Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Identifiability of piecewise constant conductivity and its stability

Semion Gutman and Junhong Ha

***Abstract*— We study the identifiability (i.e. the unique identification) problems for 1-D heat conduction in a nonhomogeneous rod. The piecewise constant conductivity of the rod can be uniquely identified from finitely many observations of the process at equidistant points. Such an identification is accomplished by the novel Marching Algorithm. In addition, the continuity of the solution and identification maps is established. An algorithm for the conductivity recovery from noisy data is proposed.**

I. INTRODUCTION

Consider the heat conduction in a nonhomogeneous insulated rod of a unit length, with the ends kept at zero temperature at all times. Our main interest is in the identification and identifiability of the discontinuous conductivity (thermal diffusivity) coefficient $a(x),\ 0 \le x \le 1$. The **identification** problem consists of finding a conductivity $a(x)$ in an admissible set K for which the temperature $u(x,t)$ fits given observations in a prescribed sense. Under a wide range of conditions one can establish the continuity of the objective function $J(a)$ representing the best fit to the observations. Then the existence of the best fit to data conductivity follows if the admissible set K is compact in the appropriate topology. However, such an approach usually does not guarantee the uniqueness of the found conductivity $a(x)$. Establishing such a uniqueness is referred to as the **identifiability** problem.

From physical considerations the conductivity coefficients $a(x)$ are assumed to be in

$$A_{\mathrm{ad}} = \{a \in L^\infty(0,1) : 0 < \nu \le a(x) \le \mu\}. \qquad (1)$$

The temperature $u(a) = u(x,t;a)$ inside the rod satisfies

$$\begin{array}{ll} u_t - (a(x)u_x)_x = 0, & Q = (0,1)\times(0,T), \\ u(0,t) = u(1,t) = 0, & t \in (0,T), \\ u(x,0) = g(x), & x \in (0,1), \end{array} \qquad (2)$$

where $g \in H = L^2(0,1)$. Suppose that one is given an observation $z(t) = u(p,t;a)$ of the heat conduction process (2) for $t_1 < t < t_2$ at some observation point $0 < p < 1$. From the series solution for (2) and the uniqueness of the Dirichlet series expansion (see Section 2), one can, in principle, recover all the eigenvalues of the associated Sturm-Lioville problem. If one also knows the eigenvalues for the heat conduction process with the same coefficient a and different boundary conditions, then classical results of Gelfand and Levitan [3] show that smooth coefficients $a(x)$ can be uniquely identified from the knowledge of the two spectral sequences. Also, if the entire spectral function is known (i.e. the eigenvalues and the values of the derivatives of the normalized eigenfunctions at $x = 0$), then the conductivity is identifiable as well. However, such results have little practical value, since the observation data $z(t)$ always contain some noise, and therefore one cannot hope to adequately identify more than just a few first eigenvalues of the problem.

A different approach is taken in [8,13,14,15]. These works show that one can identify a constant conductivity a in (2) from the measurement $z(t)$ taken at one point $p \in (0,1)$. These works also discuss problems more general than (2), including problems with a broad range of boundary conditions, non-zero forcing functions, as well as elliptic and hyperbolic problems. In [9, 2] and references therein identifiability results are obtained for elliptic and parabolic equations with discontinuous parameters in a multidimensional setting. A typical assumption there is that one knows the normal derivative of the solution at the boundary of the region for every Dirichlet boundary input.

Suppose that the conductivity a is known to be piecewise continuous with sufficiently separated points of discontinuity. More precisely, let $a \in PC(\sigma)$ defined in Section 2. Let $u(x,t;a)$ be the solution of (2). The eigenfunctions and the eigenvalues for (2) are defined from the associated Sturm-Liouville problem (5).

In our approach the identifiability is achieved in two steps: First, given finitely many equidistant observation points $\{p_m\}_{m=1}^{M-1}$ on interval $(0,1)$ (as specified in Theorem 7), we extract the first eigenvalue $\lambda_1(a)$ and a constant nonzero multiple of the first eigenfunction $G_m(a) = C(a)\psi_1(p_m;a)$ from the observations $z_m(t;a) = u(p_m,t;a)$. This defines the M-tuple

$$\mathcal{G}(a) = (\lambda_1(a), G_1(a), \cdots, G_{M-1}(a)) \in \mathbb{R}^M. \qquad (3)$$

Second, the Marching Algorithm (see Theorem 7) identifies the conductivity a from $\mathcal{G}(a)$.

We start by recalling some basic properties of (2) in Section 2. The main result of this paper is Theorem 7. It is discussed in Section 3. The continuity properties of the solution map $a \to \mathcal{G}(a)$ and the identification map $\mathcal{G}^{-1}(a)$ are established in Section 4. An algorithm for the identification of a from noisy data is presented in Section 5.

This exposition outlines main results obtained in [6], [7]. Paper [7] also contains an extension of the identifiability results to systems with nonzero boundary and external inputs.

