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Optimal Terminal Wealth under Partial Information: Both the Drift and the Volatility Driven by a Discrete Time Markov Chain

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***Abstract*— We consider a multi-stock market model. The stock price process satisfies a stochastic differential equation where both the drift and the volatility are driven by a discrete-time Markov chain of finite states. Not only the underlying Brownian motion but also the Markov chain in the stochastic differential equation are assumed to be unobservable. Investors can observe the stock price process only. The main result of this paper is that we derive the approximation of the optimal trading strategy and the corresponding optimal expected utility function from terminal wealth.**

I. INTRODUCTION

We consider an incomplete market model in which stocks are driven by an m-dimensional Geometric Brownian motion the same as in the Black-Scholes model. However, the drift and the diffusion coefficients of this process depend on a discrete time Markov chain. Such a model is called a scheme-switching model since when the Markov changes from one state to another, the market is considered to change from one scheme to another. Investors observe the stock price only. They only have 'partial information' since we assume the state of the Markov chain is not observable. The objective is to optimize trading strategy by maximizing a utility function from terminal wealth.

There are quite a few papers devoted to studies of the problem of maximizing the expected utility function from terminal wealth under partial information. Pham and Quenez [15] considered a stochastic volatility model. They solved the portfolio optimization problem under partial information by stochastic filtering techniques and adapting martingale duality methods. For more literature on partial information and stochastic volatility problems, we refer to Lakner [12],[13], Frey [8], Runggaldier [16] and Frey and Runggaldier [9].

Sass and Haussman [18] considered a multi-stock market model in continuous time. The drift is a continuous time, finite state Markov chain, and the volatility matrix is constant and nonsingular. They used Malliavin calculus and Hidden Markov Chain theory to derive an explicit expression for the optimal portfolio selection. However, their method can not be extended to the case in which the volatility is driven by a Markov chain, because the EM algorithm they used to estimate the drift does not work for the volatility due to the fact that the measures involved in their method are not equivalent if the volatility is driven by a Markov chain.

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In this paper, we consider a discrete time multi-stock market model where both the drift and the volatility are driven by a Markov chain. In our paper, we use the method of estimating volatility studied by Elliott[4], Elliott, et al. [5], developed for the discrete time setting. The algorithm enables us to estimate the states of the Markov chain and its transition matrix. We solve the problem of optimizing the expected utility from terminal wealth, and using dynamic programming we construct the optimal strategy in terms of the filter of the price process. The proofs of the results given and further analysis can be found in Taksar and Zeng[20].

II. MODELS AND PRELIMINARY RESULTS

A. Regime Switching Model: Continuous-Time

Consider an m-dimensional stock price process whose dynamics is given by a geometric Brownian motion equations:

$$dS(t) = diag(S_t)(\mu(Y(t))dt + \hat{\sigma}(Y(t))dW(t)), 0 \le t \le T. \quad (1)$$

Here $S_t = (S_t^{(1)}, S_t^{(2)}, ..., S_t^{(m)})'$, the column vector W_t is a m-dimensional standard Brownian motion. $Y(t)$ is a finite state, homogeneous Markov chain with a generator $Q = (q_{ij})_{d\times d}$, independent of $W(t)$. The distribution of $Y(0)$ is known. $Y(t)$ has a state space $\mathcal{M} = \{e_1, ..., e_d\}$, where $e_i, i = 1, 2, ..., d$ is the unit vector in $\mathcal{R}^d$.

$$Y(t) \in \mathcal{M} := \{e_1, ..., e_d\}.$$

There are different values for the drift and different matrices for the volatility corresponding to states of the Markov chain $Y(t)$. Thus $\mu(\cdot)$(resp. $\sigma(\cdot)$) is a mapping of $\mathcal{M}$ (resp. $\mathcal{N} := \{B := (b_{i,j})_{1\le i,j\le m}$ is invertible $|b_{i,j} \in \mathcal{R}^+\}$.)into R^m (resp. into $R^{m\times m}$).

Assume the short interest rate r is constant for simplicity. Then (1) may be written as follows.

$$d\ log(e^{-rt}S(t)) = (\mu(Y(t)) - r1_m - diag(\hat{\sigma}_{n-1}\hat{\sigma}'_{n-1}))dt + \hat{\sigma}(Y(t))dW(t), \quad (2)$$

where $0 \le t \le T, 1_m = (1, 1, ..., 1)' \in \mathcal{R}^{m\times 1}$,where we use the following convention:

$$log((x_1, x_2, ..., x_n)') = (log(x_1), ..., log(x_n))'.$$

B. Regime Switching Model: Discrete-Time

In this paper, we will consider a discrete approximation to the continuous time model (2).

