Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

On the genericity of the differential observability of controlled discrete-time systems

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Abstract— **In this paper, we prove the genericity of the differential observability for discrete-time systems with more outputs than inputs.**

Index Terms— **Observability, nonlinear systems, discrete-time systems, transversality theory.**

I. INTRODUCTION

In this paper, we study the genericity of the differential observability for discrete-time controlled nonlinear systems such that:

$$\begin{cases} x_{k+1} = f(x_k, u_k) \\ \quad y_k = h(x_k, u_k) \\ x_k \in X,\ u_k \in U,\ y_k \in \mathbb{R}^p \end{cases} \tag{1}$$

where:

1) X and U are C^∞ compact connected second-countable manifold with dimensions n and m respectively;
2) $f : X \times U \to X$ is a parametrized diffeomorphism: that is to say, for every $u \in U$, the mapping $f(\cdot, u)$ is a C^∞ diffeomorphism; we denote by $\mathrm{Diff_U}(X)$ the set of all parametrized diffeomorphisms;
3) $h : X \times U \to \mathbb{R}^p$ is a C^∞ mapping.

To be more specific, we shall introduce some notations; given $f \in \mathrm{Diff_U}(X)$ and $h \in C^\infty(X \times U, \mathbb{R}^p)$, we denote by $\underline{u_N}$ the finite sequence $(u_0, \ldots, u_{N-1})$ of elements of U, and we define recursively $f^k(x, \underline{u_k})$ by

$$f^1(x, \underline{u_1}) = f(x, u_0)$$
$$f^{k+1}(x, \underline{u_{k+1}}) = f(f^k(x, \underline{u_k}), u_k) \qquad \text{for } k \geq 1$$

Let us recall the notion of observability investigated in this paper. begindefinition Two initial conditions x_0 and $\bar{x}_0$ and an input u (*i.e.* a sequence $(u_k)_{k\geq 0}$ of elements of U) being given, x_k and $\bar{x}_k$ denote the points $x_k = f^k(x_0, \underline{u_k})$ and $\bar{x}_k = f^k(\bar{x}_0, \underline{u_k})$.

System (1) is said observable for input u if for any initial conditions $x_0 \neq \bar{x}_0$, there exists an index k (possibly depending on the initial conditions) such that $x_k \neq \bar{x}_k$.

System (1) is said observable if it is observable for each input. enddefinition

Below, we are introducing a stronger notion of observability. We consider the mapping $\Theta^{f,h}_{2n+1}$ from $X \times U^{2n+1}$ to $\mathbb{R}^{(2n+1)p} \times U^{2n+1}$ defined by

$$\Theta^{f,h}_{2n+1}(x, \underline{u_{2n+1}}) =$$
$$(h(x,u_0), h(f^1(x,\underline{u_1}),u_1), \ldots, h(f^{2n}(x,\underline{u_{2n}}),u_{2n}), \underline{u_{2n+1}})$$

Notice that this mapping is the discrete-time analogous of the mapping $S\Phi^\Sigma_k$ defined in [5].

Definition 1: We shall say that system (1) is *strongly observable* if the related mapping $\Theta^{f,h}_{2n+1}$ defined above is one-to-one.

In article [4], we proved that system (1) is generically strongly observable as long as $p > \dim U$; more precisely, we proved that the set of pairs (f,h) which make the mapping $\Theta^{f,h}_{2n+1}$ one-to-one is a residual.

In this article, we deal with a stronger notion of observability:

Definition 2: We shall say that system (1) is *strongly differentially observable* if, for every fixed sequence $\underline{u_{2n+1}}$, the mapping $\bar{\Theta}^{f,h}_{2n+1}$ from X to $\mathbb{R}^{(2n+1)p}$ defined by

$$\bar{\Theta}^{f,h}_{2n+1}(x) = (h(x,u_0), h(f^1(x,\underline{u_1}),u_1), \ldots, h(f^{2n}(x,\underline{u_{2n}}),u_{2n}))$$

is an embedding.

In the continuation of [4], the goal of this paper is to prove that system (1) is strongly differentially observable as long as $p > \dim U$.