II. AUXILIARY RESULTS

In this section we collect some results for the solutions $u(x,t;a)$ of (2), as well as for its associated Sturm-Liouville

S. Gutman is with the Department of Mathematics, The University of Oklahoma, Norman, Oklahoma 73019, USA sgutman@ou.edu

J. Ha is with the School of Liberal Arts, Korea University of Technology and Education, Cheonan 330-708, South Korea hjh@kut.ac.kr

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

problem. See [1,6,10,11,12] for a detailed discussion.

Definition 1: (1). Function $a(x)$ is said to belong to the class $\mathcal{PC}_N$ if $a \in A_{\rm ad} = \{a \in L^\infty(0,1) : 0 < \nu \le a(x) \le \mu\}$ for some positive constants ν and μ, and it has the form $a(x) = a_i$ for $x \in [x_{i-1}, x_i),\ i = 1, 2, \cdots, N$.

(2). Let $\mathcal{PC} = \cup_{N=1}^{\infty} \mathcal{PC}_N$.

(3). Let $\sigma > 0$. Define

$$\mathcal{PC}(\sigma) = \{a \in \mathcal{PC}\ :\ x_n - x_{n-1} \ge \sigma,\ n = 1, \cdots, N\}.$$

Note that $a \in \mathcal{PC}(\sigma)$ attains at most $N = [[1/\sigma]]$ distinct values $a_i, 0 < \nu \le a_i \le \mu$.

Let $a \in \mathcal{PC}_N$. Then the governing system (2) is

$$\begin{aligned} &u_t - a_i u_{xx} = 0, \quad x \in (x_{i-1}, x_i), \quad t \in (0, T),\\ &u(0,t) = u(1,t) = 0, \quad t \in (0,T),\\ &u(x_i+, t) = u(x_i-, t),\\ &a_{i+1}u_x(x_i+, t) = a_i u_x(x_i-, t),\\ &u(x,0) = g(x), \quad x \in (0,1), \end{aligned} \tag{4}$$

where $g \in L^2(0,1)$ and $i = 1, 2, \cdots, N-1$. Denote by $\|\cdot\|$, $<\cdot,\cdot>$ the norm and the inner product in $H = L^2(0,1)$.

Theorem 2: Let $a \in \mathcal{PC}$. Then

(i) The associated Sturm-Liouville problem

$$\begin{aligned} &(a(x)v(x)')' = -\lambda v(x), \quad x \ne x_i,\\ &v(0) = v(1) = 0,\\ &v(x_i+) = v(x_i-),\\ &a(x_i+)v_x(x_i+) = a(x_i-)v_x(x_i-) \end{aligned} \tag{5}$$

has infinitely many eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots \to \infty.$$

The normalized eigenfunctions $\{v_k\}_{k=1}^{\infty}$ form an orthonormal basis in $L^2[0,1]$. Eigenfunctions $\{v_k/\sqrt{\lambda_k}\}_{k=1}^{\infty}$ form an orthonormal basis in V_a, where V_a is $H_0^1[0,1]$ with the norm $\|v\|_a^2 = \int_0^1 a(x)|v'(x)|dx$.

(ii) Each eigenvalue is simple. For each eigenvalue λ_k there exists a unique continuous, piecewise smooth normalized eigenfunction $v_k(x)$ such that $v_k'(0+) > 0$, and the function $a(x)v_k'(x)$ is continuous on $[0,1]$.

(iii) Eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ satisfy the inequality

$$\nu\pi^2 k^2 \le \lambda_k \le \mu\pi^2 k^2.$$

(iv) First eigenfunction v_1 satisfies $v_1(x) > 0$ for any $x \in (0,1)$.

(v) First eigenfunction v_1 has a unique point of maximum $q \in (0,1) :\ v_1(x) < v_1(q)$ for any $x \ne q$.

(vi) For any fixed $t > 0$ the solution u of (2) is given by

$$u(x,t;a) = \sum_{k=1}^{\infty} < g, v_k > e^{-\lambda_k t} v_k(x),$$

and the series converges uniformly and absolutely on $[0,1]$.

(vii) For any $p \in (0,1)$ function

$$z(t) = u(p,t;a), \quad t > 0$$

is real analytic on $(0,\infty)$.

Series of the form $\sum_{k=1}^{\infty} C_k e^{-\lambda_k t}$ are known as the Dirichlet series. The following lemma shows that the Dirichlet series representation of a function is unique.

Lemma 3: Let $\mu_k > 0,\ k = 1, 2, \cdots$ be a strictly increasing sequence. Suppose that $T_1 \ge 0$ and $\sum_{k=1}^{\infty} |C_k| < \infty$. If

$$\sum_{k=1}^{\infty} C_k e^{-\mu_k t} = 0 \quad \text{for all} \quad t \in (T_1, T_2)$$

then $C_k = 0$ for $k = 1, 2, \cdots$.

The result follows at once from the observation that the series $\sum_{k=1}^{\infty} C_k e^{-\mu_k z}$ converges uniformly in $Re\ z > 0$ region of the complex plane, implying that it is an analytic function there. See Chapter 9 of [17] for additional results on Dirichlet series.