Let $\Delta t = \frac{T}{N}$, $Y_n = Y(n\Delta t)$,

$$\mu_n = \mu(Y(n\Delta t)), \hat{\sigma}_n = \hat{\sigma}(Y(n\Delta t)), S_n = S(n\Delta t),$$

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where n=0,1,...,N. Then the equation (2.2) becomes

$$\begin{aligned} y_n &:= \log(S_n e^{-r\Delta t}) - \log(S_{n-1}) \\ &= (\mu_{n-1} - r1_m - diag(\hat{\sigma}_{n-1}\hat{\sigma}'_{n-1})/2)\Delta t \\ &+\hat{\sigma}_{n-1}(W_n - W_{n-1}), \end{aligned} \tag{3}$$

for n=1,2,3,...,N.

Let

$$\begin{cases} g_n: & = (\mu_n - r1_m - diag(\hat{\sigma}_n\hat{\sigma}'_n)/2)\Delta t \\ & = (\mu(Y_n) - r - diag(\hat{\sigma}(Y_n)\hat{\sigma}(Y_n)')/2)\Delta t, \\ \sigma_n: & = \hat{\sigma}_n\sqrt{\Delta t} = \hat{\sigma}(Y_n)\sqrt{\Delta t} \end{cases} \tag{4}$$

Then

$$y_n = g_{n-1} + \sigma_{n-1}Z_n, n = 1, 2, ..., N, \tag{5}$$

where $Z_n = (W_n - W_{n-1})/\sqrt{\Delta t}, n = 1, 2, ..., N$ is a sequence of standard normal i.i.d. random variables.

Note that g_n and σ_n are functions of Y_n and can be written as $G(Y_n)$ and $H(Y_n)$ respectively which obviously satisfy

$$\begin{aligned} \frac{G(e_i)}{\Delta t} &= \mu(e_i) - r1_m - diag(\hat{\sigma}(e_i)\hat{\sigma}(e_i)')/2, \\ \frac{H(e_i)}{\sqrt{\Delta t}} &= \hat{\sigma}(e_i) \end{aligned} \tag{6}$$

In this paper, we assume only the price of stock S_n or y_n can be observed. Denote the filtration generated by S_n by $\{\mathcal{F}_n\}$. We will study an optimization of the utility function from terminal wealth in the discrete time model (5).

C. Preliminary Results

We present some preliminary results which will be used in the proofs in the subsequent. By the definition of y_n (3), we know that for each i=1,2,...,m,

$$S_n^{(i)} = S_{n-1}^{(i)} e^{y_n^{(i)}} e^{r\Delta t}.$$

For $k = 1, 2, ..., d$, denote

$$b_k := G(e_k) \in \mathcal{R}^{m\times 1}, f_k := H(e_k) \in \mathcal{R}^{m\times m}.$$

Let $b_k(i)$ stands for the i^{th} component of b_k, let $f_k(i)$ stands for the i^{th} row of f_k. Then we have

$$\begin{aligned} &Pr(y_n^{(i)} \le t|\mathcal{F}_{n-1}) \\ &= \textstyle\sum_{k=1}^d Pr(Y_{n-1} = e_k|\mathcal{F}_{n-1}) \int_{-\infty}^{t-b_k(i)} \phi_{ik}(x)dx, \end{aligned} \tag{7}$$

where $\phi_{ik}(x) = \frac{1}{\sqrt{2\pi(f_k(i)\cdot f_k(i)')}} e^{-\frac{x^2}{2f_k(i)\cdot f_k(i)'}}$, i=1,2,...,m.

