Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

On the curvature of the trajectory manifold of nonlinear systems

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Abstract— **This paper represents a preliminary contribution in the direction of characterizing geometric properties of the trajectory manifold of nonlinear systems. We introduce the notion of curvature of the trajectory manifold and define it by means of a nonlinear quadratic optimal control problem. The quadratic cost can be viewed as a weighted L_2 norm induced by a suitable inner product that provides a notion of orthogonality. The curvature at a given trajectory is defined in terms of the curves orthogonal to the tangent space at the given trajectory. We characterize the set of orthogonal curves. We show that it is a topological complement of the tangent space. We provide numerical techniques to compute orthogonal curves and to compute a lower bound of the curvature. We test these techniques on the inverted pendulum example.**

I. INTRODUCTION

A fundamental contribution to the analysis of dynamical systems and the design of control strategies has come from optimal control theory. Optimal control allows to study the trajectories of control systems in order to optimize a desired cost and possibly taking into account constraints on the state and the input of the system. Also, the well known (and quite effective) Receding Horizon (or Model Predictive) control strategy, relies on the solution of an optimal control problem. An important issue in optimal control that arises both in studying trajectories and in designing a Receding Horizon scheme, is the following. Given a desired curve, find the trajectory of the given system that is "closest" (according to given criteria) to the desired curve. It is well known that for linear systems, under reasonable assumptions, it is possible to find a unique closest trajectory for any desired curve (that has enough regularity). This is due to the linearity of the constraint enforced by the dynamics. For nonlinear systems, depending on the nature of the nonlinearity, existence and uniqueness problems may arise when the desired curve is too far from the space of trajectories. In this paper we want to define a parameter that provides information on how different the desired curves can be chosen while preserving conditions that guarantee i) existence and uniqueness of a local (at least) minimizer and (possibly) ii) convergence of numerical techniques to compute it. Equivalently, we want to provide a parameter that measures how much the nonlinear dynamics departs from its linear approximation at a given trajectory.

In the last fifty years important contributions have been done from a theoretical point of view in order to provide conditions for existence and characterization of the minimizer and from a numerical point of view to provide tools for solving optimal control problems. A detailed analysis of conditions for existence of optimal trajectories may be found, e.g., in [1] and [2]. In [3] and [4] the important role played by strong positive definiteness in the minimization of quadratic functional was emphasized.

In this paper we introduce for the first time, at the best of our knowledge, the idea of curvature of the trajectory manifold of a nonlinear system. That is, a measure of how much the space of bounded (state-input) trajectories departs from its linear approximation at a given trajectory. In analogy with a curve in a finite dimensional space, we provide a well suited definition of curvature for the trajectory manifold by use of a (nonlinear) quadratic optimal control problem. In order to define and estimate the curvature of the trajectory manifold, we introduce a suitable notion of inner product (namely a weighted L_2 product), which induces a notion of orthogonality for "state-input" curves. We characterize the space of curves orthogonal (in L_2 sense) to the tangent space at a point (namely a trajectory) on the trajectory manifold. Also, we prove that such space is a closed subspace with respect to the L_∞ norm and, therefore, an orthogonal complement for the tangent space (thus showing that the tangent space splits the space).

We provide numerical techniques to compute curves in the orthogonal complement. These techniques are based on a projection operator based Newton method for the solution of nonlinear optimal control problems [5]. Finally, we show how to provide a lower bound for the curvature along one direction in the orthogonal complement. We apply the numerical techniques to the example of an inverted pendulum.

II. CURVATURE OF AN EMBEDDED MANIFOLD IN $\mathbb{R}^n$

Consider $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ as an inner product space with associated norm and let $\mathcal{M} \subset \mathbb{R}^n$ be a C^2 manifold of dimension $k < n$. A sphere $S \subset \mathbb{R}^n$ is tangent to $\mathcal{M}$ at $\xi \in \mathcal{M}$ if $S \cap (\xi + T_\xi \mathcal{M}) = \{\xi\}$ so that $T_\xi \mathcal{M} \subset T_\xi S$. Clearly the center of each tangent sphere lies on the translated subspace $\xi + \mathcal{N}_\xi$ where the normal space $\mathcal{N}_\xi := (T_\xi \mathcal{M})^\perp$ is the orthogonal complement of the tangent space $T_\xi \mathcal{M}$. Thus the sphere $S_r(\eta) := \{x \in \mathbb{R}^n : \|x - \eta\| = r\}$ is tangent

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to $\mathcal{M}$ at ξ iff $\eta - \xi \in \mathcal{N}_\xi$ and $\|\eta - \xi\| = r$ so that every tangent sphere at $\xi \in \mathcal{M}$ is of the form $S_r(\xi + r\gamma)$ for some $\gamma \in \mathcal{N}_\xi$, $\|\gamma\| = 1$, and $r > 0$.

