Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Set Invariance Under Controlled Nonlinear Dynamics with Application to Robust RH Control

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Abstract— **This paper is concerned with the characterization of the maximal admissible uncertainty under which a nonlinear discrete-time dynamic system can be stabilized in the neighborhood of the origin by a Receding Horizon (RH) state-feedback control scheme. This topic is of great interest in both analysis and design of robust RH controllers for constrained discrete-time nonlinear systems. In particular, under mild assumptions on the nominal transition map, the robustness of the overall RH control scheme is shown to depend on the invariance properties of the terminal set constraint, which is a design parameter for the controller. In this framework, resorting to set invariance theoretic arguments, a numerical procedure is proposed which allows to evaluate the robust invariance properties of the terminal set constraint. An application example is provided to show the effectiveness of the proposed approach.**

I. Introduction

The design of reliable Receding Horizon (RH) controllers for nonlinear discrete-time systems subjected to state and control constraints is a very active area of research. In this framework, invariant set theory [6] has been exploited to provide nominal feasibility and stability conditions [10]. Changing the control objective from nominal to robust stabilization, this paper is mainly focused on the use of invariant set theory for the synthesis of robustly stabilizing RH controllers for uncertain nonlinear system. In particular, considering the class of additive transition uncertainty, the problem of evaluating an upper norm bound on admissible perturbations is of great practical interests for the synthesis of reliable controllers. For the class of RH algorithms which impose as stabilizing condition a fixed terminal constraint set, X_f, at the end of the horizon, the robustness of the overall c-l system has been shown to depend on the invariance properties of X_f, [12]. An estimate of the uncertainty bound can be obtained by computing the controllability set to X_f, denoted as $\mathcal{C}_1(X_f)$ (i.e., the set of state vectors which can be steered in X_f by an admissible control action). In general, it is very difficult to express $\mathcal{C}_1(X_f)$ analytically, therefore the conception of algorithms for its numerical approximation is needed. It must be remarked that the numerical approximation of $\mathcal{C}_1(X_f)$, for a generic nonlinear system, is a very computational demanding problem, although the underlying theory is well established and a large number of results have been proposed since the seminal paper [4]. Several algorithms exist for different classes of systems. Linear systems have been approached, among many, by [5], [11] and [17]. Procedures for the computation of approximations of the maximal invariant set for nonlinear systems have been discussed in [7]. On the other side, the problem of characterizing the maximal admissible uncertainty for robust set invariance has been approached only recently, in the framework of constrained linear systems, [16]. Significant recent advances on this subject can be found in the works of Artstein, Rakovic and co-workers [2], [15]. In this framework, the contribution of the present paper is twofold.

i) First, a novel iterative procedure is presented to evaluate, with arbitrary numerical accuracy, the robust invariance properties of X_f under a nonlinear constrained map parametrized in the control, $\hat{f}(x,u)$, $x\in X$, $u\in U$, without requiring the exact computation of $\mathcal{C}_1(X_f)$;
ii) Moreover, resorting to the notion of *finite-time controllability to a target set*, the robust stability properties of RH control policies based on constraints tightening and terminal set constraint are analyzed, showing that the developed algorithm can be used to evaluate less conservative bounds on admissible uncertainties with respect to those provided by current literature.

