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Constrained Optimal Control Theory for Differential Linear Repetitive Processes

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Abstract— **Differential repetitive processes are a distinct class of continuous-discrete two-dimensional linear systems of both systems theoretic and applications interest. These processes complete a series of sweeps termed passes through a set of dynamics defined over a finite duration known as the pass length, and once the end is reached the process is reset to its starting position before the next pass begins. Moreover the output or pass profile produced on each pass explicitly contributes to the dynamics of the next one. Applications areas include iterative learning control and iterative solution algorithms, for classes of dynamic nonlinear optimal control problems based on the maximum principle, and the modeling of numerous industrial processes such as metal rolling, long-wall cutting, etc. In this paper we develop substantial new results on optimal control of these processes in the presence of constraints where the cost function and constraints are motivated by practical application of iterative learning control to robotic manipulators and other electromechanical systems. The analysis is based on generalizing the well-known maximum and ϵ-maximum principles to them.**

I. Introduction

Repetitive processes are a distinct class of two-dimensional (2D) systems of both systems theoretic and applications interest. The unique characteristic of such a process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem in that the output sequence of pass profiles generated can contain oscillations which increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(t), 0 \leq t \leq \alpha$, generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(t),\ 0 \leq t \leq \alpha,\ k \geq 0$.

Physical examples of repetitive processes include long-wall coal cutting and metal-rolling operations (see, for example, the references cited in [16]). Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications include classes of iterative learning control (ILC) schemes (see, for example [1]–[3], [12], [13]) and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle [14]. In the case of iterative learning control for the linear dynamics case, the stability theory for differential (and discrete) linear repetitive processes is one method which can be used to undertake a stability/convergence analysis of a powerful class of such algorithms and thereby produce vital design information concerning the trade-offs required between convergence and transient performance.

Attempts to control these processes using standard (or 1D) systems theory and associated algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e., information propagation occurs from pass to pass and along a given pass. Also the initial conditions are reset before the start of each new pass, and the structure of these can be somewhat complex. For example, if they are an explicit function of points on the previous pass profile, then this alone can destroy stability. In seeking a rigorous foundation on which to develop a control theory for these processes, it is natural to attempt to exploit structural links which exist between these processes and other classes of 2D linear systems. In this paper we consider so-called differential linear repetitive processes where information propagation along the pass is governed by a matrix differential equation. The systems theory for 2D discrete linear systems (such as the optimal control results in [4]) and, in particular, the extensively studied Roesser [15] and Fornasini Marchesini [8] state-space models is not applicable.

In this paper we develop substantial new results on the optimal control of differential linear repetitive processes with constraints which we motivate from the iterative learning control application. The results themselves are obtained by extending the maximum principle and the ϵ-maximum principle [10] to them. A sensitivity analysis of the resulting optimal control is also undertaken, and some relevant differentiation properties are established. The proofs of the results given, together with a numerical example and further analysis, can be found in [7].

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II. Preliminaries

Suppose now that the plant dynamics are described by the following matrix differential equation over $0 \le t \le \alpha,\ k \ge 0$

$$\frac{dx_k(t)}{dt} = Ax_k(t) + Dx_{k-1}(t) + bu_k(t), \tag{1}$$

where, on pass k, $x_k(t)$ is the $n \times 1$ state (equal to the pass profile or output) vector, $u_k(t)$ is the scalar control input, A, D are constant $n \times n$ matrices, and b is a given $n \times 1$ vector. (This model is chosen for simplicity of presentation and is easily extended to the case when the pass profile vector is a linear combination of the current pass state, input, and previous pass profile vectors.)