Since $\mathcal{M}$ is a C^2 manifold, there is a C^2 mapping $\psi : \mathbb{R}^k \to \mathbb{R}^n$ with $\xi = \psi(0)$ providing local coordinates on a neighborhood of ξ in $\mathcal{M}$ so that $D\psi(0)$ is full rank and $T_\xi\mathcal{M} = D\psi(0) \cdot \mathbb{R}^k$ is of dimension k. In particular, there is an open neighborhood U of $0 \in \mathbb{R}^k$ such that $\psi(\cdot)$ is one-to-one on U and $D\psi(\alpha)$ is full rank for each $\alpha \in U$.

Now, for each $\gamma \in \mathcal{N}_\xi$, $\|\gamma\| = 1$, we can imagine that there is a range of radii $[0, r_0]$ for which ξ is, locally, the unique nearest point (in $\mathcal{M}$) to each of the sphere centers $\xi + r\gamma$, $r \in [0, r_0]$. To this end, we have

Proposition 2.1: Given $\xi \in \mathcal{M}$ and $\gamma \in \mathcal{N}_\xi$, $\|\gamma\| = 1$, there is an $\epsilon > 0$ and an $r_0 > 0$ such that 0 is the unique minimizer of

$$\min_{\alpha \in B_\epsilon \subset \mathbb{R}^k} h(\alpha; r)$$

for every $r \in [0, r_0]$ where

$$h(\alpha; r) := \|(\xi + r\gamma) - \psi(\alpha)\|^2/2\,.$$

That is, the minimum norm projection of $\xi + r\gamma$ onto $\psi(B_\epsilon) \subset \mathcal{M}$ is $\xi = \psi(0)$ for each $r \in [0, r_0]$.

Proof: Computing, we find

$$Dh(\alpha; r) \cdot \beta = \langle (\xi + r\gamma) - \psi(\alpha),\, -D\psi(\alpha) \cdot \beta \rangle$$

and

$$\begin{aligned} D^2h(\alpha; r) \cdot (\beta, \beta) &= \|D\psi(\alpha) \cdot \beta\|^2 \\ &\quad - \langle (\xi + r\gamma) - \psi(\alpha),\, D^2\psi(\alpha) \cdot (\beta, \beta) \rangle\,. \end{aligned}$$

We see that $\alpha = 0$ is a stationary point of $h(\cdot; r)$ for *all* $r > 0$ since

$$Dh(0; r) \cdot \beta = -r\langle \gamma,\, D\psi(0) \cdot \beta \rangle = 0$$

for all $\beta \in \mathbb{R}^k$. To ensure that $\alpha = 0$ is an isolated local minimum of $h(\cdot; r)$, we examine the second derivative

$$\begin{aligned} &D^2h(0; r) \cdot (\beta, \beta) \\ &\quad = \|D\psi(0) \cdot \beta\|^2 - r\langle \gamma,\, D^2\psi(0) \cdot (\beta, \beta) \rangle \end{aligned}$$

for positive definiteness. Since the first term is positive definite, it is clear that there is an $r_0 > 0$ such that $D^2h(0; r)$ is positive definite for every $r \in [0, r_0]$. For instance, we may take $r_0 = b_1/2b_2$ where $b_1 = \min_{\|\beta\|=1} \|D\psi(0) \cdot \beta\|^2$ and $b_2 = \max_{\|\beta\|=1} \langle \gamma,\, D^2\psi(0) \cdot (\beta, \beta) \rangle$ when $b_2 > 0$ and any $r_0 > 0$ otherwise. Noting that the minimum eigenvalue of $D^2h(0; r)$ depends continuously on r, we may choose r_0 to be any positive $r_0 < r_1$ where r_1 is the first positive zero of the function $r \mapsto \lambda_{\min}(D^2h(0; r))$ when $b_2 > 0$ and $+\infty$ when $b_2 \le 0$.

To see that there is an $\epsilon > 0$ such that $\alpha = 0$ is the unique minimizer for all $r \in [0, r_0]$, note that

$$h(\alpha; r) = h(0; r) + \tfrac{1}{2}D^2h(0; r) \cdot (\alpha, \alpha) + R(\alpha, r) \cdot (\alpha, \alpha)$$

where the bilinear operator $R(\alpha, r)$ is continuous in α and r and $\|R(\alpha, r)\| \to 0$ as $\alpha \to 0$ for *every* $r \ge 0$. It follows that the continuous function $\alpha \mapsto \max_{r \in [0, r_0]} \|R(\alpha, r)\|$ goes to zero as $\alpha \to 0$ so that there is an $\epsilon > 0$ such that $\|R(\alpha, r)\| \le \min_{s \in [0, r_0]} \lambda_{\min}(D^2h(0; s))/4$ for all $\|\alpha\| \le \epsilon$ and all $r \in [0, r_0]$. ■

Definition 2.2: Given $\xi \in \mathcal{M}$ and $\gamma \in \mathcal{N}_\xi$, $\|\gamma\| = 1$, the *radius of curvature of $\mathcal{M}$ at ξ in the direction γ* is

$$\rho(\xi, \gamma) = \min\{r > 0 : \lambda_{\min}(D^2h(0; r)) = 0\}\,,$$

where $\psi(\cdot)$ is a local parametrization with $\xi = \psi(0)$ and $h(\alpha; r)$ is as defined above. By definition, $\rho(\xi, \gamma) = +\infty$ if the positive radius zero set is empty.