On this subject, one has to mention first the important work from J.-P. Gauthier and I. Kupka. In a first

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All the auhors are involved in the PAI Utique-CMCU # 05G1507.

paper, with also H. Hammouri [6], the authors investigated the genericity of observability for uncontrolled continuous-time systems. This work was generalized by J.-P. Gauthier and I. Kupka in [7], [5] the authors proved the genericity of differential observability for systems with more outputs than inputs. To be more precise, in their paper, the authors show that if a bound on the derivatives of the control is given, the set of systems Σ such that the mapping $S\Phi_\Sigma^N$ (analogous, for continuous systems, to mapping $\Theta_{2n+1}^{f,h}$)is an embedding, is an open and dense set. When this condition is not assumed, the authors prove that this set is residual (and therefore dense) but the openness property remain an open problem. In our case, we assume that the controls belong to a compact manifold, so this difficulty disappears. Also, we don't have to consider the case of nonsmooth controls. Nevertheless there are some other difficulties, for example we have to pay a special attention to periodic points of f_u. As far as we are concerned by discrete-time systems, we have to cite several papers on the subject of the genericity of the observability : first, a paper written by Aeyels [2] in which the author considers uncontrolled continuous-time systems and the discrete-time systems obtained by discretizing the continuous ones. In this paper, the author introduced the notion of P-observability. The system

$$\begin{cases} \dot{x} &= f(x) \\ y &= h(x) \end{cases} \tag{2}$$

is said P-observable if, given a time $T > 0$ and a finite subset P of $[0,T]$, for every pair (x,y) of distinct elements in X^2, there exists a $t_i \in P$ such that $h \circ \Phi_{t_i}(x) \neq h \circ \Phi_{t_i}(y)$ where Φ denotes the flow of f. One of the results in this paper is the proof of the existence of an open and dense set of vector fields such that, a vector field f in this set being fixed, the subset of functions h belonging to $C^r(X,\mathbb{R})$ such that the system (f,h) is P-observable is open and dense in $C^r(X,\mathbb{R})$. This is true for almost any finite subset P of $(2\dim X + 1)$ points in $[0,T]$.

To an uncontrolled discrete-time systems such that

$$\begin{cases} x_{k+1} &= f(x_k) \\ y_k &= h(x_k) \\ x_k \in M, & \text{compact manifold, } y_k \in \mathbb{R} \end{cases} \tag{3}$$

is attached a map analogous to the map $\Theta_{2n+1}^{f,h}$ defined above: consider

$$\begin{array}{cccl} \Phi: & M & \longrightarrow & \mathbb{R}^{2n+1} \\ & x & \longmapsto & (h(x), h \circ f(x), \ldots, h \circ f^{2n}(x)) \end{array}$$

where n is the dimension of manifold M. In [11], the proof that, generically, Φ is an embedding is sketched while in [9] and [12], the same result is proved in greater detail (see also the concluding remarks of [2]).

In the case of controlled discrete-time systems, in article [10], the authors investigate controlled discrete-time systems and obtain some results which are similar (but not identical) to the one presented here; namely they present a result of genericity of the observability but it is not a result about observability for every input. As in the present paper, the tools used in the work of these authors belong to the transversality theory.

Before going straight to the point, we want to add some words about the fact that the observation function h depends on u. This situation is not common in automatic control theory, but the opposite assumption leads to clumsy statements. Nevertheless, as explained in the conclusion of [4], the result of genericity can be proved also for systems where h does not depend on u. The paper is organized as follows: in the next section, some facts from transversality theory are recalled, in § III, the main result is stated together with some definitions and lemmas; in § IV, our result is proved through the demonstrations of five lemmas.

II. Some facts from transversality theory

In this section we recall some theorems from differential topology which will be intensively used in the proof of the main result of this paper. For details on the C^∞ Whitney topology, the reader is referred to the book "Stable Mappings and their Singularities" [8].

If X and Y are two smooth manifolds, $J^k(X,Y)$ will denote, as usual, the set of k-jets from X to Y, $\alpha : J^k(X,Y) \to X$ is the source map and $\beta : J^k(X,Y) \to Y$ the target map; moreover we denote by $C^r(X,Y)$ ($1 \leq r \leq +\infty$) the set of C^r maps from X to Y. If f is in $C^\infty(X,Y)$ $j^k f$ denotes the k-jet of f. Recall that the set $C^\infty(X,Y)$ endowed with the Whitney topology is a Baire space and so every residual set of $C^\infty(X,Y)$ (*ie* every countable intersection of open dense subsets) is dense.

