) = \hat{x}(\cdot, \hat{x}_0, \theta_0, 0)$. ■

Preserving the terminology, often used in the literature for MJLS, in this paper we deal exclusively with stability in the following internal sense.

Definition 2: System (3) is said to be stochastically stable (SS) if, for any initial condition $\hat{x}_0 \in L^{\hat{n}}_2$ and initial distribution ν_0, we have that $\|\hat{x}_{zi}\|_{\mathbb{R}_+} < \infty$. ■

Concerning SS for this class of systems, an extensive series of results may be found in [9]. In particular, it has been proved that SS of (3) implies $z \in L^{n_z}_2(\mathbb{R}_+)$ for any $v \in L^{n_v}_2(\mathbb{R}_+)$. That is, SS leads to some kind of *external* L_2 input-output stability for this system (see also [6, Remark 2]).

If Σ is SS we may define, in the spirit of the H_∞ theory, the following *perturbation operator* $\mathbb{L} : L^{n_v}_2(\mathbb{R}_+) \to L^{n_z}_2(\mathbb{R}_+)$:

$$\mathbb{L}v(t) = \hat{C}_{\theta_t}\hat{x}_{zs}(t) + \hat{D}_{\theta_t}v(t), \tag{4}$$

which describes how input disturbances affect the output of system Σ, in such a way that $z(\cdot) = \mathbb{L}v(\cdot)$ whenever $\hat{x}_0 = 0$. The worst-case effect of such disturbances is measured by the induced norm of $\mathbb{L}$ from $L^{n_v}_2(\mathbb{R}_+)$ into $L^{n_z}_2(\mathbb{R}_+)$:

$$\|\mathbb{L}\| = \sup_{v\in L^{n_v}_2(\mathbb{R}_+),\ \|v\|_{\mathbb{R}_+}\neq 0} \frac{\|\mathbb{L}v\|_{\mathbb{R}_+}}{\|v\|_{\mathbb{R}_+}}. \tag{5}$$

We conclude this section by presenting one important tool which we shall be employing in the remainder of this paper. The so-called JBRL is the starting point towards a characterization of H_∞ controllers for the class of MJLS under consideration, when it comes to an LMI approach.

Lemma 2: (JBRL). System (3) is SS with $\|\mathbb{L}\| < \gamma$ if and only if there is $P = (P_1, P_2, \dots) \in \tilde{\mathbb{H}}^{\hat{n}-}_{\sup}$ such that, for all

$i \in \mathcal{S}$,

$$\begin{bmatrix} \hat{A}_i^* P_i + P_i \hat{A}_i + \sum_{j\in\mathcal{S}} \lambda_{ij} P_j & P_i \hat{B}_i & \hat{C}_i^* \\ \hat{B}_i^* P_i & \gamma^2 I_{n_v} & \hat{D}_i^* \\ \hat{C}_i & \hat{D}_i & I_{n_z} \end{bmatrix} \gg 0. \quad (6)$$

Proof: The proof follows directly from [6], in view of Theorem 1. ■

IV. Disturbance Attenuation

In this section we shall establish the disturbance attenuation (DA) problem in the jump setting and apply the JBRL in order to characterize the existence of solutions in terms of LMIs.

A. Problem setting

Bearing in mind the definition of $\{\theta_t\}$, consider the following MJLS,

$$\Sigma_u : \left\{ \begin{array}{rcl} \dot{x}(t) &=& A_{\theta_t} x(t) + B_{\theta_t} v(t) + G_{\theta_t} u(t) \\ z(t) &=& C_{\theta_t} x(t) + D_{\theta_t} v(t) + H_{\theta_t} u(t) \\ y(t) &=& \Gamma_{\theta_t} x(t) + L_{\theta_t} v(t), \end{array} \right. \quad (7)$$

for $t \in \mathbb{R}_+$ and $x(0) = x_0 \in L_2^n$, where $A = (A_1, A_2, \ldots) \in \mathbb{H}^n_{\text{sup}}$ in the same way as $B \in \mathbb{H}^{n_v,n}_{\text{sup}}$, $G \in \mathbb{H}^{n_u,n}_{\text{sup}}$, $C \in \mathbb{H}^{n,n_z}_{\text{sup}}$, $D \in \mathbb{H}^{n_v,n_z}_{\text{sup}}$, $H \in \mathbb{H}^{n_u,n_z}_{\text{sup}}$, $\Gamma \in \mathbb{H}^{n,n_y}_{\text{sup}}$ and $L \in \mathbb{H}^{n_v,n_y}_{\text{sup}}$.