II. Main Notations

Let $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{Z}$, and $\mathbb{Z}_{\geq 0}$ denote the real, the non-negative real, the integer, and the non-negative integer sets of numbers, respectively. The Euclidean norm is denoted as $|\cdot|$. Given a compact set $A \subseteq \mathbb{R}^n$, let ∂A denote the boundary of A. Given a vector $x\in\mathbb{R}^n$, $\mathrm{d}(x,A) \triangleq \inf\{|\xi - x|, \xi\in A\}$ is the point-to-set distance from $x \in \mathbb{R}^n$ to A, while $\Phi(x,A) \triangleq \{\mathrm{d}(x,\partial A)$ if $x \in A$, $-\mathrm{d}(x,\partial A)$ if $x \notin A\}$ denotes the signed distance function. Given two sets $A\subseteq\mathbb{R}^n$, $B \subseteq \mathbb{R}^n$, $\mathrm{dist}(A,B) \triangleq \inf\{\mathrm{d}(\zeta,A), \zeta\in B\}$ is the minimal set-to-set distance. The difference between two given sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$, with $B \subseteq A$, is denoted as $A\backslash B \triangleq \{x : x \in A, x\notin B\}$. Given two sets $A \in \mathbb{R}^n$, $B \in \mathbb{R}^n$, then the Pontryagin difference set C is defined as $C = A \backsim B \triangleq \{x\in \mathbb{R}^n : x+\xi\in A, \forall\xi\in B\}$, while the Minkowski sum set is defined as $S = A\oplus B \triangleq \{x\in\mathbb{R}^n : x=\xi+\eta,\ \xi\in A,\ \eta\in B\}$. Given a vector $\eta \in \mathbb{R}^n$ and a positive scalar $\rho \in \mathbb{R}_{>0}$, the closed ball centered in η and of radius ρ is denoted as $\mathcal{B}(\eta,\rho) \triangleq \{\xi\in\mathbb{R}^n : |\xi-\eta|\leq\rho\}$. The shorthand $\mathcal{B}(\rho)$ is used when the ball is centered in the origin. A function $\alpha : \mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing.

III. Preliminaries

Consider the nonlinear discrete-time dynamic system

$$x_{t+1} = f(x_t, u_t, \upsilon_t),\ t \in \mathbb{Z}_{>0},\ x_0 = \bar{x} \tag{1}$$

where $x_t \in \mathbb{R}^n$ denotes the state vector, $u_t \in \mathbb{R}^m$ the control vector and $\upsilon_t \in \Upsilon$ is an uncertain exogenous input vector,

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with $\Upsilon \subset \mathbb{R}^r$ compact with $0 \in \Upsilon$. Assume that state and control variables are subject to the following constraints

$$x_t \in X, \ \ t \in \mathbb{Z}_{\geq 0}\,, \tag{2}$$

$$u_t \in U, \ \ t \in \mathbb{Z}_{\geq 0}\,, \tag{3}$$

where X and U are compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, containing the origin as an interior point. Given the system (1), let $\hat{f}(x_t, u_t)$, with $\hat{f}(0,0) = 0$, denote the *nominal* model used for control design purposes.

Assumption 1: The nominal map $\hat{f}(x,u)$ is Lipschitz with respect to x in X, with L. constant $L_{f_x} \in R_{>0}$. □

Now, given (1) and a generic state feedback control law $\kappa(x_t)$, let us define the map $g(x_t, \upsilon_t) \triangleq f(x_t, \kappa(x_t), \upsilon_t)$. Then the c-l dynamics under the action of the exogenous input υ_t is described by the system

$$x_{t+1} = g(x_t, \upsilon_t),\ t \in \mathbb{Z}_{\geq 0},\ x_0 = \bar{x} \tag{4}$$

with $x_t \in X$ and $\upsilon_t \in \Upsilon$. Moreover, let $\hat{x}_{t+i|t}$, $i \in Z_{>0}$, denote the state generated by means of the nominal model on the basis of the state information at time t, x_t, and of a sequence of inputs $\mathbf{u}_{t,t+1-1} \triangleq \mathrm{col}[u_t, \ldots, u_{t+i-1}]$, such that

$$\hat{x}_{t+i|t} = \hat{f}(\hat{x}_{t+i-1|t}, u_{t+i-1}),\ \hat{x}_{t|t} = x_t,\ i \in \mathbb{Z}_{>0}. \tag{5}$$

Defining the *additive transition uncertainty* vector $d_t \triangleq f(x_t, u_t, \upsilon_t) - \hat{f}(x_t, u_t)$, we have

$$x_{t+1} = \hat{f}(x_t, u_t) + d_t\,. \tag{6}$$

In the sequel, given an initial state vector x_0, the prediction at time t obtained by (5) will be denoted as $\hat{x}(x_0, \mathbf{u}_{0,i-1}, t)$, with $t \in \{1, \ldots, i\}$. In this way, the dependency of the solution on a particular sequence of controls $\mathbf{u}_{0,i-1} \triangleq \mathrm{col}[u_0, \ldots, u_{i-1}]$ is pointed out, while the true system trajectory due to a specified realization of uncertainties $\mathbf{d}_{0,i-1} \triangleq \mathrm{col}[d_0, \ldots, d_{i-1}]$ will be denoted as $x(x_0, \mathbf{u}_{0,i-1}, \mathbf{d}_{0,i-1}, t)$.