It is easy to verify that the definition of $\rho(\xi, \gamma)$ is independent of the choice of parametrization $\psi(\cdot)$.

From the proof of Proposition 2.1, we see that, for each $r_0 \in (0, \rho(\xi, \gamma))$, there is an $\epsilon = \epsilon(r_0) > 0$ such that $r_0 = \|(\xi + r_0\gamma) - \xi\| < \|(\xi + r_0\gamma) - \zeta\|$ for all $\zeta \in \psi(B_\epsilon) \setminus \{\xi\}$. This helps support the notion that the sphere $S_{\rho(\xi,\gamma)}(\xi + \rho(\xi, \gamma)\gamma)$ is a generalization of the osculating circle to a planar curve. Furthermore, we have

Proposition 2.3: If $r > \rho(\xi, \gamma)$ then the stationary point $\alpha = 0$ is *not* a local minimizer of $h(\cdot; r)$.

Proof: Let β_0 be the unit eigenvector associated with $0 = \lambda_{\min}(D^2h(0; \rho(\xi, \gamma)))$ and note that $D^2h(0; r) \cdot (\beta_0, \beta_0) < 0$ for every $r > \rho(\xi, \gamma)$. It follows that, for each $r > \rho(\xi, \gamma)$, there is an $\epsilon_0 > 0$ such that $h(\epsilon\beta_0; r) < h(0; r)$ for all $\epsilon \in (0, \epsilon_0)$. ■

Definition 2.4: The *radius of curvature of $\mathcal{M}$ at ξ* is

$$\rho(\xi) = \min_{\gamma \in \mathcal{N}_\xi, \|\gamma\|=1} \rho(\xi, \gamma)\,.$$

The *curvature of $\mathcal{M}$ at ξ* is

$$\sigma(\xi) = 1/\rho(\xi)\,.$$

Proposition 2.5: For each $\xi \in \mathcal{M}$, there is a $r_1 > 0$ such that $\rho(\xi) \ge r_1 > 0$.

Proof: Noting that $|\langle \gamma,\, D^2\psi(0) \cdot (\beta, \beta) \rangle| \le \|D^2\psi(0)\| =: b_3$ for all $\|\gamma\| = 1$ and $\|\beta\| = 1$, we see that $\rho(\xi) \ge b_1/2b_3 =: r_1$ where $b_1 = \lambda_{\min}(D^2h(0; 0))$ (the b_1 value as above). ■

III. The Trajectory Manifold and the Projection Operator

Here, we recall some properties of the space of trajectories proved in [8] by use of the projection operator approach. The approach is based on the idea that a feedback system trajectory tracking defines a continuous nonlinear projection operator mapping curves into trajectories.

Suppose that $\xi = (\alpha(t), \mu(t))$, $t \ge 0$, is a bounded curve and let $\eta = (x(t), u(t))$, $t \ge 0$, be the trajectory determined by the nonlinear feedback system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \quad x(0) = x_0 \\ u(t) &= \mu(t) + K(t)(\alpha(t) - x(t)). \end{aligned}$$

Now, suppose that ξ_0 is a trajectory of f and that K is bounded and such that the above feedback exponentially stabilizes ξ_0. Then the feedback system defines a continuous nonlinear *projection operator*

$$\mathcal{P} : \xi = (\alpha(\cdot), \mu(\cdot)) \mapsto \eta = (x(\cdot), u(\cdot))$$

which is $\mathcal{C}^r$ (whenever f is) on an L_∞ neighborhood of ξ_0. Restricted to a finite interval $[0,T]$, the resulting operator continues to be C^r and, while stability is no longer an issue, one would like to choose a bounded K that gives $\mathcal{P}$ a reasonable modulus of continuity around ξ_0.

We let X denote the closed subspace of $L_\infty^{n+m}[0,T]$ of curves $\zeta=(\beta(\cdot),\nu(\cdot))$ with continuous $\beta(\cdot)$, $\beta(0)=0$, and bounded $\nu(\cdot)$. Equipped with the norm $\|\zeta\|_X=\|\zeta\|_{L_\infty}$, X is a Banach space. Trajectories of the nonlinear system f through x_0 belong to the affine space $\widetilde{X}:=(x_0,0)+X$.

We denote $\mathcal{T}$ the set of bounded trajectories of f on $[0,T]$.