The notion of transversality is of paramount importance for our purpose and we recall below its definition.

begindefinition Let f be a smooth mapping between two smooth manifolds X and Y, W a submanifold of Y and x a point in X. We shall say that f intersects W *transversely* at x if either

1) $f(x) \notin W$, or
2) $f(x) \in W$ and $T_{f(x)}Y = T_{f(x)}W + df_x(T_xX)$,

T_xX denoting the tangent space to X at x and df_x the Jacobian of f at x. We shall say that f intersects W transversely if it intersects W transversely at x for all x in W. We shall use of the symbol $\pitchfork$ to denote the transversality. enddefinition

The following theorem states a result of genericity [8].

Theorem 3 (Thom Transversality Theorem): Let X and Y be smooth manifold and W a submanifold of $J^k(X,Y)$ and let

$$T_W = \{f \in \mathrm{C}^\infty(X,Y) \mid j^k f \pitchfork W\}$$

Then T_W is a residual subset of $\mathrm{C}^\infty(X,Y)$ in the C^∞ topology. Moreover, if W is closed, then T_W is open.

The following result generalizes the above theorem to multijet spaces. We first define the set $X^{(s)} = \{(x_1,\dots,x_s) \in X^s \mid x_i \neq x_j \text{ for } 1 \le i < j \le s\}$ and the mapping

$$\begin{array}{rccc} \alpha^s : & \left(J^k(X,Y)\right)^s & \longrightarrow & X^s \\ & (\sigma_1,\dots,\sigma_s) & \longmapsto & \big(\alpha(\sigma_1),\dots,\alpha(\sigma_s)\big) \end{array}$$

and we let $J_s^k(X,Y) = (\alpha^s)^{-1}(X^{(s)})$, $J_s^k(X,Y)$ is a submanifold of $\left(J^k(X,Y)\right)^s$.

For $f \in C^\infty(X,Y)$, we can define

$$\begin{array}{rccc} j_s^k f : & X^{(s)} & \longrightarrow & J_s^k(X,Y) \\ & (x_1,\dots,x_s) & \longmapsto & \big(j^k f(x_1),\dots,j^k f(x_s)\big) \end{array}$$

Theorem 4 (Multijet Transversality Theorem): Let W be a submanifold of $J_s^k(X,Y)$ and let

$$T_W = \{f \in C^\infty(X,Y) \mid j_s^k f \pitchfork W\}.$$

Then T_W is a residual subset of $C^\infty(X,Y)$ in the C^∞ topology. Moreover, if W is compact, then T_W is open.

We shall use also a transversality theorem due to Abraham [1]. Let $\mathscr{A}, X$ and Y be C^r manifolds and ρ a map from $\mathscr{A}$ to $C^r(X,Y)$.

For $a \in \mathscr{A}$, we write ρ_a, the C^r map:

$$\begin{array}{rccc} \rho_a : & X & \longrightarrow & Y \\ & x & \longmapsto & \rho_a(x) = \rho(a)(x) \end{array}$$

and we say that ρ is a C^r representation if the evaluation map:

$$\begin{array}{rccc} \mathrm{ev}_\rho : & \mathscr{A} \times X & \longrightarrow & Y \\ & (a,x) & \longmapsto & \rho_a(x) = \rho(a)(x) \end{array}$$

is a C^r map from $\mathscr{A} \times X$ to Y.

Theorem 5 (Abraham Transversal Density Theorem): Let $\mathscr{A}, X, Y$ be C^r manifolds, $\rho : \mathscr{A} \to C^r(X,Y)$ a C^r representation, $W \subset Y$ a submanifold (not necessarily closed), and $\mathrm{ev}_\rho : \mathscr{A} \times X \to Y$ the evaluation map. Define $\mathscr{A}_W \subset \mathscr{A}$ by:

$$\mathscr{A}_W = \{a \in \mathscr{A} \mid \rho_a \pitchfork W\}$$

Assume that:

1) X has a finite dimension n and W has a finite codimension q in Y;
2) $\mathscr{A}$ and X are second countable;
3) $r > \max(0, n-q)$;
4) $\mathrm{ev}_\rho \pitchfork W$.