Remark. According to Theorem 2(vi) for each fixed $p \in (0,1)$ the solution $z(t) = u(p,t;a)$ of (2) is given by a Dirichlet series. However, Lemma 3 is not directly applicable since the coefficients $C_k =< g, v_k > v_k(p)$ are only square summable. Nevertheless, the conclusion of Lemma 3 remains valid since the exponents μ_k in the Dirichlet series are the eigenvalues λ_k which satisfy the growth condition stated in (iii) of Theorem 2. This allows one to conclude (Theorem 2(vii)) that the solution $z(t)$ is a real analytic function on $(0,\infty)$ and the uniqueness of such representation follows. Thus it would be a mistake to simply refer to the standard results such as Lemma 3 for the uniqueness of the Dirichlet series representation to justify the paper's conclusions.

III. Identifiability of piecewise constant conductivities from finitely many observations

The central part of the identification method is the Marching Algorithm contained in Theorem 7. Recall that it uses only the M-tuple $\mathcal{G}(a)$, see (3). That is we need only the first eigenvalue λ_1 and a nonzero multiple of the first eigenfunction v_1 of (5) for the identification of the conductivity $a(x)$.

Suppose that $p^* \in (x_{i-1}, x_i)$. Then v_1 can be expressed on (x_{i-1}, x_i) as

$$v_1(x) = A\cos\left(\sqrt{\frac{\lambda_1}{a_i}}(x - p^*) + \gamma\right), \quad -\frac{\pi}{2} < \gamma < \frac{\pi}{2}$$

with $A > 0$. The range for γ in the above representation follows from the fact that $v_1(p^*) = A\cos\gamma > 0$ by Theorem 2(iv).

The identifiability of piecewise constant conductivities is based on the following three Lemmas, see [6].

Lemma 4: Suppose that $\delta > 0$. Assume $Q_1, Q_3 \ge 0$, $Q_2 > 0$ and $0 < Q_1 + Q_3 < 2Q_2$. Let

$$\Gamma = \left\{(A, \omega, \gamma)\ :\ A > 0,\ 0 < \omega < \frac{\pi}{2\delta},\ -\frac{\pi}{2} < \gamma < \frac{\pi}{2}\right\}.$$

Then the system of equations

$$A\cos(\omega\delta - \gamma) = Q_1, \quad A\cos\gamma = Q_2, \quad A\cos(\omega\delta + \gamma) = Q_3$$

has a unique solution $(A, \omega, \gamma) \in \Gamma$ given by

$$\omega = \frac{1}{\delta}\arccos\frac{Q_1 + Q_3}{2Q_2}, \quad \gamma = \arctan\left(\frac{Q_1 - Q_3}{2Q_2\sin\omega\delta}\right),$$

$$A = \frac{Q_2}{\cos\gamma}.$$

Lemma 5: Suppose that $\delta > 0,\ 0 < p \le x_1 < p + \delta < 1,\ 0 < \omega_1, \omega_2 < \pi/2\delta$.

Let $w(x),\ v(x),\ x \in [p, p+\delta]$ be such that

$$w(x) = A_1 \cos\omega_1 x + B_1 \sin\omega_1 x,$$
$$v(x) = A_2 \cos\omega_2 x + B_2 \sin\omega_2 x.$$

Suppose that

$$v(x_1) = w(x_1), \quad \omega_1^2 v'(x_1) = \omega_2^2 w'(x_1),$$
$$v'(x_1) > 0, \quad v(x_1) > 0.$$

Then

(i) Conditions $v(p+\delta) = w(p+\delta),\ v'(p+\delta) \ge 0$ and $\omega_1 \le \omega_2$ imply $\omega_1 = \omega_2$.

(ii) Conditions $v(p+\delta) = w(p+\delta),\ w'(p+\delta) \ge 0$ and $\omega_1 \ge \omega_2$ imply $\omega_1 = \omega_2$.

Lemma 6: Let $\delta > 0,\ 0 < \eta \le 2\delta,\ \omega_1 \ne \omega_2$ with $0 < \omega_1\delta, \omega_2\delta < \pi/2$. Also let $A, B > 0,\ 0 \le p < p + \eta \le 1$ and

$$w(x) = A\cos[\omega_1(x-p) + \gamma_1],$$
$$v(x) = B\cos[\omega_2(x-p-\eta) + \gamma_2]$$

with $|\gamma_1|, |\gamma_2| < \pi/2$.