The derivative of the projection operator, $\zeta\mapsto D\mathcal{P}(\xi)\cdot\zeta$, is given by the standard linearization

$$\begin{aligned}&\dot z(t)=A(\eta(t))z(t)+B(\eta(t))v(t), z(0)=\beta(0),\\&v(t)=\nu(t)+K(t)[\beta(t)-z(t)]\ .\end{aligned}$$

where $D\mathcal{P}(\xi)\cdot\zeta=(z(\cdot),v(\cdot))$, with $\zeta=(\beta(\cdot),\nu(\cdot))$, and $A(\eta(t))=f_x(x(t),u(t))$ and $B(\eta(t))=f_u(x(t),u(t))$. A key property of $D\mathcal{P}(\xi)$ is that it is a continuous linear projection operator. In [8] it was shown, using this property, that the projection operator $\mathcal{P}$ provides a convenient parametrization of the trajectories in the neighborhood of a given trajectory. Indeed, the tangent space $T_\xi\mathcal{T}$ (of bounded trajectories of the linearization of $\dot x=f(x,u)$ about $\xi\in\mathcal{T}$) can be used to parametrize *all* nearby trajectories. That is, given $\xi\in\mathcal{T}$, there is an $\epsilon>0$ such that, for each $\eta\in\mathcal{T}$ with $\|\eta-\xi\|<\epsilon$, there is a *unique* $\zeta\in T_\xi\mathcal{T}$ such that $\eta=\mathcal{P}(\xi+\zeta)$. Using this property, a $\mathcal{C}^r$ atlas of charts, indexed by trajectories $\xi\in\mathcal{T}$, is available, so that $\mathcal{T}$ can be shown to be a $\mathcal{C}^r$ Banach manifold.

The above property also allows one to prove the following important proposition.

Proposition 3.1 (The tangent space is a split space [8]): Let $\xi_0\in\mathcal{T}$. The tangent space $T_{\xi_0}\mathcal{T}$ is the split subspace of X given by

$$T_{\xi_0}\mathcal{T}=\{\zeta\in X:\zeta=D\mathcal{P}(\xi_0)\cdot\zeta\}$$

where $\mathcal{P}$ is any projection operator defined by a bounded (or, on $[0,\infty)$, $stabilizing$) K. That is, $T_{\xi_0}\mathcal{T}$ has a closed topological complement in L_∞. □

IV. SECOND ORDER SUFFICIENCY CONDITIONS FOR OPTIMAL CONTROL

Consider the following optimal control problem

$$\begin{aligned}&\text{minimize}\int_0^T l(\tau,x(\tau),u(\tau))d\tau+m(x(T))\\&\text{subj. to } \dot x(t)=f(x(t),u(t)),\quad x(0)=x_0\end{aligned}\tag{1}$$

over the class of essentially bounded measurable inputs, where $l(t,x,u)$, $m(x)$ and $f(x,u)$ are $\mathcal{C}^2$ in $x\in\mathbb{R}^n$ and $u\in\mathbb{R}^m$, and $l(t,x,u)$ is continuous in t. If $\bar\xi=(\bar x(\cdot),\bar u(\cdot))$ is a local minimizer of (1), then there is an absolutely continuous costate trajectory $\bar p(\cdot)$ such that

$$\begin{aligned}\dot{\bar x}(t)&=f(\bar x(t),\bar u(t)),\\\dot{\bar p}(t)&=-f_x(\bar x(t),\bar u(t))^T\bar p(t)-l_x(\bar x(t),\bar u(t))^T,\\0&=f_u(\bar x(t),\bar u(t))^T\bar p(t)+l_u(\bar x(t),\bar u(t))^T\\x(0)&=x_0,\ \bar p(T)=m_x(\bar x(T))^T\end{aligned}\tag{2}$$

for (almost all) $t\in[0,T]$. The first order optimality condition expressed in equations (2) can be conveniently expressed using the pre-Hamiltonian

$$H(t,x,p,u):=l(t,x,u)+p^Tf(x,u).$$

Remark 4.1: Note that, for the optimal control problem (1), there is *no* possibility of an abnormal extremal since the only constraints present are the initial condition and the dynamics. This can be easily shown using a projection operator calculation.

Next, we recall the second order sufficiency condition (SSC) that ensures that the trajecotry $\bar\xi$ is an isolated local minimizer. For $\zeta=(z(\cdot),v(\cdot))$, let

$$q(\bar\xi)\cdot(\zeta,\zeta)=\int_0^T\begin{bmatrix}z(\tau)\\v(\tau)\end{bmatrix}^T\begin{bmatrix}H_{xx}(\tau)&H_{xu}(\tau)\\H_{ux}(\tau)&H_{uu}(\tau)\end{bmatrix}^T\begin{bmatrix}z(\tau)\\v(\tau)\end{bmatrix}d\tau+\\ z(T)^Tm_{xx}(T)z(T)$$

be the quadratic form describing the second variation of the Lagrangian. Let $\mathcal{L}_{\bar\xi}\subset L_\infty^{n+m}[0,T]$ denote the subspace of trajectories of the linearized dynamics

$$\dot z(t)=f_x(\bar x(t),\bar u(t))z(t)+f_u(\bar x(t),\bar u(t))v(t),\ z(0)=0$$

for $v\in L_\infty^m[0,T]$. Also, let $h(\xi)=\int_0^T l(\tau,x(\tau),u(\tau))d\tau+m(x(T))$ for $\xi=(x(\cdot),u(\cdot))$.

We have the following second order sufficiency condition (SSC) [6], [3], [7].