We refer to $x(t)$ as the state variable for a given $t \in \mathbb{R}_+$ and to $(x(t), \theta_t)$ as an augmented state variable. Just as before, we assume $v \in L_2^{n_v}(\mathbb{R}_+)$ is any finite energy stochastic disturbance acting on the system; the specific structure of the n_u-dimensional control input u will be defined soon.

We call z and y the *error* and *measurement* outputs, respectively. The former represents some adversely disturbed output, which should be relatively insensitive to input disturbances. In addition, the processes x and z may only be observed through the measurement y (which without any loss of generality is not directly fed by u), what makes this model rather general for both practical and theoretical reasons. Sometimes we refer to the system (7) simply as Σ_u.

To meet the ends of the H_∞ control problem, it is natural to ask the following question about Σ_u: *How does one characterize a class of (suboptimal) controllers $\mathcal{K}$ which guarantee not only stability, but also that the worse-case effect caused by the disturbance (v) on the error output (z) is smaller than some prescribed attenuation level?* (In what follows it will be meaningful to measure such effect through (5).)

In our approach, we consider the class of dynamic compensators that map the augmented measurement process (y, θ) into control policies u according to the following (k-dimensional) model:

$$\mathcal{K} : \begin{bmatrix} \dot{x}_{\mathcal{K}}(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \mathcal{K}^{11}_{\theta_t} & \mathcal{K}^{12}_{\theta_t} \\ \mathcal{K}^{21}_{\theta_t} & \mathcal{K}^{22}_{\theta_t} \end{bmatrix} \begin{bmatrix} x_{\mathcal{K}}(t) \\ y(t) \end{bmatrix}, \quad x_{\mathcal{K}}(0) = 0. \quad (8)$$

The above compensator is completely defined through the matrix $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2, \ldots) \in \mathbb{H}^{(k+n_y),(k+n_u)}_{\text{sup}}$, where $\mathcal{K}_i = [\mathcal{K}_i^{\bullet\bullet}]$, $i \in \mathcal{S}$. For this reason, the same symbol is used to denote both the system and the matrix in question without confusion.

It is possible to incorporate both systems, Σ_u and $\mathcal{K}$, into a *closed-loop* system $\Sigma_{\mathcal{K}}$, with the augmented state variable $(\hat{x}(t), \theta_t) = (x(t), x_{\mathcal{K}}(t), \theta_t)$ for any $t \in \mathbb{R}_+$. Defining $\hat{n} = n + k$ and $\hat{x}_0 = (x_0, 0)$ we have that the state and output equations for this $\hat{n}$-dimensional system may be written as an instance of (3):

$$\Sigma_{\mathcal{K}} : \left\{ \begin{array}{rcll} \dot{\hat{x}}(t) &=& \hat{A}_{\theta_t} \hat{x}(t) + \hat{B}_{\theta_t} v(t), & \hat{x}(0) = \hat{x}_0 \\ z(t) &=& \hat{C}_{\theta_t} \hat{x}(t) + \hat{D}_{\theta_t} v(t), & t \in \mathbb{R}_+ \end{array} \right. \quad (9)$$

where $\hat{A} = (\hat{A}_1, \hat{A}_2, \ldots) \in \mathbb{H}^{\hat{n}}_{\text{sup}}$, $\hat{B} = (\hat{B}_1, \hat{B}_2, \ldots) \in \mathbb{H}^{n_v,\hat{n}}_{\text{sup}}$, $\hat{C} = (\hat{C}_1, \hat{C}_2, \ldots) \in \mathbb{H}^{\hat{n},n_z}_{\text{sup}}$, $\hat{D} = (\hat{D}_1, \hat{D}_2, \ldots) \in \mathbb{H}^{n_v,n_z}_{\text{sup}}$ and, for $i \in \mathcal{S}$,