Then system

$$w(q) = v(q), \tag{6}$$
$$\omega_2^2 w'(q) = \omega_1^2 v'(q), \tag{7}$$
$$w(q) > 0, \quad v(q) > 0 \tag{8}$$

admits at most one solution q on $[p, p+\eta]$. This unique solution q can be computed as follows:

If $\gamma_1 \ge 0$ then

$$q = p + \frac{1}{\omega_1}\left[\arctan\left(\omega_1\sqrt{\left|\frac{B^2 - A^2}{A^2\omega_2^2 - B^2\omega_1^2}\right|}\right) - \gamma_1\right]. \tag{9}$$

If $\gamma_2 \le 0$ then

$$q = p + \eta + \frac{1}{\omega_2}\left[-\arctan\left(\omega_2\sqrt{\left|\frac{B^2 - A^2}{A^2\omega_2^2 - B^2\omega_1^2}\right|}\right) - \gamma_2\right]. \tag{10}$$

Otherwise compute q_1 and q_2 according to formulas (9) and (10) and discard the one that does not satisfy the conditions of the Lemma.

By the definition of $a \in \mathcal{PC}$ there exist $N \in \mathbf{N}$ and a finite sequence $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$ such that a is a constant on each subinterval $(x_{n-1}, x_n),\ n = 1, \cdots, N$. Let $\sigma > 0$.

The following Theorem is our main result.

Theorem 7: Given $\sigma > 0$ let an integer M be such that

$$M \ge \frac{3}{\sigma} \quad \text{and} \quad M > 2\sqrt{\frac{\mu}{\nu}}.$$

Suppose that the initial data $g(x) > 0,\ 0 < x < 1$ and the observations $z_m(t) = u(p_m, t; a), p_m = m/M$ for $m = 1, 2, \cdots, M-1$ and $0 \le T_1 < t < T_2$ of the heat conduction process (4) are given. Then the conductivity $a \in A_{\rm ad}$ is identifiable in the class of piecewise constant functions $\mathcal{PC}(\sigma)$.

Proof: The identification proceeds in two steps. In step I the M-tuple $\mathcal{G}(a)$ is extracted from the observations $z_m(t)$. In step II the Marching Algorithm identifies $a(x)$.

Step I. *Data extraction.*

Using (vi) of Theorem 2 we get

$$z_m(t) = \sum_{k=1}^{\infty} g_k e^{-\lambda_k t} v_k(p_m), \quad m = 1, 2, \cdots, M-1, \tag{11}$$

where $g_k = < g, v_k >$ for $k = 1, 2, \cdots$. By Theorem 2(iv) $v_1(x) > 0$ on interval $(0,1)$. Since g is positive on $(0,1)$ we conclude that $g_1 v_1(p_m) > 0$. While $z_m(t)$ is represented by a Dirichlet series, one needs an additional argument to establish the uniqueness of the coefficients and exponents λ_k in it, see Lemma 3 and the remark after it. According to Theorem 2(vii) each observation $z_m(t)$ is, in fact, a real analytic function. Therefore, if there is another representation of this type for $z_m(t)$, their difference would be an analytic function vanishing on the interval $0 \le T_1 < t < T_2$. Thus such a function would be identically equal to zero. It is easy to see that this could happen only if the corresponding nonzero coefficients and the exponents be identical in such representations.

In conclusion, one can uniquely determine the nonzero coefficients in (11) and the corresponding exponents, see Section 5 for an algorithm. In particular, one determines the first eigenvalue λ_1 and the values of

$$G_m = g_1 v_1(p_m) > 0, \quad p_m = m/M \tag{12}$$

for $m = 1, 2, \cdots, M-1$. This determines the M-tuple $\mathcal{G}(a)$, see (3). Because of the zero boundary conditions we let $G_0 = G_M = 0$.

Step II. *Marching Algorithm.*

The algorithm marches from the left end $x = 0$ to a certain observation point $p_{l-1} \in (0,1)$ and identifies the values a_n and the discontinuity points x_n of the conductivity a on $[0, p_{l-1}]$. Then the algorithm marches from the right end point $x = 1$ to the left until it reaches the observation point $p_{l+1} \in (0,1)$ identifying the values and the discontinuity points of a on $[p_{l+1}, 1]$. Finally, the values of a and its discontinuity are identified on the interval $[p_{l-1}, p_{l+1}]$. The overall goal of the algorithm is to determine the number $N-1$ of the discontinuities of a on $[0,1]$, the discontinuity points $x_n,\ n = 1, 2, \cdots, N-1$ and the values a_n of a on $[x_{n-1}, x_n],\ n = 1, 2, \cdots, N$ $(x_0 = 0,\ x_N = 1)$. As a part of the process the algorithm determines certain functions $H_n(x)$ defined on intervals $[x_{n-1}, x_n],\ n = 1, 2, \cdots N$. The resulting function $H(x)$ defined on $[0,1]$ is a multiple of the first eigenfunction v_1 over the entire interval $[0,1]$.

(i) Find $l,\ 0 < l < M$ such that $G_l = \max\{G_m : m = 1, 2, \cdots, M-1\}$ and $G_m < G_l$ for any $0 \le m < l$.

(ii) Let $i = 1,\ m = 0$.