Theorem 4.2 (SSC): Let $\bar\xi$ be a stationary trajectory for (1) with corresponding costate trajectory $\bar p(\cdot)$ and suppose that there is an $r_0>0$ such that $H_{uu}(t)\ge r_0I$ for $t\in[0,T]$. If $q(\bar\xi)\cdot(\zeta,\zeta)>0$ for all $\zeta\in\mathcal{L}_{\bar\xi}$, then $\bar\xi$ is an isolated local minimum of (1). That is, there is an $\epsilon>0$ such that $h(\xi)>h(\bar\xi)$ for any trajectory $\xi\neq\bar\xi$ with $\|\xi-\bar\xi\|_{L_\infty}<\epsilon$. □

The following theorem is useful, e.g., for numerical computations, giving an equivalent condition for positive definiteness of q. We denote $\widetilde{A}(t)=A(t)-B(t)H_{uu}(t)^{-1}H_{xu}(t)^T$ and $\widetilde{Q}(t)=H_{xx}(t)-B(t)H_{uu}(t)^{-1}H_{xu}(t)^T$.

Theorem 4.3 (SSC and Riccati equation [7]): The following statements are equivalent.

(i) $q>0$ on $\mathcal{L}_{\bar\xi}$

(ii) the Riccati equation

$$\begin{aligned}\dot P+&\widetilde{A}(t)^TP+P\widetilde{A}(t)-PB(t)R(t)^{-1}B(t)^TP\\&+\widetilde{Q}(t)=0,\ P(T)=P_1\end{aligned}$$

has a finite solution on $[0,T]$.

□

V. THE CURVATURE OF THE TRAJECTORY MANIFOLD

In this section we provide main definitions and preliminary results to answer the following question. How can we measure the extent to which the trajectory manifold of a nonlinear system departs from that of its linear approximation about a given trajectory?

A. A notion of orthogonality on X

Following the ideas in the finite dimensional case, we want to define the radius of curvature of the trajectory manifold relative to a suitable inner product and its associated norm.

We begin by considering the nonlinear quadratic optimal control problem

$$\begin{aligned} \text{minimize } & \frac{1}{2}\int_0^T \|x(t)-x_1(t)\|_Q^2 + \|u(t)-u_1(t)\|_R^2 \, dt \\ & + \frac{1}{2}\|x(T)-x_1(T)\|_{P_1}^2 \\ \text{subj. to } & \dot{x}(t) = f(x(t),u(t)), \quad x(0)=x_0, \end{aligned} \tag{3}$$

where Q, R, and P_1 are symmetric positive definite matrices.

The cost in (3) may be seen to be the square of a weighted L_2 norm of the curve $\xi - \xi_1 = (x(\cdot)-x_1(\cdot), u(\cdot)-u_1(\cdot))$. Given $\xi_1 = (\alpha_1(\cdot),\mu_1(\cdot))$ and $\xi_2 = (\alpha_2(\cdot),\mu_2(\cdot))$, define the inner product

$$\begin{aligned} \langle \xi_1, \xi_2 \rangle = & \int_0^T x_1(t)^T Q x_2(t) + u_1(t)^T R u_2(t) \, dt \\ & + x_1(T)^T P_1 x_2(T) \end{aligned} \tag{4}$$

and write $\|\xi\|_2^2 = \langle \xi, \xi \rangle$.

With this notation, we may rewrite the optimal control problem (3) as

$$\min_{\xi \in \mathcal{T}} \frac{1}{2}\|\xi - \xi_1\|_2^2. \tag{5}$$

Using this inner product, we define (analogously to the finite dimensional case) the space of curves orthogonal to the tangent space at the trajectory $\xi_0 \in \mathcal{T}$ by

$$\mathcal{N}_{\xi_0} = \{\gamma \in X \mid \langle \gamma, \zeta \rangle = 0, \text{ for all } \zeta \in T_{\xi_0}\mathcal{T}\}.$$

B. An orthogonal complement to the tangent space: main properties and parametrization

Next, we show that $\mathcal{N}_{\xi_0}$ is a closed subspace, providing an orthogonal splitting (with $T_{\xi_0}\mathcal{T}$) of X. Then, we show how to parameterize the orthogonal complement by means of state curves.

The first order necessary condition for optimality of (5) may be written as

$$\langle \xi_0 - \xi_1, \zeta \rangle = 0 \quad \forall \zeta \in T_{\xi_0}\mathcal{T}. \tag{6}$$

We will show that this is, in fact, a notion of orthogonality. That is, the vector $\gamma_1 = \xi_0 - \xi_1$ is orthogonal (in L_2 sense) to the tangent space $T_{\xi_0}\mathcal{T}$.

Using this condition we can characterize the whole set $\mathcal{N}_{\xi_0}$ of curves orthogonal (in L_2 sense) to the tangent space $T_{\xi_0}\mathcal{T}$.