Moreover, for the sake of clarity, the subscript t will be neglected when not strictly required, in particular when the transition map describing the nominal dynamics serves as a nonlinear map $\hat{f}(x,u)$ parametrized in the control u.

IV. Problem Formulation

The control objective consists in designing a state-feedback control law capable to robustly stabilize the system (1), guaranteeing the c-l invariance of a compact set contain the origin as interior point, in presence of the class of uncertainty assumed in the following.

Assumption 2 (Uncertainty): The additive transition uncertainty vector d_t belongs to the compact ball $D \triangleq \mathcal{B}(\bar{d})$, with $\bar{d} \triangleq \sup_{\upsilon \in \Upsilon} \mu(|\upsilon|)$, where μ is a $\mathcal{K}$-function. □

In order to meet the control specifications, the class of robust RH controllers based on the constraint tightening technique is employed. In this regard, our main focus consists in providing a less conservative estimate, with respect to current literature, of the bound on the uncertainty, $\bar{d}$, for which the system can be robustly stabilized by the RH policy to be described in the following.

First, we describe how the RH state-feedback control law is obtained by solving, at each time instant t, a Finite Horizon Optimal Control Problem (FHOCP).

Definition 4.1 (FHOCP): Given a positive integer $N_c \in \mathbb{Z}_{\geq 0}$, at any time $t \in \mathbb{Z}_{\geq 0}$, let $\mathbf{u}_{t,t+N_c-1|t} \triangleq \mathrm{col}[u_{t|t}, u_{t+1|t}, \ldots, u_{t+N_c-1|t}]$ denote a sequence of input variables over the control horizon N_c. Then, given a stage–cost function h, the constraint sets $X_{t+i|t} \subseteq X$, $\ i \in \{1, \ldots, N_c\}$, a terminal cost function h_f and a terminal set X_f, the *Finite Horizon Optimal Control Problem* (FHOCP) consists in minimizing, with respect to $\mathbf{u}_{t,t+N_c-1|t}$, the following cost function

$$J_{FH}(x_t, \mathbf{u}_{t,t+N_c-1|t}, N_c) \triangleq \sum_{l=t}^{t+N_c-1} h(\hat{x}_{l|t}, u_{l|t}) + h_f(\hat{x}_{t+N_c|t})$$

subject to the

i) nominal dynamics (5) ;
ii) constraints $u_{t+i-1|t} \in U, \hat{x}_{t+i|t} \in X_{t+i|t}, i \in \{1, \ldots, N_c\}$;
iii) terminal state constraint $\hat{x}_{t+N_c|t} \in X_f$. □

The usual RH control paradigm can now be stated as follows: given a time instant $t \in \mathbb{Z}_{\geq 0}$, let $\hat{x}_{t|t} = x_t$, and find the optimal control sequence $\mathbf{u}^\circ_{t,t+N_c-1|t}$ by solving the FHOCP. Then, according to the RH strategy, apply

$$u_t = \kappa_{RH}(x_t)\,, \tag{7}$$

where $\kappa_{RH}(x_t) \triangleq u^\circ_{t,t}$ and $u^\circ_{t,t}$ is the first element of the optimal sequence $\mathbf{u}^\circ_{t,t+N_c-1|t}$ (implicitly dependent on x_t).