(iii) Use Lemma 4 to find $A_i,\ \omega_i$ and γ_i from the system

$$\begin{cases} A_i \cos(\omega_i \delta - \gamma_i) = G_m, \\ A_i \cos \gamma_i = G_{m+1}, \\ A_i \cos(\omega_i \delta + \gamma_i) = G_{m+2}. \end{cases} \quad (13)$$

Let

$$H_i(x) = A_i \cos(\omega_i (x - p_{m+1}) + \gamma_i).$$

(iv) If $m + 3 \geq l$ then go to step (vii). If $H_i(p_{m+3}) \neq G_{m+3}$, or $H_i(p_{m+3}) = G_{m+3}$ and $H_i'(p_{m+3}) \leq 0$ then a has a discontinuity x_i on interval $[p_{m+2}, p_{m+3})$. Proceed to the next step (v).
If $H_i(p_{m+3}) = G_{m+3}$ and $H_i'(p_{m+3}) > 0$ then let $m := m + 1$ and repeat this step (iv).

(v) Use Lemma 4 to find A_{i+1}, ω_{i+1} and γ_{i+1} from the system

$$\begin{cases} A_{i+1} \cos(\omega_{i+1} \delta - \gamma_{i+1}) = G_{m+3}, \\ A_{i+1} \cos \gamma_{i+1} = G_{m+4}, \\ A_{i+1} \cos(\omega_{i+1} \delta + \gamma_{i+1}) = G_{m+5}. \end{cases} \quad (14)$$

Let

$$H_{i+1}(x) = A_{i+1} \cos(\omega_{i+1} (x - p_{m+4}) + \gamma_{i+1}).$$

(vi) Use formulas in Lemma 6 to find the unique discontinuity point $x_i \in [p_{m+2}, p_{m+3})$. The parameters and functions used in Lemma 6 are defined as follows. Let $p = p_{m+2}$, $\eta = \delta$. To avoid a confusion we are going to use the notation Ω_1, Ω_2, Γ_1, Γ_2 for the corresponding parameters ω_1, ω_2, γ_1, γ_2 required in Lemma 6. Let $\Omega_1 = \omega_i$, $\Omega_2 = \omega_{i+1}$. For $w(x)$ use function $H_i(x)$ recentered at $p = p_{m+2}$, i.e. rewrite $H_i(x)$ in the form

$$w(x) = H_i(x) = A \cos(\Omega_1 (x - p_{m+2}) + \Gamma_1), \quad |\Gamma_1| < \pi/2.$$

For $v(x)$ use function H_{i+1} recentered at $p + \eta = p_{m+3}$, i.e.

$$v(x) = H_{i+1}(x) = B \cos(\Omega_2 (x - p_{m+3}) + \Gamma_2), \quad |\Gamma_2| < \pi/2.$$

Let $i := i + 1$, $m := m + 3$. If $m < l$ then return to step (iv). If $m \geq l$ then go to the next step (vii).

(vii) Do steps (ii)-(vi) in the reverse direction of x, advancing from $x = 1$ to $x = p_{l+1}$. Identify the values and the discontinuity points of a on $[p_{l+1}, 1]$, as well as determine the corresponding functions $H_i(x)$.

(viii) Using the notation introduced in (vi) let $H_j(x)$ be the previously determined function H on interval $[p_{l-2}, p_{l-1}]$. Recenter it at $p = p_{l-1}$, i.e. $w(x) = H_j(x) = A \cos(\Omega_1 (x - p_{l-1}) + \Gamma_1)$. Let $H_{j+1}(x)$ be the previously determined function H on interval $[p_{l+1}, p_{l+2}]$. Recenter it at p_{l+1}: $v(x) = H_{j+1}(x) = B \cos(\Omega_2 (x - p_{l+1}) + \Gamma_2)$. If $\Omega_1 = \Omega_2$ then stop, otherwise use Lemma 6 with $\eta = 2\delta$, and the above parameters to find the discontinuity $x_j \in [p_{l-1}, p_{l+1}]$. Stop.

The justification of the Marching Algorithm is given in [6]. ∎

The Marching Algorithm of Theorem 7 requires measurements of the system at possibly large number of observation points. Our next Theorem shows that if a piecewise constant conductivity a is known to have just one point of discontinuity x_1, and its values a_1 and a_2 are known beforehand, then the discontinuity point x_1 can be determined from just one measurement of the heat conduction process.

Theorem 8: Let $p \in (0, 1)$ be an observation point, $g(x) > 0$ on $(0, 1)$, and the observation $z_p(t) = u(x_p, t; a)$, $t \in (T_1, T_2)$ of the heat conduction process (4) be given. Suppose that the conductivity $a \in A_{\text{ad}}$ is piecewise constant and has only one (unknown) point of discontinuity $x_1 \in (0, 1)$. Given positive values $a_1 \neq a_2$ such that $a(x) = a_1$ for $0 \leq x < x_1$ and $a(x) = a_2$ for $x_1 \leq x < 1$ the point of discontinuity x_1 is identifiable.

IV. Continuity of the solution and the identification maps

The following results are established in [7] in a somewhat more general setting.