In each practical application only a finite number of passes will actually be completed. Hence one approach to the control of these processes is to formulate an optimal control problem where the cost function to be minimized is actually the sum of the cost function for each pass. Suppose therefore that $N < \infty$ denotes the number of passes actually completed, introduce the set $K = \{1, 2, \dots, N\}$, and let T denote the finite interval (the pass length) $[0, \alpha]$. Then, with the above observations in mind, consider (1) with boundary conditions

$$x_k(0) = d_k, \quad k \in K, \quad x_0(t) = f(t), \quad t \in T, \tag{2}$$

where d_k is an $n \times 1$ vector with constant entries and $f(t)$ is a known function $t \in T$. Then the optimal control problem considered is

$$\max_{u_k} J(u),\ J(u) = \sum_{k \in K} p_k^T x_k(\alpha), \tag{3}$$

where $p_k,\ k = 1, \dots, N$, is a given $n \times 1$ vector subject to an end of pass (or terminal) constraint of the form

$$H_k x_k(\alpha) = o_k, \quad k \in K, \tag{4}$$

where o_k is an $m \times 1$ vector and H_k is an $m \times n$ matrix, and the control inputs satisfy the following admissibility condition.

Definition 1: For each pass number $k \in K$ the piecewise continuous function $u_k : T \to R$ is termed an admissible control for this pass if it satisfies

$$|u_k(t)| \le 1,\ t \in T, \tag{5}$$

and the corresponding state vector $x_k(t),\ t \in T$, of (1) satisfies the boundary conditions

$$x_k(0) = d_k,\ H_k x_k(\alpha) = o_k.$$

Also, without loss of generality, we assume that the matrix A has simple eigenvalues $\lambda_i,\ 1 \le i \le n$, and that it is stable in the sense that Re $\lambda_i < 0,\ 1 \le i \le n$. (Stability of the matrix A is a necessary condition for so-called stability along the pass (essentially bounded input bounded output stability) independent of the pass length [16].)

III. Optimality conditions for the supporting control functions

Consider first (1)–(2) in the absence of the terminal conditions (4). Then it has been shown elsewhere [5] that the solution of these equations can be written as

$$\begin{aligned} x_k(t) &= \sum_{j=1}^{k} K_j(t) d_{k+1-j} + \int_0^t K_k(t-\tau) D f(\tau) d\tau \\ &+ \sum_{j=1}^{k} \int_0^t K_j(t-\tau) b u_{k+1-j}(\tau) d\tau, \\ k &= 1, \dots, N, \end{aligned} \tag{6}$$

where the $K_i(t)$ are the solutions of the following $n \times n$ matrix differential equations:

$$\begin{aligned} \dot{K}_1(t) &= AK_1(t), \\ \dot{K}_i(t) &= AK_i(t) + DK_{i-1}(t), \quad i = 2, \dots, N, \end{aligned} \tag{7}$$

with initial conditions

$$K_1(0) = I_n, \quad K_i(0) = 0, \quad i = 2, \dots, N. \tag{8}$$

ow by using (6) we can rewrite the optimal problem considered here in the following integral form:

$$\max_{u_1, \dots, u_N} J(u),\ J(u) = \sum_{j=1}^{N} \int_0^\alpha c_j(\tau) u_j(\tau) d\tau + \gamma, \tag{9}$$

subject to the terminal conditions (4) and the control constraint (5). Also we can write

$$\begin{aligned} \int_0^\alpha g_{11}(\tau) u_1(\tau) d\tau &= h_1, \\ \int_0^\alpha \Big[g_{21}(\tau) u_1(\tau) + g_{22}(\tau) u_2(\tau) \Big] d\tau &= h_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ \int_0^\alpha \Big[g_{N1}(\tau) u_1(\tau) + \dots + g_{NN}(\tau) u_N(\tau) \Big] d\tau &= h_N, \end{aligned} \tag{10}$$