Proof: of (7) $Pr(y_n^{(i)} \le t|\mathcal{F}_{n-1}) = Pr(g_{n-1}^{(i)} + \sigma_{n-1}^{(i)} Z_n \le t|\mathcal{F}_{n-1})$

$$= \textstyle\sum_{k=1}^d Pr(b_k(i) + f_k(i)Z_n \le t, Y_{n-1} = e_k|\mathcal{F}_{n-1})$$

$$= \textstyle\sum_{k=1}^d Pr(b_k(i) + f_k(i)Z_n \le t)Pr(Y_{n-1} = e_k|\mathcal{F}_{n-1})$$

$$= \textstyle\sum_{k=1}^d Pr(Y_{n-1} = e_k|\mathcal{F}_{n-1}) \int_{-\infty}^{t-b_k(i)} \phi_{ik}(x)dx.$$

■

Similarly, for the multi-dimensional case, $x \in R^{m\times 1}$, we have

$$\begin{aligned} &Pr(y_n \le x|\mathcal{F}_{n-1}) \\ &= \textstyle\sum_{k=1}^d Pr(Y_{n-1} = e_k|\mathcal{F}_{n-1}) \\ &\cdot \int_{-\infty}^{x_1-b_k(1)} \dots \int_{-\infty}^{x_m-b_k(m)} \phi_k(z)dz, \end{aligned} \tag{8}$$

where $\phi_k(x) = (2\pi|f_k f_k'|)^{\frac{-1}{2}} e^{-x'(f_k f_k')^{-1}x/2}, x \in R^{m\times 1}$

And we have a recursive filter :

$$\begin{aligned} &Pr(Y_n = e_k|\mathcal{F}_n) \\ &= \frac{\sum_{i=1}^d Pr(Y_{n-1}=e_i|\mathcal{F}_{n-1})\phi_i(y_n-b_i)p_{ki}}{\sum_{i=1}^d Pr(Y_{n-1}=e_i|\mathcal{F}_{n-1})\phi_i(y_n-b_i)}, \end{aligned} \tag{9}$$

where p_{ki} is the (k,i) entry of the transition matrix P.

In the sequel, we will use the notation: $E_n[\zeta] := E[\zeta|\mathcal{F}_n]$.

Let $\alpha < 1$. For $i = 1, 2, ...m$, we have

$$\begin{aligned} &(i)\ |E_{n-1}[e^{y_n^{(i)}} - 1]| = O(\Delta t), \\ &(ii)\ |E_{n-1}[(e^{y_n^{(i)}} - 1)^2]| = O(\Delta t), \\ &(iii)\ |E_{n-1}[(e^{y_n^{(i)}} - 1)^3]| = O(\Delta t)^2, \\ &(iv)\ |E_{n-1}[(1 - e^{-y_n^{(i)}})^3]| = O(\Delta t)^2, \\ &(v) |E_{n-1}[e^{\alpha y_n^{(i)}}(1 - e^{-y_n^{(i)}})^3]| = O(\Delta t)^2. \end{aligned}$$

III. Definition and Optimization Problem

A. Wealth Process and Admissible Strategies

In this section we describe a discrete time optimization model which approximates the original continuous time model (2).

Definition $h_{n-1} \in \mathcal{F}_{n-1}, n = 1, ..., N$ are column vectors $\in R^{m\times 1}$. A wealth process $\{X_n^{h_{n-1}}\}_{n=1,2,...,N}, X_0 = x_0$ is defined as

$$X_n^{h_{n-1}} = X_{n-1}^{h_{n-2}} e^{r\Delta t}(1 - \sum_{i=1}^m h_{n-1}^{(i)}) + X_{n-1}^{h_{n-2}} \sum_{i=1}^m h_{n-1}^{(i)} \frac{S_n^{(i)}}{S_{n-1}^{(i)}},$$

where $h_{n-1}^{(i)}$ or $S_{n-1}^{(i)}$ denotes the i^{th} component of the vector h_{n-1} or S_{n-1}. Using the notations defined in Section II, the wealth process has a simpler expression:

$$X_n^{h_{n-1}} = X_{n-1}^{h_{n-2}} e^{r\Delta t}(1 + h_{n-1}\cdot(e^{y_n} - 1)),$$

where "·" stands for the vectors or matrices multiplication. Generally, we have

$$X_n^{h_{n-1}} = X_0 e^{rn\Delta t}(1 + h_0\cdot(e^{y_1} - 1))...(1 + h_{n-1}\cdot(e^{y_n} - 1)),$$

where n=1,...,N.

Definition A vector sequence $h = \{h_i\}_{i=0}^{N-1}$ is an **admissible** strategy if $Pr(X_n^{h_{n-1}} > 0$, for all n=1,2...,N)=1.

We use $\mathcal{H}$ to denote the set of all admissible strategies.

One can check that if h is admissible, then

$$\|h_{n-1}\|_1 \le 1, 1 \ge h_{n-1}^{(i)} \ge 0, \tag{10}$$

for each $i \in \{1, 2, ...m\}$.