$$\begin{array}{rclrcl} \hat{A}_i &=& A_i^0 + \hat{G}_i \mathcal{K}_i \hat{\Gamma}_i, & \hat{B}_i &=& B_i^0 + \hat{G}_i \mathcal{K}_i \hat{L}_i, \\ \hat{C}_i &=& C_i^0 + \hat{H}_i \mathcal{K}_i \hat{\Gamma}_i, & \hat{D}_i &=& D_i^0 + \hat{H}_i \mathcal{K}_i \hat{L}_i, \end{array}$$

with

$$\begin{bmatrix} A_i^0 & B_i^0 & \hat{G}_i \\ C_i^0 & D_i^0 & \hat{H}_i \\ \hat{\Gamma}_i & \hat{L}_i & \star \end{bmatrix} = \left[\begin{array}{cc|c|cc} A_i & 0 & B_i & 0 & G_i \\ 0 & 0_k & 0_{k\times n_v} & I_k & 0 \\ \hline C_i & 0_{n_z\times k} & D_i & 0 & H_i \\ \hline 0 & I_k & 0 & \star & \star \\ \Gamma_i & 0 & L_i & \star & \star \end{array} \right],$$

where $\star$ denotes an entry which has no importance in the rest of the paper. For the sake of simplicity we also introduce the following definition.

Definition 3: A compensator $\mathcal{K} \in \mathbb{H}^{(k+n_y),(k+n_u)}_{\text{sup}}$ such as (8) is said to be H_∞ *of level* γ whenever the closed loop system $\Sigma_{\mathcal{K}}$ is SS and $\|\mathbb{L}\| < \gamma$, in accordance to (4)–(5). ■

B. Characterization results (general case)

In this subsection we shall derive conditions which relate the existence of H_∞ controllers of a given level γ to the solvability of linear matrix inequalities. The main results here will be further explored in the next section, where the full-order case is dealt with.

The first result we prove shows that a given compensator $\mathcal{K}$ guarantees SS of system $\Sigma_{\mathcal{K}}$ with a desired DA level γ if and only if a specific LMI feasibility problem in the variable P possesses an adequate solution. It should be noted that *any* H_∞ controller must satisfy this condition in order to solve the DA problem at hand.

Proposition 1: Given a compensator $\mathcal{K}$ and a desired disturbance attenuation level $\gamma > 0$ to be achieved, the following are equivalent:

(i) System $\Sigma_{\mathcal{K}}$ is SS with $\|\mathbb{L}\| < \gamma$;
(ii) There exists $P = (P_1, P_2, \ldots) \in \tilde{\mathbb{H}}^{\hat{n}-}_{\text{sup}}$ such that

$$\mathcal{M}^0 + (\mathcal{H}\mathcal{P})^* \mathcal{K} \mathcal{J} + \mathcal{J}^* \mathcal{K}^* (\mathcal{H}\mathcal{P}) \gg 0, \quad (10)$$

where $\mathcal{M}^0 = (\mathcal{M}_1^0, \mathcal{M}_2^0, \ldots) \in \mathbb{H}^{(\hat{n}+n_v+n_z)*}_{\text{sup}}$, with

$$\mathcal{M}_i^0 = \begin{bmatrix} P_i A_i^0 + (A_i^0)^* P_i + \sum_{j\in\mathcal{S}} \lambda_{ij} P_j & P_i B_i^0 & (C_i^0)^* \\ (B_i^0)^* P_i & \gamma^2 I_{n_v} & D_i^* \\ C_i^0 & D_i & I_{n_z} \end{bmatrix}$$

and $\mathcal{P} \in \mathbb{H}_{\sup}^{(\hat{n}+n_v+n_z)*}$, $\mathcal{H} \in \mathbb{H}_{\sup}^{(\hat{n}+n_v+n_z),(k+n_u)}$ and $\mathcal{J} \in \mathbb{H}_{\sup}^{(\hat{n}+n_v+n_z),(k+n_y)}$ given by

$$\mathcal{P} = \begin{bmatrix} P & 0 & 0 \\ 0 & I_{n_v} & 0 \\ 0 & 0 & I_{n_z} \end{bmatrix}, \qquad \begin{aligned} \mathcal{H} &= \begin{bmatrix} \hat{G}^* & 0_{(k+n_u)\times n_v} & \hat{H}^* \end{bmatrix}, \\ \mathcal{J} &= \begin{bmatrix} \hat{\Gamma} & \hat{L} & 0_{(k+n_y)\times n_z} \end{bmatrix}. \end{aligned}$$

Proof: See [4, Proposition 4.2]. ∎

The following proposition states that the existence of any compensator *at all* which is H_∞ of level γ depends on the feasibility of two specific sets of matrix inequalities.