and

$$|u_k(\tau)| \le 1,\ \tau \in T, \quad k = 1, \dots, N,$$

where the scalar γ and the scalar functions $c_j(\tau)$ are defined as follows

$$\begin{aligned} \gamma &= \sum_{k=1}^{N} \sum_{j=1}^{k} p_k^T K_j(\alpha) d_{k+1-j} \\ &+ \sum_{k=1}^{N} \int_0^\alpha p_k^T K_k(\alpha-\tau) D f(\tau) d\tau, \\ c_j(\tau) &= \sum_{k=j}^{N} p_k^T K_{k+1-j}(\alpha-\tau) b, \quad j = 1, \dots, N, \\ g_{kj}(\tau) &= H_k K_{k+1-j}(\alpha-\tau) b,\ j \le k, \\ h_k &= o_k - \sum_{j=1}^{k} H_k K_j(\alpha) d_{k+1-j} \\ &- \int_0^\alpha H_k K_k(\alpha-\tau) D f(\tau) d\tau, k = 1, \dots, N. \end{aligned}$$

Also we require the following.

Definition 2: For each fixed $k,\ 1 \le k \le N$, the time instances $\tau_{ki},\ 1 \le i \le m :\ 0 < \tau_{k1} < \tau_{k2} < \cdots < \tau_{km} < \alpha$ are termed supporting, and their collection $\tau_{sup}^k := \{\tau_{k1}, \ldots, \tau_{km}\}$ is termed the support of pass k for (1)–(4) if the matrix $G_{sup}^k := \{g_{kk}(\tau_{k1}), \ldots, g_{kk}(\tau_{km})\}$ (i.e., the jth column of the matrix here is the $m \times 1$ vector $g_{kk}(\tau_{kj})$) is nonsingular.

By using (7) we have that $g_{kk}(\tau) = H_k e^{A(\alpha-\tau)}b$. Therefore the existence of the support τ_{sup}^k is guaranteed by controllability of the pair $\{H_k A, b\}$.

Definition 3: A pair $\{\tau_{sup}^k,\ u_k(t), k = 1, \ldots, N\}$ consisting of a support τ_{sup}^k and admissible control functions $u_k(t), t \in T$ is termed a supporting control function for (1)–(4).

Remark 1: These last two definitions are motivated as follows. Often an optimal control problem solution has the so-called bang-bang form; i.e., the control function takes only boundary values in the admissible set U. If $U = \{-1 \le u \le +1\}$, then $u^0(t) = \pm 1$ (the "switch-on/switch-off" regime). Also the switching times are constructive elements in the design of the optimal controller. Hence, our goal is to apply these key elements directly to the optimality conditions, and consequently we use the supporting time instances and control.

Let $\{\tau_{sup}^k, u_k(t), k = 1, \ldots, N\}$ be a support control function and construct a sequence of $m \times 1$ vectors $\{\nu^{(k)},\ k = 1, \ldots, N\}$ by solving the following set of linear algebraic equations:

$$(\nu^{(N)})^T G_{sup}^N - c_{sup}^{(N)} = 0,$$

$$(\nu^{(N-1)})^T G_{sup}^{N-1} + (\nu^{(N)})^T F_{(N-1)sup}^N - c_{sup}^{(N-1)} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(\nu^{(1)})^T G_{sup}^1 + (\nu^{(2)})^T F_{1sup}^2 + \cdots + (\nu^{(N)})^T F_{1sup}^N - c_{sup}^{(1)} = 0,$$

where the $1 \times m$ vectors $c_{sup}^{(k)}$ and the $m \times m$ matrices F_{jsup}^k are given by

$$c_{sup}^{(k)} := \big(c_k(\tau_{k1}), \ldots, c_k(\tau_{km})\big), \qquad k = 1, \ldots, N,$$

and for $k > j, \quad j = 1, \ldots, N-1$,

$$F_{jsup}^k := \bigg(g_{kj}(\tau_{j1}), \ldots, g_{kj}(\tau_{jm})\bigg),$$

respectively.