The inequalities above imply no stock shorting as well as no money borrowing in our model. Rogers also mentioned such a restriction on the portfolio in his h-investor model([17]).

B. HARA Utility Functions

Definition A function $u : (x_u, \infty) \to \mathcal{R}, x_u \in \mathcal{R} \cup \{-\infty\}$, is called a utility function, if u is strictly increasing, strictly concave, twice continuously differentiable on (x_u, ∞), satisfies $lim_{x\to\infty} u'(x) = 0$ and $lim_{x\to x_u^+} u'(x) = \infty$.

Next we define the coefficient of absolute risk aversion ,

$$R_a(x) = -\frac{u''(x)}{u'(x)}.$$

Definition If $R_a^{-1}(x)$ is a linear function, i.e. $R_a^{-1}(x) = a + bx$, then we say that u(x) is of the hyperbolic absolute risk aversion (HARA) class.

The most popular HARA utility functions are Constant Relative Risk Aversion (CRRA): $u(x) = x^\beta/\beta$, where $\beta < 1$.
Constant Absolute Risk Aversion (CARA): $u(x) = -e^{-\beta x}/\beta$, where $\beta > 0$.

C. Optimization Problem

Let u(x) be a utility function, and $\{X_n^h\}$ be the wealth process. The objective is to calculate

$$V^* = \sup_{h\in\mathcal{H}} \{E[u(X_N^h)]\} \tag{11}$$

and to find an admissible trading strategy h^* s.t. $E[u(X_N^{h^*})] = V^*$.

IV. Dynamic Programming

Define

$$U_n(x) = \sup_{h\in\mathcal{H}} E[u(X_N^h)|\mathcal{F}_n],\ n = 0, 1, ..., N. \tag{12}$$

From this definition $U_0 = V^*$ as defined in the optimization problem of Section III. The dynamic programming equation for the sequence $u_0, u_1, ..., u_N$ is

$$\begin{cases} U_n(x) \\ = \sup_{h_n} E[U_{n+1}(xe^{r\Delta t} + xe^{r\Delta t}h_n \cdot (e^{y_{n+1}} - 1))|\mathcal{F}_n], \\ U_N(x) = u(x). \end{cases} \tag{13}$$

In what follows we derive a numerical scheme for the solution of these dynamic programming equations and approximations for the optimal strategies for the original optimization problem.

For each $\eta_n \in \mathcal{F}_n, n = 0, 1, ...N-1$,define

$$A_{n-1,\eta_n} = E_{n-1}[\eta_n(e^{y_n} - 1) \cdot (e^{y_n} - 1)'].$$

Proposition 4.1: Let $u(x) = x^\alpha/\alpha, 0 \neq \alpha < 1$. Let

$\lambda_N := 1$, $\eta_n := E_n[\prod_{k=n}^N \lambda_k]$,
$\lambda_{n-1} := (1 + \frac{1}{1-\alpha}(e^{y_n} - 1)' A_{n-1,\eta_n}^{-1} E_{n-1}[(e^{y_n} - 1)\eta_n])^\alpha$,

n=1,..,N. Then

(i) $\eta_n \in \mathcal{F}_n$ is bounded,

(ii)A_{n-1,η_n} is invertible and there exists a constant C, such that $\Delta t\|A_{n-1,\eta_n}^{-1}\| < C$ for all n=1,2,...N.

(iii)$\|E_{n-1}[\eta_n(e^{y_n} - 1)]\| = o(\sqrt{\Delta t})$.

The proof of this proposition follows from Taksar and Zeng[20] Theorem D.1 (iii), (ii) and Theorem D.2 in Appendix D.

For each $\eta_n \in \mathcal{F}_n, n = 0, 1, ...N-1$, define

$$\begin{aligned} &W_n(x_n, h_n) := E[\eta_{n+1}u(x_n^{h_n})|\mathcal{F}_n] \\ &= E[\eta_{n+1}u(x_ne^{r\Delta t} + x_ne^{r\Delta t}h_n \cdot (e^{y_{n+1}} - 1))|\mathcal{F}_n], \\ &V_n(x_n, h_n) \\ &:= E[\eta_{n+1}(u(x_ne^{r\Delta t}) + u'(x_ne^{r\Delta t})x_ne^{r\Delta t}h_n \cdot (e^{y_{n+1}} - 1) \\ &+\tfrac{1}{2}u''(x_ne^{r\Delta t})(x_ne^{r\Delta t}h_n \cdot (e^{y_{n+1}} - 1))^2)|\mathcal{F}_n], \end{aligned} \tag{14}$$