Proposition 2: There exists a compensator $\mathcal{K}$ which is H_∞ of level γ if and only if the following conditions are satisfied, for some $P = (P_1, P_2, \ldots) \in \tilde{\mathbb{H}}_{\sup}^{\hat{n}-}$:

(i) $\mathcal{M}^0 \gg 0$ on $\mathcal{N}(\mathcal{J})$;
(ii) $\mathcal{P}^{-1}\mathcal{M}^0\mathcal{P}^{-1} \gg 0$ on $\mathcal{N}(\mathcal{H})$.

Proof: See [4, Proposition 4.3]. ∎

It should be noted that, even though the first condition of the last proposition corresponds to an LMI feasibility problem, the same doesn't hold for the second one. Besides, it will be shown that the restrictions on $\mathcal{N}(\mathcal{J})$ and $\mathcal{N}(\mathcal{H})$ can be expressed under simpler forms. With the aid of Corollary 1 we shall restate the last result in a more suitable way, in the sequel.

Proposition 3: For $X, Y \in \mathbb{H}_{\sup}^{n*}$, $P_2 = (P_{21}, P_{22}, \ldots)$, $S_2 = (S_{21}, S_{22}, \ldots) \in \mathbb{H}_{\sup}^{k,n}$, $P_3 = (P_{31}, P_{32}, \ldots) \in \mathbb{H}_{\sup}^{k}$ and $S_3 = (S_{31}, S_{32}, \ldots) \in \mathbb{H}_{\sup}^{k}$, let

$$P_i := \begin{bmatrix} X_i & P_{2i} \\ P_{2i}^* & P_{3i} \end{bmatrix}, \qquad S_i := P_i^{-1} = \begin{bmatrix} Y_i & S_{2i} \\ S_{2i}^* & S_{3i} \end{bmatrix}.$$

Then:

(i) $\mathcal{M}^0 \gg 0$ on $\mathcal{N}(\mathcal{J})$ if and only if

$$\begin{bmatrix} \mathcal{T}_i(X) & X_i B_i & C_i^* \\ B_i^* X_i & \gamma^2 I_{n_v} & D_i^* \\ C_i & D_i & I_{n_z} \end{bmatrix} \gg 0 \quad \text{on} \quad \mathcal{N}\begin{bmatrix} \Gamma_i & L_i & 0_{n_y\times n_z} \end{bmatrix}, \tag{11}$$

(ii) $\mathcal{P}^{-1}\mathcal{M}^0\mathcal{P}^{-1} \gg 0$ on $\mathcal{N}(\mathcal{H})$ if and only if

$$\begin{bmatrix} Q_i(S) & Y_i C_i^* & B_i \\ C_i Y_i & I_{n_z} & D_i \\ B_i^* & D_i^* & \gamma^2 I_{n_v} \end{bmatrix} \gg 0 \quad \text{on} \quad \mathcal{N}\begin{bmatrix} G_i^* & H_i^* & 0_{n_u\times n_v} \end{bmatrix}, \tag{12}$$

where $\mathcal{T}_i(X) = A_i^* X_i + X_i A_i + \sum_{j\in\mathcal{S}} \lambda_{ij} X_j$, and $Q_i(S) = Y_i A_i^* + A_i Y_i + \lambda_{ii} Y_i + \sum_{j\neq i} \lambda_{ij} R_{ij}$, with $R_{ij} := Y_i X_j Y_i + Y_i P_{2j} S_{2i}^* + S_{2i} P_{2j}^* Y_i + S_{2i} P_{3j} S_{2i}^*$.