Introduce the $1 \times mN$ vectors $(\hat{\nu})^T$ and c_{sup} as

$$\begin{aligned} (\hat{\nu})^T &= ((\nu^{(1)})^T, \ldots, (\nu^{(N)})^T), \\ c_{sup} &= (c_{sup}^{(1)}, c_{sup}^{(2)}, \ldots, c_{sup}^{(N)}), \end{aligned}$$

Then the linear algebraic equations above can be rewritten in the form

$$\hat{\nu}^T \tilde{G}_{sup} - c_{sup} = 0, \tag{11}$$

where the $mN \times mN$ lower triangular matrix $\tilde{G}_{sup}$ is (Definition 2) nonsingular and therefore $\hat{\nu}^T = c_{sup}\tilde{G}_{sup}^{-1}$.

To establish the new optimality conditions, define the so-called co-control $1 \times N$ vector function

$$\Delta(t) = (\Delta_1(t), \ldots, \Delta_N(t)),$$

as

$$\begin{aligned} \Delta_1(t) &= \nu^{(1)T} g_{11}(t) + \nu^{(2)T} g_{21}(t) + \cdots \\ &+ \nu^{(N)T} g_{N1}(t) - c_1(t), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta_{N-1}(t) &= \nu^{(N-1)T} g_{N-1N-1}(t) + \nu^{(N)T} g_{NN-1}(t) \\ &- c_{N-1}(t), \\ \Delta_N(t) &= \nu^{(N)T} g_{NN}(t) - c_N(t), \end{aligned}$$

or, on introducing the $1 \times N$ vector function,

$$c(t) = (c_1(t), \ldots, c_N(t)),$$

$$\Delta(t) = \hat{\nu}^T \hat{G}(t) - c(t),$$

where $\hat{G}(t)$ is an $mN \times N$ matrix of the form

$$\hat{G}(t) = \begin{pmatrix} g_{11}(t) & 0_{m\times 1} & \ldots & 0_{m\times 1} \\ g_{21}(t) & g_{22}(t) & \ldots & 0_{m\times 1} \\ \ldots & \ldots & \ldots & \ldots \\ g_{N1}(t) & g_{N2}(t) & \ldots & g_{NN}(t) \end{pmatrix}. \tag{12}$$

Note also that the $mN \times mN$ matrix $\tilde{G}_{sup}$ is obtained from $\hat{G}(t)$ in an obvious manner by evaluating the rows of the matrix $\hat{G}(t)$ at the supporting moments $t \in \tau_{sup}^k,\ k = 1, \ldots, N$.

Definition 4: We say that the supporting control function $\{\tau_{sup}^k, u_k(t), k = 1, \ldots, N\}$ is non-degenerate for the problem (1)–(3) if

$$\frac{d\Delta_k(\tau_j)}{dt} \neq 0 \quad \forall\ \tau_j \in \tau_{sup}^k, \quad k = 1, \ldots, N.$$

Remark 2: Here nondegeneracy means that in a small neighborhood of the supporting points the admissible control can be replaced by constant functions whose values are less than those on the control constraint boundary and satisfy (10); i.e., the support control function is nonsingular if there exist numbers $\lambda_0 > 0,\ \mu_0 > 0,\ u_j^k(\lambda),\ j = 1, \ldots, m,\ k = 1, \ldots, N$, such that the following equalities:

$$\begin{aligned} \sum_{j=1}^{k}\sum_{i=1}^{m} u_j^i(\lambda)\int_{\tau_{ij}-\lambda}^{\tau_{ij}+\lambda} g_{kj}(t)dt &= \sum_{j=1}^{k}\sum_{i=1}^{m}\int_{\tau_{ij}-\lambda}^{\tau_{ij}+\lambda} X \\ X &= g_{kj}(t)u_j(t)dt, \\ |u_j^k| &\le 1 - \mu_0, \end{aligned} \tag{13}$$

$j = 1, \ldots, m,\ k = 1, \ldots, N$, hold for all $\lambda,\ 0 < \lambda < \lambda_0$, and $k,\ 1 \le k \le N$. This fact is used in the proof of the optimality conditions.