Assume for any $x_{n-1} \geq 0$, there exists $h_{n-1}^*, h_{n-1}^{**} \in [0, 1]^m \subset \mathcal{R}^m, s.t.$

$$W_{n-1}(x_{n-1}, h_{n-1}^*) = \sup_{h_{n-1}} W_{n-1}(x_{n-1}, h_{n-1}), \tag{15}$$

and

$$V_{n-1}(x_{n-1}, h_{n-1}^{**}) = \sup_{h_{n-1}} V_{n-1}(x_{n-1}, h_{n-1}). \tag{16}$$

As is seen from (14), V_n is obtained from W_n by taking a Taylor expansion up to the second term. Thus the function V_n approximates W_n when Δt is small(and as a result y_n is close to 0).

Lemma 4.2: Let $u(x) = x^\alpha/\alpha, 0 \neq \alpha < 1$. Let η_n be defined as in Prop. 4.1. Then

$$|W_{n-1}(x_{n-1}, h_{n-1}^{**}) - W_{n-1}(x_{n-1}, h_{n-1}^*)| = x^\alpha o(\Delta t), \tag{17}$$

where

$$h_{n-1}^{**} = \frac{1}{1-\alpha} A_{n-1,\eta_n}^{-1} \cdot E_{n-1}[\eta_n(e^{y_n} - 1)]. \tag{18}$$

Using the same route we can prove that Lemma 4.2 holds for any η_n satisfying the three conditions in Proposition 1. The motivation to define W_n and V_n can be seen as follows. Let $n = N-1$. Then in (13) we have

$$\begin{aligned} &U_{N-1}(x) \\ &= \sup_{h_{N-1}} E[u(xe^{r\Delta t} + xe^{r\Delta t}h_{N-1} \cdot (e^{y_N} - 1))|\mathcal{F}_{N-1}]. \end{aligned}$$

Clearly, $U_{N-1}(x_{N-1})$ coincides with $W_{N-1}(x_{N-1}, h_{N-1}^*)$ defined in (15) with $\eta_N = 1$. Below, we write $\mathcal{E}_{n-1}(x_{n-1}) := W_{n-1}(x_{n-1}, h_{n-1}^*) - W_{n-1}(x_{n-1}, h_{n-1}^{**})$ for convenience.

In other words,

$$\begin{aligned} &U_{N-1}(x_{N-1}) = W_{N-1}(x_{N-1}, h_{N-1}^*) \\ &= W_{N-1}(x_{N-1}, h_{N-1}^{**}) + \mathcal{E}_{N-1} \\ &\approx W_{N-1}(x_{N-1}, h_{N-1}^{**}) \\ &= E_{N-1}[u(x_{N-1}e^{r\Delta t}(1 + h_{N-1}^{**} \cdot (e^{y_N} - 1)))] \end{aligned} \tag{19}$$

since $\mathcal{E}_{N-1}$ is small by Lemma 4.2.

As a result, one can find an approximation for the optimal strategy in a recursive way. Moreover the expected utility of the terminal wealth associated with $h^{**} = \{h_k^{**}\}_{k=0}^{N-1}$ is an approximation for the value function V^* in the optimization problem. The next two theorems show that in a limit as Δt tends to zero the value of V^{**} converges to that of V^*. Thus h_n^{**} can serve as an approximation for the optimal strategy.

Hence Lemma 4.2 shows that the difference between the wealth associated with optimal portfolio h^* and the portfolio h^{**} is small when Δt is small.

Theorem 4.3: Let $u(x) = \frac{x^\alpha}{\alpha}, 0 \neq \alpha < 1$. Define

$$\begin{aligned} &\lambda_N := 1, \eta_N = 1, \\ &h_{n-1}^{**} = \tfrac{1}{1-\alpha} A_{n-1,\eta_n}^{-1} \cdot E_{n-1}[(e^{y_n}-1)\eta_n], \\ &\lambda_{n-1} := e^{r\Delta t \alpha}(1 + \tfrac{1}{1-\alpha} h_{n-1}^{**} \cdot (e^{y_n}-1))^\alpha, \\ &\eta_n := E_n[\textstyle\prod_{i=n}^N \lambda_i], 1 \leq n \leq N. \end{aligned} \tag{20}$$

Then

$$|U_n^{**}(x_n) - U_n(x_n)| = u(x_n)o(1),$$

where

$U_n^{**}(x_n) := E_n[u(x_{n,N}^{**})]$, and

$x_{n,k}^{**} := x_n e^{(k-n)r\Delta t} \cdot \prod_{i=n}^{k-1}(1 + h_i^{**} \cdot (e^{y_{i+1}} - 1))$.