Theorem 9: Let $a \in \mathcal{PC} \subset A_{\text{ad}}$ equipped with the $L^1[0, 1]$ topology, and $u(a)$ be the solution of the heat conduction process (2), and $0 < t_0 < T$. Then

1) The mapping $a \rightarrow u(a)$ from $\mathcal{PC}$ into $C([0, T]; L^2[0, 1])$ is continuous.
2) The mapping $a \rightarrow u(a)$ from $\mathcal{PC}$ into $C([t_0, T]; C[0, 1])$ is continuous.

Theorem 10: Let A_{ad} be equipped with the $L^1[0, 1]$ topology. Let $N \in \mathbb{N}$ and $\sigma > 0$. Then

1) Set $\mathcal{PC}_N \subset A_{\text{ad}}$ is compact.
2) Set $\mathcal{PC}(\sigma) \subset A_{\text{ad}}$ is compact.

Theorem 11: Let A_{ad} be equipped with the $L^1[0, 1]$ topology, and the solution map $\mathcal{G} : \mathcal{PC}(\sigma) \rightarrow \mathbb{R}^M$ be defined as in (3). Then the identification map $\mathcal{G}^{-1} : \mathcal{G}(\mathcal{PC}(\sigma)) \rightarrow \mathcal{PC}(\sigma)$ is continuous.

Proof: It is established in [7] that the eigenvalues and the eigenfunctions are continuously dependent on the conductivities a in the $L^1[0, 1]$ topology. This and 10 imply that the solution map $\mathcal{G}(a)$ is continuous on a compact set $\mathcal{PC}(\sigma)$. By Theorem 7, the inverse mapping $\mathcal{G}^{-1}$ is well defined on the set $\mathcal{G}(\mathcal{PC}(\sigma))$. Thus it is continuous. ∎

V. Numerical results

The main objective of this paper is the development of a theoretical framework for the parameters' identifiability described in previous sections. Nevertheless, from a practical perspective it is desirable to develop an algorithm for such an identifiability incorporating the new insights gained in the theoretical part. The main new element of it is the separation of the identification process into the following two parts. First, the observation data is used to recover the M-tuple $\mathcal{G}(a)$, see (3). In the second step this input is used to recover the conductivity by the Marching Algorithm. We emphasize that only one (first) eigenvalue and the eigenfunction are needed for the identification.

Before considering noise contaminated observation data $z_m(t)$, let us assume that $z_m(t)$ are known precisely on an interval $I = (t_0, T)$, $t_0 \geq 0$. In this case the observations are given by the Dirichlet series

$$z_m(t) = \sum_{k=1}^{\infty} < g, \psi_k > e^{-\lambda_k t} \psi_k(p_m). \quad (15)$$

The functions $z_m(t)$ are analytic for $t > 0$ and, therefore, can be uniquely extended to $(0, \infty)$ from I. The first eigenvalue λ_1 and the data sequence $\{G_m =< g, \psi_1 > \psi_1(p_m)\}_{m=1}^{M-1}$ can be recovered from the Dirichlet series representing $z_m(t)$ by

$$\lambda_1 = -\frac{1}{h}\lim_{t\to\infty} \ln \frac{z_m(t+h)}{z_m(t)}, \quad G_m = \lim_{t\to\infty} e^{\lambda_1 t} z_m(t), \tag{16}$$

where $h > 0$.

The second step in the algorithm, i.e. the identification of the conductivity a is accomplished by the Marching Algorithm. Numerical experiments show the perfect identification for noiseless data. The identification rapidly deteriorates even for small noise in the data.

Hence a different algorithm is needed for the practically important case of noise contaminated data. It should also take into an account the severe ill-posedness of the identification of data from Dirichlet series. It is the distinct advantage of the proposed algorithm that it uses only the first eigenvalue λ_1 for the conductivity identification. In what follows LMA refers to the Levenberg-Marquardt algorithm for the nonlinear least squares minimization, and BA to the Brent algorithm for a single variable nonlinear minimization, see [16] for details.

The algorithm proceeds as follows (see details below)

1). For each $m = 1, 2, ..., M$ find the best fit for the data $z_m(t_j)$ by minimizing $\Psi(\lambda, c; m)$ defined in (17). Call the results of these minimizations for the exponents λ by $\mu^{(m)}$ and for the coefficients c by $c_m(\lambda)$. They give the best fit to the data $z_m(t)$ by only one term of the Dirichlet series (15). Numerical results show that these values are not sufficiently good for the final identification, but they are appropriate as a first guess for the best fit to $z_m(t)$ by two terms of the Dirichlet series.

2). Apply the LMA to minimize $\Phi(\mu, \nu, c, b; m)$ defined in (18). Use the initial guess $\mu^{(m)}, 4\mu^{(m)}, c_m(\lambda), 0$ for the variables μ, ν, c, b correspondingly. Call the results of these minimizations for the variable μ by $\lambda_1^{(m)}$. The initial value $4\mu^{(m)}$ for the second eigenvalue is used because of Theorem 2(3). A direct application of the LMA without the initial values obtained in Step 1 did not produce consistent results. Now the data $z_m(t)$ is approximated by the first two terms of the Dirichlet series (15). Thus, for each m there is an estimate $\lambda_1^{(m)}$ for the first eigenvalue λ_1.