Proof: See [4, Proposition 4.4]. ∎

Remark 1: In the single-mode case (when $\mathcal{S} = \{1\}$) the above result reconciles with the LMI characterization results stated, for instance, in [10], [8], [11]. ∎

Although the last result applies to the general-order case (when k is arbitrary), it has two major drawbacks:

(i) Relation (12) depends on every entry of S (and P, consequently) through the term $S_i S_j^{-1} S_i$;
(ii) The equality $S = P^{-1}$ leads to the coupling condition $X = (Y - S_2 S_3^{-1} S_2^*)^{-1}$, which is highly non-linear.

In an effort to overcome such drawbacks, we consider in the next subsection the full-order case. The main idea is that, by restricting ourselves to a specific class of Lyapunov functions, a fairly complete LMI characterization may be derived along the lines of Proposition 3.

C. Characterization results (full-order case)

This subsection deals with the so-called *full-order case*, in which $k = n$ (that is, we consider controllers of the same order as the to-be-controlled system). The main result (Theorem 3) states an equivalent condition to the existence of H_∞ controllers of such type in terms of two distinct LMI problems (see also Algorithm 1).

The basic idea here is to restrict ourselves to the class of quadratic Lyapunov functions parametrized by

$$P_i = \begin{bmatrix} X_i & Y_i^{-1} - X_i \\ Y_i^{-1} - X_i & X_i - Y_i^{-1} \end{bmatrix}, \quad i \in \mathcal{S}. \tag{13}$$

Notice that, in this case, $S_i := P_i^{-1} = \left[\begin{smallmatrix} Y_i & Y_i \\ Y_i & \star \end{smallmatrix}\right]$ for every such i, and hence $S_i S_j^{-1} S_i = \left[\begin{smallmatrix} Y_i Y_j^{-1} Y_i & \star \\ \star & \star \end{smallmatrix}\right]$.

In what follows we shall investigate what conditions must $X, Y \in \mathbb{H}_{\sup}^{n*}$ satisfy so that a quadratic Lyapunov function may be defined with the aid of (13). First, we derive a sufficient condition in terms of LMIs. This result, in conjunction with an auxiliary lemma, will be germane to the proof of Theorem 3, which shows that the sufficient condition is also necessary.

Theorem 2: There exists a full-order H_∞ compensator $\mathcal{K}$ of level γ whenever there exist $X, Y \in \mathbb{H}_{\sup}^{n*}$ such that the following set of LMIs is satisfied for every $i \in \mathcal{S}$:

$$\begin{bmatrix} A_i^* X_i + X_i A_i + \sum_{j\in\mathcal{S}} \lambda_{ij} X_j & X_i B_i & C_i^* \\ B_i^* X_i & \gamma^2 I_{n_v} & D_i^* \\ C_i & D_i & I_{n_z} \end{bmatrix} \gg 0 \quad \text{on} \quad \mathcal{N}\begin{bmatrix} \Gamma_i & L_i & 0_{n_y\times n_z} \end{bmatrix}, \tag{14a}$$

$$\begin{bmatrix} Y_i A_i^* + A_i Y_i + \lambda_{ii} Y_i & Y_i C_i^* & B_i & \underline{\lambda}_i' \otimes Y_i \\ C_i Y_i & I_{n_z} & D_i & 0 \\ B_i^* & D_i^* & \gamma^2 I_{n_v} & 0 \\ \underline{\lambda}_i \otimes Y_i & 0 & 0 & \mathbb{D}_i(Y) \end{bmatrix} \gg 0 \quad \text{on} \quad \mathcal{N}\begin{bmatrix} G_i^* & H_i^* & 0_{n_u\times n_v} & 0_{n_u\times\infty} \end{bmatrix}, \tag{14b}$$

and

$$\begin{bmatrix} Y_i & I \\ I & X_i \end{bmatrix} \ll 0, \tag{14c}$$

in which $0_{\bullet\times\infty}$ is a zero matrix with an infinite number of columns and, for any such i,

$$\begin{aligned} \underline{\lambda}_i &:= \operatorname{col}\left(\lambda_{i1}^{1/2}, \ldots, \lambda_{i(i-1)}^{1/2}, \lambda_{i(i+1)}^{1/2}, \ldots\right) \\ \mathbb{D}_i(Y) &:= -\operatorname{diag}(Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots). \end{aligned} \tag{15}$$

Proof: See [4, Theorem 4.6]. ∎

Before presenting our main result, we state the following lemma. It is proven that the feasibility of a specific LMI problem is necessary for the existence of full-order controllers, in the spirit of [12, Theorem 4].