Let us now introduce the following further

Assumption 3 (κ_f, h_f, X_f): Assume that there exist an auxiliary control law $\kappa_f(x) : X \to U$, a function $h_f(x) : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a positive constant $L_{h_f} \in \mathbb{R}_{>0}$, a level set of h_f, $X_f \subset X$ and a positive constant $\nu \in \mathbb{R}_{>0}$ such that the following properties hold:

i) $X_f \subset X$, X_f closed, $\{0\} \in X_f$;
ii) $\kappa_f(x) \in U, \ \ \forall x \in X_f$;
iii) $h_f(x)$ Lipschitz in X_f, with L. constant $L_{h_f} \in \mathbb{R}_{>0}$;
iv) $h_f(\hat{f}(x, \kappa_f(x))) - h_f(x) < -h(x, \kappa_f(x)), \forall x \in X_f \setminus \{0\}$;
v) $h_f(\hat{f}\,(x, \kappa_f(x))) - h_f(x) \leq -\nu,\ \nu \in \mathbb{R}_{>0}, \forall x \in \partial X_f$. □

Let us denote as X_{RH} the region in which the FHOCP is feasible. If X_{RH} is RPI (see Def. 5.2) for all the possible realizations of uncertainty, then it possible to show that, by accurately choosing the stage cost h, the constraint sets $X_{t+j|t}$, $j \in \{1, \ldots, N_c\}$, the terminal cost function h_f, and by imposing a terminal constraint X_f at the end of the control horizon such that Assumption 3 holds, the c-l system is Regional Input to State Stable in X_{RH} w.r.t. norm-bounded uncertainties [14]. In this connection, we are going to derive a bound on the uncertainty for which the invariance X_{RH} is guaranteed.

V. Invariant sets and feasibility of RH Control

In this section, the interplay between invariant sets and robust RH control will be addressed. This will allow us to properly design the parameters of the FHOCP and to characterize the robustness of the c-l system under the RH control law. The following definitions aim to introduce the basic ingredients of invariant set theory that will be needed.

Definition 5.1 (RCI): A set $\Xi \subseteq X$ is a *Robust Controlled Invariant* (RCI) set for system (6) if $\exists u \in U$ such that $\hat{f}(x,u) + d \in \Xi$, $\forall x \in \Xi$ and $\forall d \in D$. □

Definition 5.2 (RPI): A set $\Xi \subset X$ is a *Robust Positively Invariant* (RPI) under the map $\hat{g}(x,d)$ if $\hat{g}(x,d) \in \Xi$, $\forall x \in \Xi$ and $\forall d \in D$. □

Definition 5.3 ($\mathcal{C}_i(X,\Xi)$): Given a set $\Xi \subseteq X$, the *i-step Controllability Set* to Ξ, $\mathcal{C}_i(X,\Xi)$, is the set of states which can be steered to Ξ by an admissible control sequence of length i, $\mathbf{u}_{0,i-1}$, under the nominal map $\hat{f}(x,u)$, subject to constraints (2) and (3), i.e.

$$\mathcal{C}_i(X,\Xi) \triangleq \Big\{ x_0 \in X : \exists \mathbf{u}_{0,i-1} \in U^i \text{ such that } \hat{x}(x_0,\mathbf{u}_{0,i-1},t) \in X,\ \forall t \in \{1,\dots,i-1\},\ \hat{x}(x_0,\mathbf{u}_{0,i-1},i) \in \Xi \Big\}.$$ □

In the sequel, the shorthand $\mathcal{C}_i(\Xi)$ will be used in place of $\mathcal{C}_i(\mathbb{R}^n,\Xi)$. Moreover, according to the literature, the set $\mathcal{C}_1(X_f)$ will be addressed as *predecessor* set of X_f.

The following Lemma (proven in the Appendix) introduces a method to compute the constraint sets of the FHOCP.

Lemma 5.1 (State Constraints Tightening): Under Assumptions 1 and 2, suppose [1], without loss of generality, $L_{f_x} \neq 1$. If the state constraints at the $j-$th prediction step of the FHOCP $X_{t+j|t}$, are computed as

$$X_{t+j|t} \triangleq X \backsim \mathcal{B}\left(\frac{L_{f_x}^i - 1}{L_{f_x} - 1}\bar{d}\right),\ \forall i \in \{1,\dots,N_c\} \quad (8)$$

then each input sequence which is feasible for the FHOCP guarantees that the true state will satisfy $x_{t+j} \in X, \forall j \in \{1,\dots,N_c\}, \forall x_t \in X_{RH}, \forall \upsilon \in \mathcal{M}_\Upsilon$. □

Resorting to feasibility arguments, the main stability result for the described robust RH controller is asserted by the following Theorem (see Appendix II for the proof).