Associate with each supporting time instance τ_{kj} a small subinterval T_{kj} from T such that the matrix $G_{gen}^k := \{\int_{T_{kj}} g_{kk}(\tau)d\tau, j = 1, \ldots, m\}$ is nonsingular. Also without loss of generality we can assume that τ_{kj} is one or the other of the end points of T_{kj} and the supporting control functions $u_k(t) = u_j^k$ for $t \in T_{kj},\ j = 1, \ldots, N$, are constant over the segments T_{kj}. Then we have the following result.

Theorem 1: A supporting control function $\{\tau_{sup}^k,\ u_k^0(t),\ k = 1,\dots,N\}$ is an optimal solution of the problem (1)–(4) if

$$u_k^0(t) = -\mathrm{sgn}\big(\Delta_k(t)\big), \qquad k = 1,\dots,N,\ t \in T. \tag{14}$$

Moreover, if this supporting control function is non-degenerate, then the above condition is necessary and sufficient.

Remark 3: The analysis which now follows shows that the above result can be reformulated in the traditional maximum principle form. In particular, it will be shown that the co-control functions $\Delta_k(t),\ t \in T$, here are connected directly to the adjoint (dual) variables $\psi_k(t),\ t \in T$, as $\Delta_k(t) = -\psi_k^T(t)b$. Note also that the term $\psi_k^T(t)b$ is part of the Hamiltonian function which arises in the maximum principle statement of the result here. Moreover, the vectors $\{\nu^{(k)},\ k = 1,\dots,N\}$ (termed Lagrange multipliers in some literature) will be used as the boundary conditions for the corresponding differential equations describing the adjoint (dual) variables $\psi_k(t)$ (in contrast to the classic maximum principle, where such boundary conditions are not specified).

Let $\psi_N(t)$ be the solution of

$$\frac{d\psi_N(t)}{dt} = -A^T\psi_N(t), \quad \psi_N(\alpha) = p_N - H_N^T\nu^N, \quad t \in T, \tag{15}$$

or

$$\psi_N(t) = K_1^T(\alpha - t)\psi(\alpha), \quad t \in T. \tag{16}$$

Hence

$$\begin{aligned}\psi_N^T(t)b &= \big(p_N^T - (\nu^N)^T H_N\big)K_1(\alpha - t)b,\\ &= p_N^T K_1(\alpha - t)b - (\nu^N)^T H_N K_1(\alpha - t)b,\\ &= c_N(t) - (\nu^N)^T g_{NN}(t) = -\Delta_N(t).\end{aligned} \tag{17}$$

In order to verify the validity of the corresponding conditions for subsequent passes, let $\psi_{N-1}(t),\ t \in T$, be a solution of the differential equation

$$\begin{aligned}\frac{d\psi_{N-1}(t)}{dt} &= -A^T\psi_{N-1}(t) - D^T\psi_N(t),\\ \psi_{N-1}(\alpha) &= p_{N-1} - H_{N-1}^T\nu^{N-1},\ t \in T.\end{aligned} \tag{18}$$

Then it follow that

$$\psi_k^T(t)b = -\Delta_k(t), \quad k = 2,\dots,N, \tag{19}$$

where $\psi_k(t),\ t \in T$, are the solutions of the following differential equations:

$$\begin{aligned}\frac{d\psi_k(t)}{dt} &= -A^T\psi_k(t) - D^T\psi_{k+1}(t),\\ \psi_k(\alpha) &= p_k - H_k^T\nu^k, \quad t \in T.\end{aligned} \tag{20}$$

For each $k = 1,\dots,N$ introduce the associated Hamilton function as

$$H_k(x_{k-1}, x_k, \psi_k, u_k) = \psi_k^T\big(Ax_k + Dx_{k-1} + bu_k\big), \quad t \in T. \tag{21}$$

Then the use of (19) yields that the optimality conditions (14) can be reformulated in maximum principle form as the following corollary to Theorem 1.