3). Let λ_1 be an average of the computed values $\lambda_1^{(m)}$. We used the middle third of the indices m since the maximum of our initial data $g(x)$ was attained in the middle of the interval $[0, 1]$. Hence these observations were relatively less affected by the noise.

4-5). Repeat the minimizations of Steps 1 and 2, but keep λ_1 frozen. Let G_m be the values of the coefficients c that minimize $\Phi(\lambda_1, \nu, c, b; m)$. This is the best fit to the data $z_m(t)$ by the first two terms of the Dirichlet series (15) with the fixed first eigenvalue λ_1. By now the first part of the identification algorithm is completed, since we have recovered the first eigenvalue λ_1 and a multiple G_m of the first eigenfunction $\psi_1(p_m), m = 1, 2, ..., M$.

6). Let an integer $N > 0$ be fixed. Choose a partition $0 < x_1 < x_2 < \cdots < x_{N-1} < 1$ of the interval $[0, 1]$, and a vector $(a_1, a_2, \dots, a_N)$ corresponding to the piecewise constant conductivity $a(x) = a_i$ for $x_{i-1} < x < x_i$. This notation is consistent with the definitions of the previous sections. Let $\psi_1(x; a)$ be the corresponding first eigenfunction of (5). Form the objective function $\Pi(a)$ by (19). The minimization in c in (19) is done directly as in Step 1. Then minimize $\Pi(a)$ in all $2N - 1$ variables using Powell's minimization method, see [16], [5]. The method is slightly modified to keep the conductivities within the class $\mathcal{PC}(\sigma)$. The resulting minimizer $\bar{a}$ is the sought conductivity. Here are the exact steps of the algorithm.

Algorithm for the conductivity identification. Let the data consists of the observations $z_m(t_j), \quad j = 1, 2, \dots J, \ m = 1, 2, \dots, M$.

1) Let $\lambda, \ c \in \mathbb{R}$ and

$$\Psi(\lambda, c; m) = \sum_{j=1}^{J} (ce^{-\lambda t_j} - z_m(t_j))^2. \tag{17}$$

Let

$$\Psi(\lambda, c_m(\lambda); m) = \min_{c\in\mathbb{R}} \Psi(\lambda, c; m).$$

Note that such a minimizer $c_m(\lambda)$ can be found directly by

$$c_m(\lambda) = \frac{\sum_{j=1}^{J} z_m(j)e^{-\lambda t_j}}{\sum_{j=1}^{J} e^{-2\lambda t_j}}.$$

Apply BA to find a $\mu^{(m)}$ such that

$$\Psi(\mu^{(m)}, c_m(\mu^{(m)}); m) = \min_{\lambda\in\mathbb{R}} \Psi(\lambda, c_m(\lambda); m).$$

2) Let

$$\Phi(\mu, \nu, c, b; m) = \sum_{j=1}^{J} (ce^{-\mu t_j} + be^{-\nu t_j} - z_m(t_j))^2. \tag{18}$$

Apply the LMA to minimize $\Phi(\mu, \nu, c, b; m)$ using the initial guess $\mu^{(m)}, 4\mu^{(m)}, c_m(\mu^{(m)}), 0$ for the variables μ, ν, c, b correspondingly. Let

$$\Phi(\lambda_1^{(m)}, \nu_m, c_m, b_m; m) = \min_{\mu,\nu,c,b} \Phi(\mu, \nu, c, b; m).$$

3) Let $k = card\{[[M/3]], \dots, [[2M/3]]\}$ and

$$\lambda_1 = \frac{1}{k} \sum_{m=[[M/3]]}^{[[2M/3]]} \lambda_1^{(m)}.$$

4) Find $c_m(\lambda_1), \ m = 1, 2, \dots, M$ (as in Step 1) such that

$$\Psi(\lambda_1, c_m(\lambda_1); m) = \min_{c\in\mathbb{R}} \Psi(\lambda_1, c; m).$$

5) Apply the LMA to minimize $\Phi(\lambda_1, \nu, c, b; m)$ in variables ν, c, b using the initial guess $4\lambda_1, c_m(\lambda_1), 0$ for the variables ν, c, b correspondingly. Let

$$\Phi(\lambda_1, \nu_m, G_m, b_m; m) = \min_{\nu,c,b} \Phi(\lambda_1, \nu, c, b; m).$$

6) Fix an integer $N > 0$. For any partition $0 < x_1 < x_2 < \cdots < x_{N-1} < 1$ of the interval $[0, 1]$, and a vector $(a_1, a_2, \ldots, a_N)$ corresponding to the piecewise constant conductivity $a(x) = a_i$ for $x_{i-1} < x < x_i$ let $\psi_1(x; a)$ be the corresponding first eigenfunction of (5). Form the objective function $\Pi(a)$ by

$$\Pi(a) = \min_{c\in\mathbb{R}} \sum_{m=1}^{M} (cG_m - \psi_1(p_m; a))^2. \qquad (19)$$

Use Powell's minimization method (see [16], [5]) in $2N - 1$ variables ($N - 1$ discontinuity points and N conductivity values) to find

$$\Pi(\bar{a}) = \min_{a\in\mathcal{PC}(\sigma)} \Pi(a).$$

The minimizer $\bar{a}$ is the sought conductivity.