Theorem 5.1 (Robust Positive Invariance of X_{RH}): Let $\bar{i}$ be the largest $i \in \mathbb{Z}_{>0}$ such that $\sum_{j=1}^{i} j \leq N_c$. Under Assumptions 1-3, the set X_{RH} is RPI under the c-l dynamics w.r.t. additive uncertainties $d \in \mathcal{B}(\bar{d})$, where

$$\bar{d} \triangleq \min\left\{ \max_{i\in\{1,\dots,\bar{i}\}}\{\bar{d}_i\}, \frac{L_{f_x}-1}{L_{f_x}^{N_c}-1}\operatorname{dist}(\mathbb{R}^n \backslash X, X_f) \right\},$$

with

$$\bar{d}_i \triangleq L_{f_x}^{i-N_c}\left(\prod_{k=1}^{i}\frac{L_{f_x}-1}{L_{f_x}^{k}-1}\right)\operatorname{dist}(\mathbb{R}^n\backslash\mathcal{C}_i(X_f),X_f)\,. \quad (9)$$ □

The result stated by Theorem 5.1 motivates us in formulating an affordable method to compute the euclidean metric $\operatorname{dist}(\mathbb{R}^n\backslash\mathcal{C}_i(X_f),X_f)$, $i \in \{1,\dots,N_c-1\}$.

VI. Approximation of $\operatorname{dist}(\mathbb{R}^n\backslash\mathcal{C}_i(X_f),X_f)$

At a first glance, the exact evaluation of $\operatorname{dist}(\mathbb{R}^n\backslash\mathcal{C}_i(X_f),X_f)$ can be carried out by directly computing $\mathcal{C}_i(X_f)$. However, in most situations, only an inner approximation $\hat{\mathcal{C}}_i(X_f)$ can be obtained numerically. In this respect, an iterative procedure will be described to compute a lower approximation of $\operatorname{dist}(\mathbb{R}^n\backslash\mathcal{C}_i(X_f),X_f)$. The algorithm is based on the following recursion

$$\begin{cases} \hat{\mathcal{C}}_i(\Xi)_1 = X_f, \\ \hat{\mathcal{C}}_i(\Xi)_{j+1} = \vartheta(\hat{\mathcal{C}}_i(\Xi)_j, X_f),\ j \in \mathbb{Z}_{\geq 1}, \end{cases} \quad (10)$$

where ϑ is a suitable set-valued function and $\hat{\mathcal{C}}_i(\Xi)_j$ is the convex inner approximation of $\mathcal{C}_i(X_f)$ computed at the j-th iteration. Referring to a numerical example reported in Section VIII, Figure 1 shows a graphical representation of two sequences of sets generated by such a recursion to approximate $\mathcal{C}_1(X_f)$ and $\mathcal{C}_8(X_f)$.

[1]The very special case $L_{f_x} = 1$ can be trivially addressed by a few suitable modifications to the proof of Lemma 5.1.

The main objective of the following analysis consists in designing the above set-valued operator ϑ, such as to guarantee the convergence the algorithm toward the desired euclidean metric $\operatorname{dist}(\mathbb{R}^n\backslash\mathcal{C}_i(X_f),X_f)$. In order to determine a function ϑ capable to satisfy the above requirements, we need to address the issue of numerical computability of set-valued operators, which poses indeed some constraints on the structure of the approximation algorithm. A detailed analysis of computability of set-valued operators for nonlinear discrete-time autonomous and controlled systems is given in [2] and [9]. In order to present a key result on the computability of controllability sets, some notions of set-valued analysis [3] must be introduced.