The case of a logarithmic utility function can be treated as the same way as the power utility function. The same results hold, although with a higher rate of convergence.

Lemma 4.4: Let $u(x) = log(x)$, choose $\eta_n = 1, n = 1, 2, ..., N$. Then we have

$$|W_{n-1}(x_{n-1}, h_{n-1}^{**}) - W_{n-1}(x_{n-1}, h_{n-1}^*)| = O(\Delta t)^2,$$

where

$$h_{n-1}^{**} = A_{n-1,1}^{-1} \cdot E_{n-1}[e^{y_n} - 1].$$

The proof is the same as that of Lemma 4.2. Moreover when we repeat the proof, we see that the resulting convergence rate is $O(\Delta t)^2$, which is higher that $O(\Delta t)$ obtained in the case of the power utility function.

Theorem 4.5: Let $u(x) = log(x)$, choose $\eta_n = 1, n = 1, 2, ..., N$.. Then

$$|U_n(x_n) - U_n^{**}(x_n)| = O(\Delta t),$$

where

$U_n^{**}(x_n) := E_n[u(x_{n,N}^{**})]$,

$x_{n,k}^{**} := x_n e^{(k-n)r\Delta t} \cdot \prod_{i=n}^{k-1}(1 + h_i^{**} \cdot (e^{y_{i+1}} - 1))$,

and h_i^{**} is defined in Lemma 4.4.

Remark There are particular cases depending on the structure of the transition matrix P, when we have the convergence rate $O(\Delta t)$ as above even for the power utility function. One of those cases is when the transition matrix has identical columns. We also have another case in [20] Appendix D, Theorem D.3 (ii) when the convergence rate is of a higher rate of $O(\Delta t)$.

Remark Let $u(x) = \frac{1}{\gamma} e^{-x\gamma}$. Define

$$\begin{aligned} &\eta_N := 1, \\ &\eta_{n-1} := e^{-\gamma h_{n-1}^{**}(e^{y_n}-1)}, \\ &h_{n-1}^{**} := A_{n-1,\eta_n}^{-1} \cdot E_{n-1}[(e^{y_n}-1)\eta_n](\tfrac{1}{\gamma x_{n-1}} e^{-r(N-n)\Delta t}), \\ &U_{n-1}^{**} \\ &:= u(x_{n-1} e^{r(N-n+1)\Delta t} \textstyle\prod_{k=n-1}^{N-1}(1 + h_i^{**} \cdot (e^{y_{i+1}} - 1))), \end{aligned}$$

where n=N,...,1. Then

$$|U_0(x_0) - U_0^{**}(x_0)| = o(1).$$

Remark From the definition (20) of η_n, we can see that η_n is the expected utility under strategy h^{**}, given $\mathcal{F}_n$, and given $x_n = 1$. That is

$$\eta_n = E[u(X_N^{h^{**}})|\mathcal{F}_n, x_n = 1].$$

V. SIMULATIONS

Generally, as a result it is not easy to compute the approximate optimal strategy $\{h_n^{**}\}_{n=0}^{N-1}$ in (18) in the case of power utility function. However we can get an estimation using a simplified strategy:

$$\bar{h}_n^{**} := -\frac{1}{1-\alpha} A_{n-1,1}^{-1} \cdot E_{n-1}[(e^{y_n} - 1)], n = 0, 1, ..., N-1,$$

From the definition of V^*, we see that the expected utility $E[u(x_N^{\bar{h}^{**}})]$ associated with $\bar{h}^{**}$ is a lower bound for V^*.

There are cases, however, where (18) is relatively easy to evaluate. For example, if the transition matrix has identical columns, then the conditional probability $Pr(Y_n = e_k|\mathcal{F}_n), n \geq 1$, would be a constant regardless of n, k. Thus η_n is a constant, and it can be excluded from the expression for h_{n-1}^{**} (18). Therefore the strategy h^{**} is the same as $\bar{h}^{**}$.