Corollary 1: The admissible supporting control $\{\tau_{sup}^k,\ u_k^0(t),\ k = 1,\dots,N\}$ is optimal if along the corresponding trajectories $x_k^0(t),\ \psi_k(t)$ of (1)–(2) and (20) the Hamiltonian function has maximum value, i.e.,

$$\begin{aligned}H_k(x_{k-1}^0(t), x_k^0(t), \psi_k, u_k^0(t)) = Y, \quad t \in T,\\ Y = \max_{|v|\le 1} H_k(x_{k-1}^0(t), x_k^0(t), \psi_k, v),\end{aligned} \tag{22}$$

for $k = 1,\dots,N$. If the admissible supporting control is non-degenerate, then this condition is necessary and sufficient.

Remark 4: In order to further emphasize the relationship between the support elements and the control function, note that the optimality conditions given by Theorem 1 can be equivalently stated in the form for $t \in T$

$$\begin{aligned}\Delta_k(t) &> 0 \text{ at } u_k^0(t) = -1, \quad \Delta_k(t) < 0 \text{ at } u_k^0(t) = 1,\\ \Delta_k(t) &= 0 \text{ at } -1 < u_k^0(t) < 1,\ k = 1, 2, \dots, N,\end{aligned} \tag{23}$$

Hence the supporting elements and control function of optimal solution are interconnected such that the supporting instances are the switching moments for optimal bang-bang control functions.

In the next section, the maximum principle for arbitrary admissible control functions of (1)–(4) is established using the sub-optimality conditions.

A. ϵ-optimality conditions

Often in the numerical implementation of optimal control algorithms it is beneficial to exploit approximate solutions with corresponding error estimation. Hence it is necessary to introduce the "sub-optimality" concept as it is often sufficient to stop the numerical computations when a satisfactory accuracy level has been achieved.

Assume that $\{u_k^0(t),\ k \in K\}$ is the optimal control for (1)–(4), and let $J(u^0)$ denote the corresponding optimal cost function value.

Definition 5: We say that the admissible control function $\{u_k^\epsilon(t),\ k \in K\}$ is ϵ-optimal if the corresponding solution $\{x_k^\epsilon(t),\ t \in T,\ k \in K\}$ of (1)–(4) satisfies $J(u^0) - J(u^\epsilon) \le \epsilon$.

Now we proceed to calculate an estimate of a supporting control function

$$\{u_k,\ \tau_{sup}^k,\ k \in K,\ t \in T\},$$

i.e., a measure of the non-optimality of the control. Note also that this estimate can be partitioned into two principal parts: one of which evaluates the degree of non-optimality of the chosen admissible control functions $u_k(t)$, and the second the error produced by non-optimality of the support τ_{sup}^k. This partition is a major advantage in the design of numerically applicable solution algorithms.

Introduce an estimate of optimality $\beta = \beta(\tau_{sup}, u)$ as the value of the maximum increment for the cost function here calculated in the absence of the principal constraints (4); i.e., this estimate is given by the solution of the following relaxed optimization problem:

$$\max_{\Delta u_k}\ \Delta J(u),\ |u_k(t) + \Delta_k u(t)| \le 1,\ t \in T,\ k = 1,\dots,N. \tag{24}$$

It is easy to see that

$$\beta = \beta(\tau_{sup}, u) = \sum_{k=1}^{N} \int_{T_k^+} \Delta_k(t)(u_k(t) + 1)dt + \sum_{k=1}^{N} \int_{T_k^-} \Delta_k(t)(u_k(t) - 1)dt, \quad (25)$$

where

$$T_k^+ = \{t \in T : \Delta_k(t) > 0\}, \quad T_k^- = \{t \in T : \Delta_k(t) < 0\},$$

and we have the following result.