Lemma 3: Suppose there exists a full-order H_∞ compensator $\mathcal{K}$ of level γ. Then there exist $X = (X_1, X_2, \ldots)$,

$Y = (Y_1, Y_2, \dots) \in \tilde{\mathbb{H}}^{n-}_{\sup}$, $J = (J_1, J_2, \dots) \in \mathbb{H}^{n_y,n}_{\sup}$, $F = (F_1, F_2, \dots) \in \mathbb{H}^{n,n_u}_{\sup}$ and $U = (U_1, U_2, \dots) \in \mathbb{H}^{n_y,n_u}_{\sup}$ such that the following LMIs are satisfied for every $i \in \mathcal{S}$:

$$\begin{bmatrix} \mathcal{T}_i(X) + \mathrm{Her}(J_i\Gamma_i) & * & * \\ (X_iB_i + J_iL_i)^* & \gamma^2 I_{n_v} & * \\ C_i + H_iU_i\Gamma_i & D_i + H_iU_iL_i & I_{n_z} \end{bmatrix} \gg 0, \quad (16a)$$

$$\begin{bmatrix} \mathcal{Q}_i(Y,F) & * & * & * \\ (B_i + G_iU_iL_i)^* & \gamma^2 I_{n_v} & * & * \\ C_iY_i + H_iF_i & D_i + H_iU_iL_i & I_{n_z} & * \\ \underline{\lambda}_i \otimes Y_i & 0 & 0 & \mathbb{D}_i(Y) \end{bmatrix} \gg 0, \quad (16b)$$

$$\begin{bmatrix} Y_i & I \\ I & X_i \end{bmatrix} \ll 0, \quad (16c)$$

where $\mathcal{T}_i(X) := A_i^*X_i + X_iA_i + \sum_{j\in\mathcal{S}} \lambda_{ij}X_j$, and $\mathcal{Q}_i(Y,F) := \mathrm{Her}(A_iY_i + G_iF_i) + \lambda_{ii}Y_i$.

Proof: See [4, Lemma 4.7]. ■

The main result of this section unifies the results obtained so far by giving equivalent conditions to the existence of full-order H_∞ compensators.

Theorem 3: (full-order characterization). The following statements are equivalent:

(i) There exists a full-order H_∞ compensator $\mathcal{K}$ of level γ;

(ii) There exist suitable X, Y such that (14) is satisfied for every $i \in \mathcal{S}$;

(iii) There exist suitable X, Y, J, F and U such that (16) is satisfied for every $i \in \mathcal{S}$.

Moreover, given any X, Y satisfying *(ii)* there always exist suitable J, F and U such that *(iii)* is satisfied.

Proof: See [4, Theorem 4.8]. ■

Bearing in mind the above characterization result we present the following algorithm, as a solution to the *existence* problem. It is noteworthy that both of the design procedures we shall present in the next subsection (Algorithms 2 and 3) depend, to some extent, on it.

Algorithm 1: (existence of compensators). The existence of some H_∞ compensator of given level $\gamma > 0$ is guaranteed by solving any one of the following convex feasibility problems:

e$_1$: Find $X = (X_1, X_2, \dots)$, $Y = (Y_1, Y_2, \dots) \in \mathbb{H}^{n*}_{\sup}$, $J = (J_1, J_2, \dots) \in \mathbb{H}^{n_y,n}_{\sup}$, $F = (F_1, F_2, \dots) \in \mathbb{H}^{n,n_u}_{\sup}$ and $U = (U_1, U_2, \dots) \in \mathbb{H}^{n_y,n_u}_{\sup}$ such that (16) is satisfied for every $i \in \mathcal{S}$.

e$_2$: Find $X = (X_1, X_2, \dots)$, $Y = (Y_1, Y_2, \dots) \in \mathbb{H}^{n*}_{\sup}$ such that (14) is satisfied for every $i \in \mathcal{S}$;

On the other hand, whenever it may be proved that either of these problems doesn't have a solution then such compensator does not exist at all. ■

Finally, we would like to point out that the above algorithm is of immediate practical interest when it comes to the finite case, in the sense that it can be efficiently implemented by widely available convex programming software (see, for instance, [5] and references therein).