We apply the results of Section IV to obtain an applicable representation of the strategy for the case of $m = 1$,

$$\bar{h}_{n-1}^{**} = \frac{\sum_{k=1}^d Pr(Y_{n-1} = e_k|\mathcal{F}_{n-1})(\mu(e_k) - r)}{(1-\alpha)\sum_{k=1}^d Pr(Y_{n-1} = e_k|\mathcal{F}_{n-1})\hat{\sigma}(e_k)^2} + O(\Delta t), \tag{21}$$

where n=1,2,...N. The simulations in this section deal with (21). We use (9) to calculate $Pr(Y_n = e_k|\mathcal{F}_n)$ recursively.

Comparing the strategies (21) with Merton strategies, we can see that in our case the constant drift or the constant volatility in the expression for Merton strategies are replaced by linear combinations of the drifts or of the volatilities corresponding to different states of the Markov chain. The weights of the linear combinations are probabilities that the Markov chain is in those states. We divide the time interval [0,T] into N parts, and assume the transition of the Markov chain occurs only at those points of time. Hence we have the Merton model on each interval. The consequence is that we might obtain a solution directly like (21) in the case of the logarithmic utility function. However, it is not true for the power or the exponential utility functions.

To illustrate this point, let us assume that the transition matrix does not have identical columns. Then in (18) we can not choose $\eta_n = 1$ as in the case of logarithmic utility function, nor can we simplify the expression by cancelling η_n as in the case, when all the columns of the transition matrix are identical. A representation as simple as (21) can not be obtained. We have to employ the Monte Carlo method to calculate the portfolio (18). However, we may use (21) to get a lower bound for V^*.

In our simulations, W_0 stands for the initial wealth. The default value of W_0 is 1. P denotes the transition matrix. The interest rate r is equal to 0.06. The time horizon T is 1, and it is divided in $N = 1000$ parts, i.e. $\Delta t = 10^{-3}$.

We compare our optimal strategy with the Merton strategy. Since the Markov chain has several states, we use the Merton strategy replacing the drift and the volatility in it with those obtained from taking average of the drift and volatility respectively over different states of the Markov chain. The resulting Merton's strategy is

$$h_{n-1} = \frac{\bar{\mu} - r}{(1-\alpha)\bar{\sigma}^2}, n = 1, ..., N, \tag{22}$$

where $\bar{\mu} = \sum_i \mu(e_i)/d, \bar{\sigma} = \sum_i \hat{\sigma}(e_i)/d$. One may think of using $\sum_i \sigma(e_i)^2/d$ instead of $\bar{\sigma}^2$ in the formula. However, our simulation shows that it dose not provide a better result.

We also compare our optimal strategy with the buy-and-hold strategy which is denoted as "b/h". The buy-and-hold strategy means to buy the stock using all cash available at the beginning , then hold the stock until the end. We generate the wealth process 1000 times and calculate the average of the utilities from the terminal wealth.

Table 1 lists the result for a Makov chain which has a transition matrix with identical columns. For the power utility function, we can obtain that our optimal strategy is a constant 0.1133 while the Merton strategy is also a constant 0.3636 different from ours as seen from (21). Table I &II show that our optimal strategy on average gives better utilities with smaller standard deviations for both the logarithm and the power law utility functions. The last lines of Table I&II show the number of simulations in which our optimal strategy generates a better utility than the Merton strategy or the "b/h" strategy. Note that in the case of the power utility function, even though our optimal strategy only generates 487 better than the Merton strategy among 1000 simulations, the average (-0.2748) is still significantly higher than the one (-0.2974) generated by the Merton strategy, and the standard deviation (0.0380) is also significantly less than 0.1348.

In Table III&IV, we use the same parameters except the transition matrix is replaced by a matrix with non-identical columns. In this case for the power utility function, the Merton strategy is still a constant (h=0.3636). However, our optimal strategies h^{**} varies from 0.1294 to 0.3324 with a mean 0.1450 while it is a constant 0.1133 in the previous case. The result are similar to those of Table I& II though.

The average utility may vary slightly if more wealth processes are generated in the simulation. However, we always find our optimal strategy generates on average the best utilities. The results in Table V&VI for 5000 and 10000 simulations show that although the utilities vary slightly, our optimal strategy still has the best performance among three strategies.