Theorem 2 (ϵ-maximum principle): Given any $\epsilon \geq 0$, the admissible control $\{u_k(t),\ t \in T,\ k \in K\}$ is ϵ-optimal for (1)–(4) if and only if there exists a support $\{\tau_{sup}^k,\ k \in K\}$ such that along the solutions $x_k(t), \psi_k(t), t \in T,\ k \in K$, of (1)–(4) and (20) the Hamiltonian attains its ϵ-maximum value, i.e.,

$$H_k(x_{k-1}(t), x_k(t), \psi_k, u_k(t)) = \hat{Y}, \quad t \in T, \quad (26)$$

$$\hat{Y} = \max_{|v| \leq 1} H_k(x_{k-1}(t), x_k(t), \psi_k, v) - \epsilon_k(t),$$

where the functions $\epsilon_k(t),\ k \in K$, satisfy the following inequality:

$$\sum_{k \in K} \int_T \epsilon_k(t)dt \leq \epsilon. \quad (27)$$

The maximum principle now follows from this last result on setting $\epsilon = 0$ as stated formally in the following corollary.

Corollary 2: The admissible control $\{u_k^0(t), k \in K,\ t \in T\}$ is optimal if and only if there exists a support $\{\tau_{sup}^{0k},\ k \in K\}$ such that the supporting control $\{u_k^0(t), \tau_{sup}^{0k},\ t \in T,\ k \in K\}$ satisfies the maximum conditions

$$\max_{|v| \leq 1} H_k\big(x_{k-1}^0(t), x_k^0(t), \psi_k, v\big) = \hat{H}_k,$$

$$\hat{H}_k = H_k\big(x_{k-1}^0(t), x_k^0(t), \psi_k, u_k^0(t)\big),$$

for all $k \in K,\ t \in T$, where $\psi_k(t)$ are the corresponding solutions of (20).

IV. Differentiable properties of the optimal solutions

An important aspect of the optimization theory is sensitivity analysis of optimal controls since, in practice, the system considered can be subject to disturbances or parameters in the available model can easily arise. Mathematically, perturbations can, for example, be described by some parameters in the initial data, boundary conditions, and control and state constraints. Hence it is clearly important to know how a problem solution depends on these parameters, and in this section we aim to characterize the changes in the solutions developed due to "small" perturbations in the parameters. This could, in turn, enable us to design fast and reliable real-time algorithms to correct the solutions for these effects. As shown next, the major advantage of the constructive approach developed in this paper is that the sensitivity analysis and some differential properties of the optimal controls under disturbances can be analyzed.

Suppose that disturbances influence the initial data for (1)–(4). In particular, consider the system (1)–(4) on the interval $T_s = [s, \alpha]$ with the initial data $x_k(s) = z_k, z_k \in G_k, k \in K$, where $G_k \subset \mathbb{R}^n$ is some neighborhood of the point $x_k = d_k$ and s belongs to the neighborhood G_0 of $t = 0$. We also assume that the following regularity condition holds: For the given disturbance domain $G_k, k \in K \cup \{0\}$, the structure of the optimal control functions for the non-disturbed data is preserved; i.e., the number of switching instances together with their order is constant.

Using Theorem 1, the optimal controls $\{u_k^0(t, s, z), k \in K\}$ are determined by the supporting time instances $\tau_{kj} = \tau_{kj}(s, z),\ k \in K,\ j = 1, \ldots, m$, which are dependent on the disturbances $(s, z_k), s \in G_0,\ z_k \in G_k,\ k \in K$. Here we study the differential properties of the functions $\tau_{kj} = \tau_{kj}(s, z),\ k \in K,\ j = 1, \ldots, m$, and for ease of notation we set $\tau \equiv \tau(s, z) = \{\tau_{kj}(s, z),\ \ k \in K,\ j = 1, \ldots, m\},\ z = \{z_k, k \in K\}$ in what follows.