V. Design

In this section we present some tools for the design of *full-order H_∞* compensators. The main theoretical result, whose *statement* has been inspired in [2, Theorem 4.2] (see also [10]), provides the aforementioned formulas for construction of full-order H_∞ compensators, as follows.

Theorem 4: Suppose suitable X, Y, J, F and U satisfying the conditions of Theorem 3 may be found. Then the following full-order compensator guarantees that SS of the closed-loop system $\Sigma_{\mathcal{K}}$ is achieved along with a DA level γ:

$$\mathcal{K}^{12} = \left(Y^{-1} - X\right)^{-1}(J - XGU), \quad (17)$$

$$\mathcal{K}^{21} = (F - U\Gamma Y)\,Y^{-1}, \quad (18)$$

$$\mathcal{K}^{22} = U, \quad (19)$$

and, for every $i \in \mathcal{S}$,

$$\begin{aligned} \mathcal{K}_i^{11} = &-\left(Y_i^{-1} - X_i\right)^{-1}\Big\{X_i(A_iY_i + G_iF_i) \\ &+ (J_i - X_iG_iU_i)\Gamma_iY_i + \tilde{A}_i^* + \sum_{j\in\mathcal{S}} \lambda_{ij}Y_j^{-1}Y_i \\ &- \tilde{C}_i^*(C_iY_i + H_iF_i) - \left[X_iB_i + J_iL_i - \tilde{C}_i^*\tilde{D}_i\right] \\ &\times \Upsilon_i^{-1}\left[\tilde{B}_i - (C_iY_i + H_iF_i)^*\tilde{D}_i\right]^*\Big\}Y_i^{-1}, \end{aligned} \quad (20)$$

where $\Upsilon_i := \gamma^2 I - \tilde{D}_i^*\tilde{D}_i$, and $\left[\begin{smallmatrix} \tilde{A}_i & \tilde{B}_i \\ \tilde{C}_i & \tilde{D}_i \end{smallmatrix}\right] = \left[\begin{smallmatrix} A_i & B_i \\ C_i & D_i \end{smallmatrix}\right] + \left[\begin{smallmatrix} G_i \\ H_i \end{smallmatrix}\right] U_i \left[\begin{smallmatrix} \Gamma_i & L_i \end{smallmatrix}\right]$.

Proof: See [4, Theorem 5.1]. ■

Remark 2: Suppose D and $\mathcal{K}^{22} = U$ are identically zero. Then, it is not difficult to see that (17), (18) and (20) reduce to the result from Theorem 4.2 in [2], in case all data are real and the set $\mathcal{S}$ is finite. ■

A. Some algorithms

In the sequel we shall present some design procedures, in order to put our results in a more practical basis. It should be noted that this whole design framework provides a collection of tools which may be readily implemented on convex programming software, at least in the finite case.

The next algorithm provides one possible way of computing a full-order controller such as the one presented in Theorem 4.

Algorithm 2: (two-step design procedure). An H_∞ compensator of given level $\gamma > 0$ may be constructed according to Theorem 4 by the following steps:

d$_1$: Solve the existence problem by means of **e$_1$**;
$\hookrightarrow$ If such a solution can't be found, then **stop**.

d$_2$: Bearing X, Y, J, F and U from the previous step build a compensator by means of relations (17)–(20). ■

Looking back at Theorem 3 it is possible to propose the following alternative to Algorithm 2. The main advantage here is that the existence of solutions depends on the feasibility of a problem of relatively smaller dimension.

Algorithm 3: (three-step design procedure). An H_∞ compensator of given level $\gamma > 0$ may be constructed according to Theorem 4 by the following steps:

D_1: Solve the existence problem by means of $\mathbf{e}_2$;
$\hookrightarrow$ If such a solution can't be found, then **stop**.

D_2: Bearing such X and Y from the above step, find $J = (J_1, J_2, \ldots) \in \mathbb{H}^{n_y,n}_{\sup}$, $F = (F_1, F_2, \ldots) \in \mathbb{H}^{n,n_u}_{\sup}$ and $U = (U_1, U_2, \ldots) \in \mathbb{H}^{n_y,n_u}_{\sup}$ such that (16a) and (16b) are satisfied (from Theorem 3 we have that there always exist a solution to this problem);

D_3: With X, Y, J, F and U obtained from the previous steps, build a compensator by means of relations (17)–(20). ∎

An explicit implementation of the above design procedures is presented in the sequel. It is noteworthy that the example under consideration does not satisfy the simplifying assumptions of [2] and, by consequence, cannot be tackled by the results therein.