TABLE I

$\mu = [0.1, 0.9]', \hat{\sigma} = [0.4, 0.7]', P = [0.95, 0.95; 0.05, 0.05], \Delta t = 10^{-3}.1000\ simulations.$

u(x)	$log(x)$		
strategy	opt	$Merton$	b/h
av. u(x)	0.0772	0.0041	0.0498
med u(x)	0.0781	0.0084	0.0535
std u(x)	0.1871	0.5592	0.4039
opt better		574	548

TABLE II

$\mu = [0.1, 0.9]', \hat{\sigma} = [0.4, 0.7]', P = [0.95, 0.95; 0.05, 0.05], \Delta t = 10^{-3}.1000\ simulations.$

u(x)	$-x^{-3}/3$		
strategy	opt	$Merton$	b/h
av. u(x)	-0.2748	-0.2974	-0.6352
med u(x)	-0.2719	-0.2687	-0.2839
std u(x)	0.0380	0.1348	0.9696
opt better		487	515

For one more example, we choose the same parameters as in the example 1 of Sass and Haussmann [18]. The results for this example are listed in Table VII&VIII. Table IX&X are copies of Table 2 of Sass and Haussmann [18]. One can see that our optimal strategy generates the average utilities(0.3969 for the logarithm, -0.1128 for the power) very close to theirs(0.399 for the logarithm, -0.121 for the power). It is not surprising because our model can be viewed as an extension of theirs in an approximate sense. Therefore similar results are expected when the same parameters are employed.

Finally, we provide standard deviations in Table I&II and Table III&IV. One can see that the standard deviation associated with the optimal strategy is always the smallest one.

TABLE III

$\mu = [0.1, 0.9]', \hat{\sigma} = [0.4, 0.7]', P = [0.95, 0.5; 0.05, 0.5], \Delta t = 10^{-3}.1000\ simulations.$

u(x)	$log(x)$		
Strategy	opt	$Merton$	b/h
av. u(x)	0.0946	0.0360	0.0760
med u(x)	0.0930	0.0177	0.0644
std u(x)	0.2703	0.6012	0.4441
opt better		565	552

TABLE IV

$\mu = [0.1, 0.9]', \hat{\sigma} = [0.4, 0.7]', P = [0.95, 0.5; 0.05, 0.5], \Delta t = 10^{-3}.1000\ simulations.$

u(x)	$-x^{-3}/3$		
Strategy	opt	$Merton$	b/h
av. u(x)	-0.2715	-0.2877	-0.6236
med u(x)	-0.2664	-0.2587	-0.2747
std u(x)	0.0557	0.1450	1.1093
opt better		470	512

TABLE V

5000, 10000 SIMULATIONS

u(x)	$log(x)$		
Strategy	*opt*	$Merton$	b/h
5000 sim.	0.0923	0.0338	0.0743
10000 sim.	0.0966	0.0422	0.0805

TABLE VI

5000, 10000 SIMULATIONS

u(x)	$-x^{-3}/3$		
Strategy	*opt*	$Merton$	b/h
5000 sim.	−0.2719	− 0.2883	− 0.6541
10000 sim.	−0.2709	− 0.2857	− 0.6284

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TABLE VII

$\mu = [0.8, -0.4]', \hat{\sigma} = [0.2, 0.2]', Q = [-30, 24; 30, -24], P = e^{Q/250}, \Delta t = 1/250.$ 500 $simulations.$

u(x)	$log(x)$		
Strategy	*opt*	$Merton$	b/h
av. u(x)	0.3969	0.1125	0.1205
med u(x)	0.2965	0.1033	0.1186
opt better		319	313

TABLE VIII

$\mu = [0.8, -0.4]', \hat{\sigma} = [0.2, 0.2]', Q = [-30, 24; 30, -24], P = e^{Q/250}, \Delta t = 1/250.$ 500 $simulations.$

u(x)	$-x^{-5}/5$		
Strategy	*opt*	$Merton$	b/h
av. u(x)	−0.1128	− 0.1525	− 0.2180
med u(x)	−0.0902	− 0.1218	− 0.1105
opt better		338	288

TABLE IX

SASS AND HAUSSMANN'S TABLE 2, KNOWN PARAMETERS

u(x)	$log(x)$		
Strategy	*opt*	$Merton$	b/h
av. u(x)	0.399	0.136	0.118
med u(x)	0.305	0.150	0.126
opt better		296	288

TABLE X

SASS AND HAUSSMANN'S TABLE 2, KNOWN PARAMETERS

u(x)	$-x^{-5}/5$		
Strategy	*opt*	$Merton$	b/h
av. u(x)	−0.121	− 0.141	− 0.214
med u(x)	−0.091	− 0.131	− 0.107
opt better		359	292

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