Then $\mathscr{A}_W$ is residual in $\mathscr{A}$.

Notice that manifold $\mathscr{A}$ is not necessarily finite dimensional; it may be a Banach space or an open subset of a Banach space.

Finally, we shall need the following theorem that can also be found in [1].

Theorem 6 (Openness of transversal intersection): Let $\mathscr{A}$, X and Y be C^r manifolds with X finite dimensional, $W \subset Y$ a closed C^r submanifold, K a compact subset of X, and $\rho : \mathscr{A} \to C^r(X,Y)$ a C^r representation. Then the subset $\mathscr{A}_{KW} \subset \mathscr{A}$ defined by

$$\mathscr{A}_{KW} = \{a \in \mathscr{A} \mid \rho_a \pitchfork_x W \text{ for } x \in K\}$$

is open.

III. MAIN RESULT

We state here our main result and some lemmas used in the proof of our theorem. Our framework is the set $\mathrm{Diff}_{\mathrm{U}}(X) \times C^\infty(X \times U, \mathbb{R}^p)$ equipped with the Whitney topology; obviously $\mathrm{Diff}_{\mathrm{U}}(X)$ is open in $C^\infty(X \times U, X)$ for this topology. In the theorem below, we assume that $\dim U < p$.

Theorem 7: The set of mappings $(f,h) \in \mathrm{Diff}_{\mathrm{U}}(X) \times C^\infty(X \times U, \mathbb{R}^p)$ such that the mapping $\bar{\Theta}_{2n+1}^{f,h}$ is an embedding, is open and dense in $\mathrm{Diff}_{\mathrm{U}}(X) \times C^\infty(X \times U, \mathbb{R}^p)$ equipped with the Whitney topology.

We begin by proving the easiest part of this result: the openness of the set of mappings (f,h) such that $\Theta_{2n+1}^{f,h}$ is an embedding.

Proof: Consider the mapping Φ from $X \times U^{2n+1}$ to $C^\infty(X \times U^{2n+1}, (\mathbb{R}^p)^{2n+1} \times U^{2n+1})$ defined by

$$\Phi(f, \underline{u_{2n+1}}) = (\Theta_{2n+1}^{f,h}, \underline{u_{2n+1}})$$

which is obviously continuous for the Whitney topology. Clearly $\Phi(f, \underline{u_{2n+1}})$ is an embedding iff the mapping $\Theta_{2n+1}^{f,h}(\cdot, \underline{u_{2n+1}})$ is an embedding for every finite sequence $\underline{u_{2n+1}} \in U^{2n+1}$. Now, since X and U are compact manifolds, the set of embeddings from $X \times U^{2n+1}$ to $(\mathbb{R}^p)^{2n+1} \times U^{2n+1}$ is open for the Whitney topology, so, due to the continuity of Φ, the set set of mappings $\Theta^{f,h}_{2n+1}(\cdot,\underline{u_{2n+1}})$ which are embeddings for every $\underline{u_{2n+1}}$ is open. ■

We shall prove now the density part of the theorem. Notice that in the continuous-time case, the set of pairs (f,h) (with f a parametrized vector field) is a Banach space for the C^r topology ($r<+\infty$) but this is not the case for the set of pairs (f,h) where f is a parametrized diffeomorphism. So, it is not possible to copy directly the reasoning of [7]. The proof of this theorem will be somewhat awkward and will be based on several technical lemmas. Before stating these lemmas, we describe below our global strategy.