B. A nominal example

Let $\mathcal{S} = \{1, 2\}$, and consider system (7) in the form

$$\left[\begin{array}{ccc|ccc} A_1 & B_1 & G_1 & A_2 & B_2 & G_2 \\ C_1 & D_1 & H_1 & C_2 & D_2 & H_2 \\ \Gamma_1 & L_1 & \star & \Gamma_2 & L_2 & \star \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & -1 & \star & 1 & 1 & \star \end{array}\right]$$

with Markov switching governed by $\left[\begin{smallmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 & +1 \\ +1 & -1 \end{smallmatrix}\right]$

Notice that the results of [2], [3] do not immediately apply here, since $D_2 \neq 0$. In the sequel we shall design a controller which ensures stochastic stability of the closed-loop system together with a prescribed disturbance attenuation level γ.

Let $\gamma = 5$. By employing Algorithm 1 we obtain that a feasible solution to (14)–(16) is given by

i	X_i	Y_i	J_i	F_i	U_i
1	-4.1633	-0.5227	19.9958	1.5509	-0.0727
2	-5.0761	-0.2459	19.4919	-0.7369	-1.0000

either by performing $\mathbf{d}_1$ (in Algorithm 2) or $\mathbf{D}_1$–$\mathbf{D}_2$ (Algorithm 3). A suboptimal controller is then given by

$$\mathcal{K}_1 = \begin{bmatrix} \mathcal{K}_1^{11} & \mathcal{K}_1^{12} \\ \mathcal{K}_1^{21} & \mathcal{K}_1^{22} \end{bmatrix} = \begin{bmatrix} -9.3202 & 8.7519 \\ -2.8944 & -0.0727 \end{bmatrix}$$
$$\mathcal{K}_2 = \begin{bmatrix} \mathcal{K}_2^{11} & \mathcal{K}_2^{12} \\ \mathcal{K}_2^{21} & \mathcal{K}_2^{22} \end{bmatrix} = \begin{bmatrix} -28.4562 & 24.3391 \\ 3.9967 & -1.0000 \end{bmatrix}$$

Finally, by performing the H_∞ optimization procedure of [4, Algorithm 5.5] with initial condition $\gamma_{\max} = 5$ we obtain, after 22 iterations and with precision $\varepsilon = 10^{-6}$, the optimal H_∞ performance $\gamma_* \approx 3.42735$.

One final remark goes as follows. For the example under consideration, the obtained results for the optimal case (γ close to γ_*) are such that $XY \approx 1$, what gives rise to unbounded controller entries in (17) and (20). Although being a drawback of the presented method, we remark that the same kind of problem arises in the LTI case (see [8, Section 9.4], for example).

VI. Conclusions

In this paper, the output feedback H_∞ control has been addressed for a class of continuous-time Markov jump linear systems with the Markov process taking values in an infinite countable set $\mathcal{S}$. We have obtained the following results:

- A theorem which characterizes whether there exists a full-order solution to the disturbance attenuation problem in terms of two distinct sets of LMIs (Theorem 3). This result connects a certain projection approach to an LMI problem which is more suitable for design.
- Extensions of Schur complements and of the projection lemma to a wider context. We remark here that one is faced with the same *uniformity* problems if dealing with, say, time-variant systems, (as, e.g., in [12]) considering the case where the system parameters are time functions with uniform bounds.
- An LMI algorithm (Algorithm 1), which allows one to check whether there is a solution for the DA problem.
- A two-step design method (Algorithm 2) which provides explicit formulas for the construction of a controller.
- An alternative three-step design method, Algorithm 3. The main issue here is that one can first check if a smaller (projected) LMI problem is feasible, which amounts to verifying whether the DA problem has a solution or not. This partial solution is then fed into the two-step procedure.
- A nominal example, for the finite case, which illustrates how the obtained results may be employed in a situation where the hypotheses of [2], [3] are not satisfied.

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