Definition 6.1 ($\hat{F}(x)$): Given the nominal transition function $\hat{f}(x,u)$, the set-valued map $\hat{F}: X \mapsto Y, Y \subseteq \mathbb{R}^n$ is defined as

$$\hat{F}(x) \triangleq \bigcup_{u\in U} \hat{f}(x,u). \quad (11)$$ □

Definition 6.2 (LSC): A set-valued map $\hat{F}: X \mapsto Y$ is called *Lower Semi-Continuous* (LSC) in X if $\forall x' \in X$, given $\epsilon \in \mathbb{R}_{>0}$, $\exists \delta \in \mathbb{R}_{>0}$ such that inequality $|x' - x''| < \delta$ implies $\hat{F}(x') \subseteq \hat{F}(x'') \oplus \mathcal{B}(\epsilon)$. □

Noting that the predecessor operator generates the natural *weak set-valued preimage* [8] of a given set $\Xi \subseteq Y$ under $\hat{F}$, $\mathcal{C}_1(\Xi) = \hat{F}^{-1}(\Xi) \triangleq \{x \in X : \hat{F}(x) \cap \Xi \neq \emptyset\}$, a computability result for $\hat{F}^{-1}(\Xi)$ would imply the computability of $\mathcal{C}_1(\Xi)$.

To this end, let us introduce the notion of robust robust controllability set, which plays a key role in the computability theory for set-valued operators, since it represents the best computable approximation of the true predecessor set [9].

Definition 6.3 ($\mathcal{RC}_i(X,\Xi)$): Given a set $\Xi \subseteq X$ and the nominal map $\hat{f}(x,u)$, the *i-step Robust Controllability* set to Ξ is defined as $\mathcal{RC}_i(X,\Xi) \triangleq \mathcal{C}_i(X,\operatorname{int}(\Xi))$. □

In the following, the shorthand $\mathcal{RC}_i(\Xi)$ will be used to denote $\mathcal{RC}_i(\mathbb{R}^n,\Xi)$.

The possibility to obtain an arbitrary accurate numerical approximation of the robust predecessor set is guaranteed by the following approximate computability result.

Theorem 6.1 ([9]): Given a set $\Xi \subseteq X$, if the map $\hat{F}(x)$ is LSC in x, $\forall x \in X$, then $\mathcal{C}_1(\Xi)$ is open whenever Ξ is open. Hence, the operator $\Xi \mapsto \mathcal{RC}_1(\Xi)$ is always *lower semicomputable*, i.e., it can be approximated arbitrarily well by a sequence of compact sets $\{\hat{\mathcal{C}}_1(\Xi)_j\}$, $j \in \mathbb{Z}_{\geq 1}$, with $\hat{\mathcal{C}}_1(\Xi)_j \supset \hat{\mathcal{C}}_1(\Xi)_{j-1}$, given an initial lower approximation $\hat{\mathcal{C}}_1(\Xi)_1 \subset \mathcal{RP}(\Xi) \subseteq \mathcal{C}_1(\Xi)$. □

Noting that a map defined as in (11) is LSC under Assumption 1, Theorem 6.1 can be readily extended to characterize the computability of the i-step robust controllability set $\mathcal{RC}_i(\Xi)$. In this regard, let us introduce the following problem.

Problem 6.1: Given a finite integer $i \in \mathbb{Z}_{>0}$ and a compact set X_f such that Assumption 3 holds, we look for a numerical set-iterative procedure, in the form of (10), capable to generate a sequence $\{\hat{\mathcal{C}}_i(X_f)_j,\ j \in \mathbb{Z}_{\geq 1}\}$ of compact sets lower approximating $\mathcal{RC}_i(X_f)$, such that

i) $\hat{\mathcal{C}}(X_f)_j \subset \mathcal{C}_i(X_f),\ \forall j \in \mathbb{Z}_{>0}$;

ii) if $\operatorname{dist}(\mathbb{R}^n\backslash\hat{\mathcal{C}}_i(X_f)_j,X_f) < \operatorname{dist}(\mathbb{R}^n\backslash\mathcal{RC}_i(X_f),X_f) \Rightarrow \operatorname{dist}(\mathbb{R}^n\backslash\hat{\mathcal{C}}_i(X_f)_{j+1},X_f) > \operatorname{dist}(\mathbb{R}^n\backslash\hat{\mathcal{C}}_i(X_f)_j,X_f), \forall j \in \mathbb{Z}_{>0}$;

iii) $\operatorname{dist}(\mathbb{R}^n\backslash\mathcal{RC}_i(X_f),X_f) \leq \lim_{j\to\infty}\operatorname{dist}(\mathbb{R}^n\backslash\hat{\mathcal{C}}_i(X_f)_j,X_f) \leq \operatorname{dist}(\mathbb{R}^n\backslash\mathcal{C}_i(X_f),X_f)$.