Suppose that $\mathscr{P}_1(f,h)$ and $\mathscr{P}_2(f,h)$ are two properties depending on $(f,h)\in \mathrm{Diff}_{\mathrm{U}}(X)\times C^\infty(X\times U,\mathbb{R}^p)$ whose conjunction is equivalent to the fact that $\bar{\Theta}^{f,h}_{2n+1}$ is an immersion. In Prop. 11, we shall prove that, for a given $f\in \mathrm{Diff}_{\mathrm{U}}(X)$, a given integer $r\geq 1$, and for every integer l, there exists a subset $U^r_l(f)$ of $C^\infty(X\times U,\mathbb{R}^p)$, open and dense for the C^r topology, such that if h belongs to the intersection $\bigcap_{l\geq 0} U^r_l(f)$, the pair (f,h) satisfies property $\mathscr{P}_1$. Moreover, we shall prove that, for every integer l, the set

$$\mathscr{U}^r_l=\bigcup_{f\in\mathscr{D}_U}\{f\}\times U^r_l(f)$$

($\mathscr{D}_U$ open dense set of $\mathrm{Diff}_{\mathrm{U}}$) is open dense in $\mathrm{Diff}_{\mathrm{U}}(X)\times C^\infty(X\times U,\mathbb{R}^p)$ equipped with the C^r topology. In Prop. 12, we shall prove that the set

$$E=\{\,(f,h)\in \mathrm{Diff}_{\mathrm{U}}(X)\times C^\infty(X\times U,\mathbb{R}^p)\mid \mathscr{P}_2(f,h)\text{ is true}\,\}$$

contains a residual set of $\mathrm{Diff}_{\mathrm{U}}(X)\times C^\infty(X\times U,\mathbb{R}^p)$. Hence, clearly, the set $E\cap(\bigcap_{\substack{k\geq 0\\ r\geq 1}}\mathscr{U}^r_k)$ contains a residual set for the C^∞ topology and a pair (f,h) belonging to this set satisfies both properties $\mathscr{P}_1$ and $\mathscr{P}_2$.

We shall give the definition of periodic points before stating our propositions.

Definition 8: Let $f\in \mathrm{Diff}_{\mathrm{U}}(X)$, we shall say that the point $(x,\underline{u_{2n+1}})\in X\times U^{2n+1}$ is periodic for f if there exist two different integers $s'<s$ in $\{0,\dots,2n\}$ such that $f^{s'}(x,\underline{u_{s'}})=f^s(x,\underline{u_s})$. If $(x,\underline{u_{2n+1}})$ is a periodic point, its period is the smallest integer s such that the above equality is satisfied.

Notations We denote by $\mathscr{P}_f$ the set of all periodic points of f; obviously, $\mathscr{P}_f$ is a closed subset of $X\times U^{2n+1}$. We denote also by $\mathscr{P}^c_f$ the set complement of $\mathscr{P}_f$: $\mathscr{P}^c_f=X\times U^{2n+1}\setminus\mathscr{P}_f$.

First, we want to state a lemma about a property of continuity of the sets of periodic points; before that, we recall the definition of the Hausdorff distance between sets.

Definition 9: Let (E,d) be a metric space, if A and B are subsets of E, the Hausdorf's distance between A and B is defined by

$$\delta(A,B)=\sup_{x\in A}d(x,B)+\sup_{y\in B}d(y,A)$$

We suppose that X and U are equipped with distances which are compatible with their topologies, so we can speak of Hausdorf's distance on $X\times U^{2n+1}$ and we state the following lemma.

Lemma 10: There exists an open and dense set in $\mathrm{Diff}_{\mathrm{U}}(X)$, denoted by $\mathscr{D}_{\mathrm{U}}$, such that for each $f\in\mathscr{D}_{\mathrm{U}}$:

1) if $\mathscr{P}_f=\varnothing$, then $\mathscr{P}_g=\varnothing$ for every g in some neighborhood of f;
2) if $\mathscr{P}_f\neq\varnothing$, then $\delta(\mathscr{P}_f,\mathscr{P}_g)$ tends to 0 as g tends to f for the C^∞ topology.

Property $\mathscr{P}_1(f,h)$ is related to the periodic points of f and is the object of the following proposition.

Proposition 11: Let $f\in\mathscr{D}_{\mathrm{U}}$ be given, for each $r>0$, there exists a sequence $(U^r_l(f))_{l\geq 1}$ of open and dense sets for the C^r topology included in $C^\infty(X\times U,\mathbb{R}^p)$ such that for every mapping h in $\cap_{l\geq 1}U^r_l(f)$, the mapping $\bar{\Theta}^{f,h}_{2n+1}$ is an immersion at each point of $\mathscr{P}^c_f$.