We start considering the following linear quadratic minimization problem. Given $\gamma \in X$

$$\min_{\zeta \in T_{\xi_0}\mathcal{T}} \frac{1}{2}\|\zeta - \gamma\|_2^2. \tag{7}$$

Note that the above minimization problem is equivalent to the following one

$$\min_{\zeta \in T_{\xi_0}\mathcal{T}} -\langle \gamma, \zeta \rangle + \frac{1}{2}\langle \zeta, \zeta \rangle. \tag{8}$$

We consider the mapping $\Gamma_{\xi_0} : X \to T_{\xi_0}\mathcal{T} : \gamma \mapsto \gamma^\top$ defined as

$$\gamma^\top = \arg \min_{\zeta \in T_{\xi_0}\mathcal{T}} \frac{1}{2}\|\zeta - \gamma\|_2^2. \tag{9}$$

It is easy to see that Γ_{ξ_0} is a projection. In fact, $\|\cdot\|_2$ is strongly positive definite. Therefore for $\gamma \in T_{\xi_0}\mathcal{T}$ the minimum is attained at $\zeta = \gamma$, so that $\Gamma_{\xi_0}(\gamma) = \gamma$.

In the next lemma we recall an important result in linear quadratic optimal control on existence and uniqueness of the minimizer for the minimization problem in (7).

Lemma 5.1 (Existence of unique minimizer for LQ case): Let $\gamma \in X$ be arbitrary. The quadratic minimization problem in (7) has a unique (global) minimizer $\gamma^\top \in T_{\xi_0}\mathcal{T}$.

From the previous lemma it follows that the unique minimizer is also the only one satisfying the first order necessary condition. Therefore, the mapping Γ_{ξ_0} may be also defined implicitly as

$$\langle \gamma - \gamma^\top, \zeta \rangle = 0, \quad \forall \zeta \in T_{\xi_0}\mathcal{T}.$$

Before stating our next result, we need some more notation. Let us denote $\gamma = (\beta(\cdot),\nu(\cdot))$. Using standard notation from optimal control the problem (8) may be written as

$$\begin{aligned} \text{minimize } & \int_0^T \beta(\tau)^T Q z(\tau) + \nu(\tau)^T R v(\tau) + \frac{1}{2}\|z(\tau)\|_Q^2 \\ & + \frac{1}{2}\|v(\tau)\|_R^2 d\tau + \beta(T)^T P_1 z(T) + \frac{1}{2}\|z(T)\|_{P_1}^2 \\ \text{subj. to } & \dot{z}(t) = A(\xi_0(t))z(t) + B(\xi_0(t))v(t), \quad z(0)=0. \end{aligned}$$

The solution of (7) may be written as $\zeta = (z(\cdot), v(\cdot))$ with

$$\begin{aligned} \dot{z}(t) =& A(\xi_0(t))z(t) + B(\xi_0(t))v(t), \qquad z(0)=0, \\ v(t) =& -R(t)^{-1}B(\xi_0(t))^T P(t) z(t) \\ & - R(t)^{-1}B(\xi_0(t))^T r(t) + R(t)^{-1}R\nu(t). \end{aligned} \tag{10}$$

where P and r satisfy (suppressing the t and $\xi_0(t)$ arguments)

$$\begin{aligned} -\dot{P} =& A^T P + PA - PBR^{-1}B^T P + Q, \\ -\dot{r} =& (A - BR^{-1}B^T P)^T r - Q\beta(\cdot) + PBR^{-1}R\nu(t), \\ & P(T) = P_1, \; r(T) = -P_1\beta(T). \end{aligned} \tag{11}$$

In the next lemma we prove that Γ_{ξ_0} is a (continuous) linear projection.

Lemma 5.2 (An LQ problem defines a linear projection): The mapping $\Gamma_{\xi_0} : X \to T_{\xi_0}\mathcal{T}$ defined in (9) is a (continuous) linear projection. □

Proof: We have already seen that Γ_{ξ_0} is a projection. We need to show that it is a bounded linear mapping.

From equation (11), $r(\cdot)$ is a linear function of $(\beta(\cdot), \nu(\cdot))$. Furthermore, in the linear differential equation for $r(\cdot)$, the matrices $P(\cdot)$, $A(\cdot)$ and $B(\cdot)$ are bounded. Therefore, $r(\cdot)$ is bounded. This implies that $(z(\cdot), v(\cdot))$ is a bounded linear function of $(\beta(\cdot), \nu(\cdot))$. ■

Using the result in Lemma 5.2 we may prove the following result.

Proposition 5.3 (Orthogonal splitting of X): Let $\xi_0 \in \mathcal{T}$ be given. Any $\gamma \in X$ may be written as

$$\gamma = \gamma^\top + \gamma^\perp,$$

where $\gamma^\top = \Gamma_{\xi_0} \cdot \gamma \in T_{\xi_0}\mathcal{T}$ and $\gamma^\perp = (I - \Gamma_{\xi_0}) \cdot \gamma \in \mathcal{N}_{\xi_0}$.