Theorem 3: If (1)–(4) is regular, then for any $k \in K$ and $j = 1, \ldots, m$ the functions $\tau_{kj} = \tau_{kj}(s, z)$ are differentiable in the domain $G_0 \times G_k \subset \mathbb{R} \times \mathbb{R}^n$.

The differential properties of the optimal controls developed above can be used for sensitivity analysis and the solution of the synthesis problem considered here. In particular, the supporting control approach [9] can be used to produce the differential equations for the switching time functions $\tau(s, z)$ necessary to design the optimal controllers. In a similar manner to [6] it can be shown that these satisfy the following differential equations:

$$G\frac{\partial \tau}{\partial s} + Q = \frac{\partial h}{\partial s}, \quad P\frac{\partial \tau}{\partial z} = \frac{\partial h}{\partial z}, \quad (28)$$

where $h(s, z) = (h_1(s, z), \ldots, h_m(t, s))$ is an $mN \times 1$-vector and the matrices G, Q, P are defined (see [6]) by those defining the process dynamics and information associated with the disturbance-free optimal solution. For example, $G = \Lambda\tilde{G}_{sup}$, where the compatibly dimensioned block matrix Λ is constructed by the disturbance-free optimal control values $u_k^0(t); k = 1, \ldots, N$ calculated in the supporting moments τ_{kj} from τ_{sup}^0. Also, by Theorem 1, these values are equivalent to the values of $\frac{d\Delta_i(\tau_{kj})}{dt}$ evaluated for the corresponding indexes $i; j; k$, where the functions $\Delta_i(t); i = 1, \ldots, N$ are designed using the switching times of the basic optimal control function. Note also that analogous differential equations can be established for the optimal values of the cost function (treated as the function $J(s, z) \equiv J(u(\tau(s, z)))$.

Remark 5: The equations (28) are (sometimes) termed Pfaff differential equations and model an essentially distinct class of continuous nD systems. The main characteristic feature of this model is that it is over-determined (the number of equations exceeds the unknown functions). It can also be shown that if the non-degenerate assumption on the supporting control functions holds, then so do the so-called Frobenious conditions which guarantee the existence and uniqueness of solutions of the Pfaff differential equations.

V. Conclusions

In this paper the supporting control functions approach has been applied to study an optimal control problem for differential linear repetitive processes. The main contribution is the development of constructive necessary and sufficient optimality conditions in forms which can be effectively used for the design of numerical algorithms. The iterative method developed in this work is based on the principle of decrease of the sub-optimality estimate; i.e., the iteration $\{\tau^k_{sup},\ u_k(t),\ k = 1,\ldots,N\} \rightarrow \{\hat{\tau}^k_{sup},\ \hat{u}_k(t),\ k = 1,\ldots,N\}$ is performed in such a way as to achieve $\beta(\hat{\tau}_{sup},\hat{u}) < \beta(\tau_{sup},u)$. Moreover, it is proved in [7] that the sub-optimality estimate is the sum of and hence the iteration procedure can be separated into two stages: (i) transformation of the admissible control functions $\{u_k(t),\ k = 1,\ldots,N\} \rightarrow \{\hat{u}_k(t),\ k = 1,\ldots,N\}$, which decreases the non-optimality measure of the admissible controls $\beta(\hat{u}) < \beta(u)$, and (ii) variation of the support $\{\tau^k_{sup},\ k = 1,\ldots,N\} \rightarrow \{\hat{\tau}^k_{sup},\ k = 1,\ldots,N\}$ to again decrease the non-optimality measure of the support, i.e., $\beta(\hat{\tau}_{sup}) < \beta(\tau_{sup})$. These transformations involve, in effect, the duality theory for the problems defined in this work and exploit the ϵ-optimality conditions also developed in this work. These results are the first in this general area, and work is currently proceeding in a number of followup areas. One such area is sensitivity analysis of optimal control in the presence of disturbances, where in the case of the ordinary linear control systems some work on this topic can be found in, for example, [11].

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