Moreover, for every nonzero integer l, the set

$$\mathscr{U}^r_l=\bigcup_{f\in\mathscr{D}_{\mathrm{U}}}\{f\}\times U^r_l(f)$$

is open and dense in $\mathrm{Diff}_{\mathrm{U}}(X)\times C^\infty(X\times U,\mathbb{R}^p)$ for the C^r topology.

The second proposition is concerned with property $\mathscr{P}_2(f,h)$, before stating it, we introduce some sets of covectors. We denote by π the canonical projection from T^*X, the cotangent bundle of X, to X and, given a integer $k>n$, we define the set $(T^*X)^{\otimes k}$ by

$$(T^*X)^{\otimes k}=\\ \{\,(p_1,\dots,p_k)\in(T^*X)^k\mid \pi(p_1)=\dots=\pi(p_k)\,\}$$

and the set $V(k,T^*X)$ by

$$V(k,T^*X)=\\ \{\,(p_1,\dots,p_k)\in(T^*X)^{\otimes k}\mid \mathrm{rank}(p_1,\dots,p_k)<n\}.$$

Clearly, $(T^*X)^{\otimes k}$ is a submanifold of $(T^*X)^k$ and $V(k,T^*X)$ is a finite union of submanifolds of $(T^*X)^{\otimes k}$ whose codimension (the codimension of the highest dimensional submanifold of the union) is equal to $k-n+1$ (see [6]).

We state now our second proposition.

Proposition 12: The set of pairs $(f,h) \in \mathrm{Diff}_{\mathrm{U}}(X) \times C^\infty(X \times U, \mathbb{R}^p)$ such that the mapping $\bar{\Theta}_{2n+1}^{f,h}$ is an immersion at each point of $\mathscr{P}_f$ is residual.

Notation If p is a point of a manifold M, hereafter we shall denote by $\mathrm{T}_p M$ the tangent space to M at p.

IV. Proof of the main result

In this section, we give a sketch of the proof of our main result.

A. Proof of Lemma 10

For the proof of this result, we need the following lemma which can be proven by using the multijet transversality theorem.

Lemma 13: There exists a residual subset, denoted by $\mathscr{R}$, of $\mathrm{Diff}_{\mathrm{U}}(X)$ such that if f is in this subset, $\mathscr{P}_f$ is either the empty set or a finite union of submanifolds of $X \times U^{2n+1}$ of codimension greatest or equal to n.

For the existence of $\mathscr{D}_{\mathrm{U}}$, we shall prove that in fact the set $\mathscr{R}$ of the above lemma is open. If $f \in \mathscr{R}$ is such that $\mathscr{P}_f$ is empty, one can prove easily that there exists a neighborhood of f such that for every parametrized diffeomorphism g in this neighborhood, $\mathscr{P}_g$ is also empty. If $\mathscr{P}_f \neq \varnothing$, by reasoning by contradiction, we prove that the existence of the open and dense set $\mathscr{D}_U$ such that if f is in $\mathscr{D}_{\mathrm{U}}$, $\mathscr{P}_f$ is either empty or a finite union of submanifolds of codimension at least n If we take f in $\mathscr{D}_{\mathrm{U}}$, we prove easily that there exists a neighborhood $\mathscr{V}$ of f such that if $g \in \mathscr{V}$ and $v \in \mathscr{P}_g$, the distance $d(v, \mathscr{P}_f)$ can be made arbitrarily small, provided that $\mathscr{V}$ is chosen small enough. The converse, *i.e.* given $w \in \mathscr{P}_f$, the distance $d(w, \mathscr{P}_g)$ can be made arbitrarily small is harder to prove. If $(x_0, \underline{u^0_{2n+1}})$ is a periodic point of f such that $f^s(x_0, \underline{u^0_s}) = f^{s'}\overline{(x_0, \underline{u^0_{s'}})}$, we prove first that there exist mappings u_i, locally defined, and such that the mapping $f^{s',s}$ defined as

$$f^{s',s}(x) = f(f \ldots (f(x, u_{s'}(x)), u_{s'+1}(x)), \ldots, u_{s-1}(x))$$

has a fixed point at $f^s(x_0, \underline{u^0_s})$ and is transverse to the diagonal $\Delta X = \{ (x,x) \mid x \in X \}$. Again, we carry out this task by using transversality arguments.