Also, $\mathcal{N}_{\xi_0}$ is a closed linear subspace of X and, thus, a topological orthogonal complement of $T_{\xi_0}\mathcal{T}$ in X. Therefore, X may be written as the direct sum of $T_{\xi_0}\mathcal{T}$ and $\mathcal{N}_{\xi_0}$, that is,

$$X = T_{\xi_0}\mathcal{T} \oplus \mathcal{N}_{\xi_0}.$$

Proof: The proof follows by using the result in Lemma 5.2 combined with Propositions 2, 5 and 6 of Section 3.9 in [10]. ■

Remark 5.4: The above proposition proves that the space of bounded curves orthogonal to the tangent space is a closed set and in fact a topological complement of $T_{\xi_0}\mathcal{T}$. From Proposition 3.1 we knew that using a suitable projection operator we could show that the tangent space splits the space X and therefore it has a closed topological complement. Here we have the further property that $\mathcal{N}_{\xi_0}$ is an orthogonal complement. □

Next, we investigate how we can parameterize $\mathcal{N}_{\xi_0}$.

In order to get conditions on $\beta(\cdot)$ and $\nu(\cdot)$ for $\gamma^\perp$ to be in $\mathcal{N}_{\xi_0}$, we just impose $(z(\cdot), v(\cdot)) = 0$ in equation (10). We get

$$\nu^\perp(t) = -R^{-1}B(\xi_0(t))^T r^\perp(t)$$

with

$$-\dot{r}^\perp = A^T r^\perp - Q\beta^\perp, \quad r^\perp(T) = -P_1\beta^\perp(T). \tag{12}$$

That is, we get

$$\gamma^\perp = (\beta^\perp(\cdot), -R^{-1}B(\xi)^T r^\perp(\cdot)).$$

This means that $\gamma^\perp$ may be parameterized by (bounded) state curves.

Remark 5.5: The input portion of the curve $\gamma^\perp \in \mathcal{N}_{\xi_0}$ is obtained by integrating an open loop differential equation that may be unstable (as e.g. in the case of the inverted pendulum shown in the next section). Numerically it is not suitable to compute it by directly integrating equation (12). In order to compute it, as we do in the next section, we use numerically robust methods, based on the solution of a suitable optimal control problem. □

C. Definition of the curvature of the trajectory manifold

Informally, we define the radius of curvature at $\xi_0 \in \mathcal{T}$ as the minimum norm of any curve $\gamma \in X$ orthogonal to $T_{\xi_0}\mathcal{T}$, such that SSC is *not* satisfied for the problem in (5) with $\xi_1 = \xi_0 + \gamma$. Notice that here, and in the rest of the paper, we are playing with two norms. To define the space of curves we use L_∞ because we want to deal with bounded curves, whereas to measure the radius of curvature we use the weighted L_2 norm defined above.

Formally, we have the following definition. In order to highlight the dependence of $q(\xi_0) \cdot (\zeta, \zeta)$ from a desired curve ξ_1, we use the notation $q(\xi_0; \xi_1) \cdot (\zeta, \zeta)$.

Definition 5.6 (Trajectory manifold radius of curvature): Let $\xi_0 \in \mathcal{T}$ be given. The *radius of curvature of $\mathcal{T}$ at ξ_0* is defined as

$$\begin{aligned}\rho(\xi_0)^2 = \inf_{\gamma \in \mathcal{N}_{\xi_0}} \; & \|\gamma\|_2^2 \\ \text{subj. to } & q(\xi_0; \xi_0 + \gamma) \cdot (\zeta, \zeta) = 0, \\ & \text{for some } \zeta \in T_{\xi_0}\mathcal{T}\end{aligned}$$ □

The *curvature at* ξ_0 is then given by $\sigma(\xi_0) = 1/\rho(\xi_0)$ for $\rho(\xi_0) < +\infty$ and 0 otherwise. From now on we will omit the words minimum and maximum respectively.

We define also the radius of curvature along a given direction as follows.

Definition 5.7 (Radius of curvature along a direction): Let $\xi_0 \in \mathcal{T}$ and $\gamma \in \mathcal{N}_{\xi_0}$, $\|\gamma\|_2 = 1$, be given. The *radius of curvature of $\mathcal{T}$ at ξ_0 along γ* is defined as

$$\rho(\xi_0, \gamma) = \sup\{\rho > 0 \mid q(\xi_0; \xi_0 + \rho\gamma) \cdot (\zeta, \zeta) > 0, \text{ for all } \zeta \in T_{\xi_0}\mathcal{T}\}$$ □

First, we show that the Definition 5.6 is well posed. In particular we show that the radius of curvature is a strictly positive number.

Theorem 5.8 (The radius of curvature is positive): The radius of curvature is strictly positive. □

Proof: The proof is similar to the finite dimensional case. ■

Second, the radius of curvature of a linear system is $+\infty$ at any trajectory. This can be seen observing that, for a linear system, problem (3) or (5) is a linear quadratic optimal control problem. It is well known that, for any $\xi_1 = (x_1(\cdot), u_1(\cdot))$, the problem has a unique global minimizer (SSC is always satisfied). It is worth noting that the viceversa is not true in general. That is, if the radius of curvature at a given trajectory is $+\infty$ the system may also be nonlinear. As a counter example, consider the trajectory manifold of the scalar nonlinear system $\dot{x} = x^3 + u$. It is easy to show that the radius of curvature at $\xi_0 = (x(\cdot), u(\cdot)) \equiv (0, 0)$ is $+\infty$. However, for any trajectory arbitrarily close to ξ_0 the radius of curvature is less than $+\infty$. Our conjecture is that if the radius of curvature is $+\infty$ at any trajectory, then the system is linear.