VI. Conclusions

The prevalent approach to parameter identification (estimation) problems is to find such parameters from the best fit to data minimization. However such an approach usually does not guarantee the uniqueness of the identified parameters. Identifiability problem consists of finding sufficient conditions assuring such a uniqueness, and there have been just a few results for the identifiability in distributed parameter systems.

In this paper we have shown that in some cases a variable conductivity in a $1D$ heat conduction process can be uniquely identified from observations of this process.

In this study it is assumed that the conductivity is piecewise constant with sufficiently separated points of discontinuity. The observations of the process are taken at equidistant points $p_m \in (0, 1)$. The total number of points needed for the unique conductivity identification can be computed from a priori known parameters of the process as specified in Theorem 7. The identification is achieved in two steps. First the data is used to recover an M-tuple $\mathcal{G}(a)$, see (3). The processed data $\mathcal{G}(a)$ contains only the first eigenvalue and a nonzero multiple of the first eigenfunction at the observation points. Then the Marching Algorithm (Theorem 7) is applied to $\mathcal{G}(a)$ to recover the sought piecewise constant conductivity $a(x)$. Both the solution and the identification maps are shown to be continuous.

The methods described in this paper can be extended to identifiability problems for heat conduction processes admitting various boundary (e.g. periodic) inputs and to other cases, see [7]. A numerical implementation shows that the Marching Algorithm achieves a perfect identification for observations with low noise levels. A satisfactory identification for higher noise levels can be achieved by the numerical algorithm presented in Section 5.

References

[1] G. Birkhoff and G. Rota, *Ordinary differential equations*, Wiley & Sons, New York, NY; 1989.

[2] A. Elayyan and V. Isakov, On uniqueness of recovery of the discontinuous coefficient of a parabolic equation, *SIAM J. Math. Anal.*, 28-1, 1997, pp. 49–59.

[3] I. M. Gelfand and B. M. Levitan, On the determination of a differential equation from its spectral function(English translation), *Amer. Math. Soc. Transl.*, 2-1, 1955, pp. 253–304.

[4] S. Gutman, Identification of discontinuous parameters in flow equations, *SIAM J. Control and Optimization*, 28-5, 1990, pp. 1049–1060.

[5] Ha J and Gutman S, Parameter estimation problem for a damped sine-Gordon equation, *International Journal of. Appl. Math. and Mech.*, 2-1, 2006, pp. 11-23.

[6] Gutman S and Ha J, Identifiability of piecewise constant conductivity in a heat conduction process, *SIAM J. Control and Optimization*, 46-2, 2007, pp 694–713.

[7] Gutman S and Ha J, Parameter Identifiability for heat conduction with a boundary input, submitted; available at http://www.math.ou.edu/ sgutman/papers.htm.

[8] S. Kitamura and S. Nakagiri, Identifiability of spatial-varying and constant parameters in distributed systems of parabolic type, *SIAM J. Control and Optimization*, 15-5, 1977, pp. 785–802.

[9] R. Kohn and M. Vogelius, Determining conductivity by boundary measurements II: Interior results, *Comm. Pure Appl. Math.*, 41, 1988, pp. 865–877.

[10] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Uralceva, *Linear and quasi-linear equations of parabolic type*, Translations of Mathematical Monographs, 23, AMS; 1968.

[11] S. Larsson and V. Thomée, *Partial differential equations with numerical methods*, Springer-Verlag, New York; 2003.

[12] J. L. Lions, *Optimal control of systems governed by partial differential equations*, Springer-Verlag, New York; 1971.

[13] S. Nakagiri, Review of Japanese work of the last 10 years on identifiability in distributed parameter systems, *Inverse Problems*, 9-2, 1993, pp. 143–191.

[14] Y. Orlov and J. Bentman, Adaptive distributed parameter systems identification with enforceable identifiability conditions and reduced-order spatial differentiation, *IEEE Transactions on Automatic Control*, 45-2, 2000, pp. 203–216.

[15] A. Pierce, Unique identification of eigenvalues and coefficients in a parabolic problem, *SIAM J. Control and Optimization*, 17-4, 1979, pp. 494–499.

[16] W. H. Press, S. A. Teukolsky, W. T. Vetterling, B. P. Flannery *Numerical Recepies in FORTRAN (2nd Ed.)*, Cambridge University Press, Cambridge; 1992.

[17] S. Saks and A. Zygmund, *Analytic functions*, Monografie Matematyczne, Warsaw; 1965.