Now if g is closed to f, $g^{s',s}$ (this mapping is built on the same u_i's than $f^{s',s}$) is closed to $f^{s',s}$. So we have the following situation: the n-dimensional submanifold V_f constituted by the points $(x, f^{s',s}(x))$ intersects transversally the diagonalΔX, if g is closed to f, $V_g = \{(x, g^{s',s}(x))\}$ is closed to V_f and as $\dim V_g = n$, V_g intersects also ΔX and as point belonging to the intersection $V_g \cap \Delta X$ is a periodic point for g, we are done.

B. Proof of proposition 11

A parametrized diffeomorphism $f \in \mathscr{D}_{\mathrm{U}}$ being given, we first prove, thanks to the Abraham transversal density theorem, that there exists a residual set $\mathscr{R}(f) \subset C^\infty(X \times U, \mathbb{R}^p)$ such that if h is in $\mathscr{R}(f)$, the mapping $\bar{\Theta}_{2n+1}^{f,h}$ is an immersion at each point of $\mathscr{P}_f^c$. Then, for every integer l and $f \in \mathscr{D}_{\mathrm{U}}$, we define the compact set $K_l(f)$ as

$$K_l(f) = \begin{cases} X \times U^{2n+1} \\ \quad \text{if } \mathscr{P}_f \text{ is empty} \\ \{ v \in X \times U^{2n+1} \mid d(v, \mathscr{P}_f) \geq 1/l \} \\ \quad \text{if } \mathscr{P}_f \neq \varnothing \end{cases}$$

and $U_l^r(f)$ denotes the set of mappings $h \in C^\infty(X \times U, \mathbb{R}^p)$ such that $\bar{\Theta}_{2n+1}^{f,h}$ is an immersion at each point of $\mathscr{P}_f^c$. By using the theorem of the openness of transversal intersection, it is easily seen that $U_l^r(f)$ is open and since $\mathscr{R}(f) \subset U_l^r(f)$, $U_l^r(f)$ is dense.

It remains to prove that $\mathscr{U}_l^r = \bigcup_{f \in \mathscr{D}_{\mathrm{U}}} \{f\} \times U_l^r(f)$ is open. First, reasoning by contradiction, we prove that, given $\varepsilon > 0$, and $f_0 \in \mathscr{D}_{\mathrm{U}}$, there exists a neighborhhod $\mathscr{V}_{f_0}$ of f_0 such that if $f \in \mathscr{V}_{f_0}$, for all $v \in K_l(f)$, we have $d(w, K_l(f_0)) < \varepsilon$. Now let $f_0, h_0) \in \mathscr{U}_l^r$, take a neighborhood $\mathscr{V}_{f_0}$ of f_0 such that if $f \in \mathscr{V}_{f_0}$, the distance $d(v, K_l(f_0))$ is less than ε. If ε is chosen small enough, there exists a neighborhood $\mathscr{W}_{h_0}$ of h_0 such that if (f,h) belongs to $\mathscr{V}_{f_0} \times \mathscr{W}_{h_0}$, $\bar{\Theta}_{2n+1}^{f,h}$ is an immersion at each point of $K_l(f)$.

C. Proof of proposition 12

The proof of Proposition 12 is based on the following three lemmas. The proofs of these lemmas, which rely on the transversality results stated in the introduction, are omitted. Let x_0 be a periodic point of order $s \leq 2n$, that is to say there exists $s' < s$ such that $f^{s'}(x_0, \underline{u_{s'}}) = f^s(x_0, \underline{u_s})$ and $f^i(x_0, \underline{u_i}) \neq f^j(x_0, \underline{u_j})$ if $i, j < s$; we denote by x_i the iterated of x_0 by f, to be more precise, $x_i = f^i(x_0, \underline{u_i})$, we put also $z_i, = f(x_i, u_i)$ and $y_i = h(x_i, u_i)$. We consider the list L