From the analysis done so far, there is no information on the structure of $\mathcal{N}_{\xi_0}$. The structure of this space influences the minimization problem. In the next section we characterize it.

VI. Numerical computations

In this section we deal with numerical techniques to compute curves in the orthogonal complement and to compute the radius of curvature along a given direction. We apply these techniques to the case of an inverted pendulum.

A. Estimating the radius of curvature along a direction

From Definition 5.7 we know that, given $\xi_0 \in \mathcal{T}$ and $\gamma \in \mathcal{N}_{\xi_0}$,

$$\rho(\xi_0, \gamma) = \sup\{\rho > 0 \mid q(\xi_0; \xi_0 + r\gamma) \cdot (\zeta, \zeta) > 0, \\ \text{for all } \zeta \in T_{\xi_0}\mathcal{T} \text{and } r \in [0, \rho]\}.$$

Now, let $\mathrm{Ric}(\xi_0 + \rho\gamma)$, $\rho > 0$, be the Riccati equation associated to the minimization problem

$$\min_{\xi \in \mathcal{T}} \frac{1}{2} \|\xi - (\xi_0 + \rho\gamma)\|_2^2,$$

at the stationary point ξ_0. Using the result in Theorem 4.3 we have that

$$\rho(\xi_0, \gamma) = \sup\{\rho > 0 \mid \mathrm{Ric}(\xi_0 + r\gamma), r \in [0, \rho], \\ \text{has a bounded solution}\}.$$

With this equivalent definition in hand, we may compute $\rho(\xi_0, \gamma)$ by simply checking the unboundedness of the Riccati equation.

B. The inverted pendulum example

We evaluated the numerical techniques described above on the inverted pendulum example. The dynamics of the pendulum is given by

$$\ddot{\varphi}(t) = \frac{g}{l} \sin\varphi(t) - \frac{u(t)}{l} \cos\varphi(t). \tag{13}$$

In order to get an initial trajectory of the pendulum, $\xi_0 \in \mathcal{T}$, we have chosen a desired curve and solved a nonlinear quadratic optimal control problem by using the projection operator based Newton method described in [5]. In Figure 2, the dashed red curves represent respectively the φ portion and the input of the initial trajectory.

Then, we have chosen a curve $\gamma = (\beta(\cdot), \nu(\cdot)) \in \mathcal{N}_{\xi_0}$ parametrized by $\beta(\cdot)$. The first component of $\beta(\cdot)$ is represented in Figure 1, while the second component has been chosen to be identically zero. Using a numerically robust version of equation (12) we have computed the input curve $\nu(\cdot)$ depicted in Figure 1.

For this choice of γ we have found SSC to be preserved at $\rho = 11.6$ and to fail at $\rho = 11.7$.

In Figure 2 we have plotted the desired state and input curves $\xi_0 + \rho\gamma$ for $\rho = 1, 11.7$, together with the stationary trajectory ξ_0 (the dashed red curve). We have solved the nonlinear quadratic optimal control problem by using the projection operator Newton method. For any value of $\rho > 0$, ξ_0 turns to be a stationary trajectory.

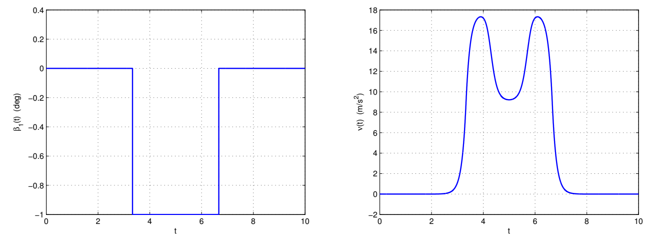

Fig. 1. The picture shows a possible choice of state curve (first component) which, combined with the depicted input curve, gives a curve in the orthogonal complement.

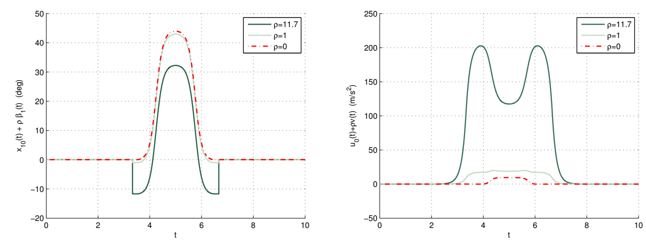

Fig. 2. The picture shows the state and control portions of the desired curves $\xi_0 + \rho\gamma$ for $\rho = 1, 11.7$, together with the corresponding stationary trajectory ξ_0.

VII. Conclusions

We have introduced the notion of curvature of the trajectory manifold for nonlinear systems. Using a suitable nonlinear quadratic optimal control problem we have defined the curvature and characterized the set of curves orthogonal to the tangent space. Using such curves we provided numerical techniques to compute a lower bound of the curvature.

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