$$(x_0, u_0, z_0, y_0), \ldots, (x_{2n}, u_{2n}, z_{2n}, y_{2n})$$

and we say that two elements (x_i, u_i, z_i, y_i) and (x_j, u_j, z_j, y_j) are equivalent if and only if $(x_i, u_i) = (x_j, u_j)$. In each equivalence class, we retain the term of least index and we obtain the following list L' extracted from L:

$$(x_{i_0}, u_{i_0}, z_{i_0}, y_{i_0}), \ldots, (x_{i_r}, u_{i_r}, z_{i_r}, y_{i_r})$$

with $i_0 < i_1 < \cdots < i_r$ (necessarily $i_0 = 0$), we claim that

Lemma 14: In the list L' above, we can find $r+1$ equalities between the terms x_i and z_i.

The two next lemmas are concerned with the derivatives of the components of $\Theta^{f,h}_{2n+1}$.

Lemma 15: Let r be a given nonnegative integer and $(i_0, \ldots, i_{n-1})$ be a given sequence of indices in $\{0, \ldots, r\}$. Given $r+1$ matrices $(A_0, \ldots, A_r)$ in $\mathrm{GL}(n, \mathbb{R})$, we consider the related sequence of matrices $(\tilde{A}_0, \ldots, \tilde{A}_{n-1})$ where

1) $\tilde{A}_0 = A_0$;
2) for $j \geq 1$, $\tilde{A}_j = A_{i_j} \tilde{A}_{j-1}$.

Let $1 \leq k \leq n-1$ and consider the subset W_k of $\mathrm{GL}(n, \mathbb{R})^{r+1} \times \mathbb{P}^{n-1}$ ($\mathbb{P}^{n-1}$ is the projective space of dimension $n-1$) constituted by the elements $(A_0, \ldots, A_r, l)$ such that, $(\tilde{A}_0, \ldots, \tilde{A}_{n-1})$ being the sequence related to $(A_0, \ldots, A_r)$,

1) the family $(l, \tilde{A}_0 l, \ldots, \tilde{A}_{k-2} l)$ is linearly independent (this family reduces to (l) if $k = 1$);
2) the family $(l, \tilde{A}_0 l, \ldots, \tilde{A}_{k-1} l)$ is linearly dependent.

The set W_k is a submanifold of $\mathrm{GL}(n, \mathbb{R})^{r+1} \times \mathscr{P}^{n-1}_f$ with codimension equal to $n-k$.

For each r from 0 to $2n$, we consider a countable family $\mathscr{F}$ of charts covering $(X \times U)^{r+1}$ and we apply the Thom transversality theorem to each chart of $\mathscr{F}$. If $(f, h) \in \mathrm{Diff}_{\mathrm{U}}(X) \times C^\infty(X, \mathbb{R}^p)$, and $(x_0, \underline{u_{2n+1}})$ is a periodic point of f with period no greater than $2n$, starting from x_0, we consider the list L' as in Lemma 14, the element $(x_0, u_0, \ldots, x_{i_r}, u_{i_r})$ constituting L' belongs to one of the charts of the family $\mathscr{F}$ and, together with the z_i's satisfy $r+1$ equalities as explained in Lemma 14. Moreover, the above reasoning shows that, the set of pairs (f, h) such that $\Theta^{f,h}_{2n+1}$ is an immersion at each periodic point of f lying in one of the charts of family $\mathscr{F}$ is residual, by considering the (countable) intersection of all the residual sets related to the charts of $\mathscr{F}$, Proposition 12 is proven.

V. CONCLUSION

As explained in section 3, the conjunction of Prop. 11 and Prop. 12 proves that the set of pairs $(f, h) \in \mathrm{Diff}_{\mathrm{U}}(X) \times C^\infty(X \times U, \mathbb{R}^p)$ such that the mapping $\Theta^{f,h}_{2n+1}$ is an immersion is residual. In [4], we proved that the set of pairs (f, h) such that $\Theta^{f,h}_{2n+1}$ is one to one is also residual, so, X and U being compact, we can conclude that the set of pairs $(f, h) \in \mathrm{Diff}_{\mathrm{U}}(X) \times C^\infty(X \times U, \mathbb{R}^p)$ such that $\Theta^{f,h}_{2n+1}$ is an embedding is residual.

Ths work has been previously published in [3].

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