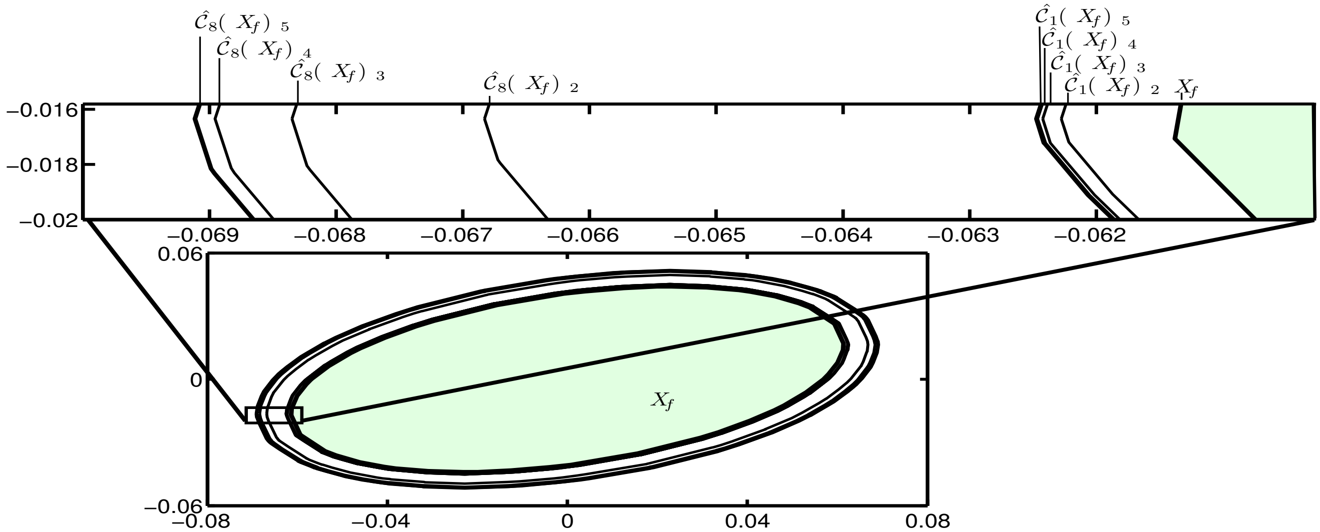

Fig. 1. Sequence of sets generated by the iterative procedure (10) for the numerical example reported in Section VIII.

□

Now, we are going to introduce a numerical framework to address the issues raised by Problem 6.1. In particular, it will be shown that the function ϑ in (10) can be designed such that all requirements are satisfied.

To this end, let us introduce the *Finite Horizon Distance Optimal Control Problem* (FHDOCP).

Problem 6.2 (FHDOCP): Given system (6), a set X_f for which Assumption 3 holds and a compact set $X \in \mathcal{RC}_i(X_f)$, consider a vector $x_0 \in \partial\Xi$. The (i-steps) *Finite Horizon Distance Optimal Control Problem* (FHDOCP) consists in finding the sequence of control moves $\mathbf{u}_{0,i-1} = \text{col}[u_0, \ldots, u_{i-1}]$, subject to (3), such that the following value function, $J_{FHD}(x_0, \mathbf{u}_{0,i-1}, X_f)$, is maximized:

$$J^\circ_{FHD}(x_0,\mathbf{u}^\circ_{0,i-1}(x_0),X_f)= \max_{\mathbf{u}_{0,i-1}\in U^i}\{\Phi\left(\hat{x}(x_0,\mathbf{u}_{0,i-1},i),X_f\right)\}.$$

□

An effective optimization-based algorithm, which satisfies the requirements raised by Problem 6.1, is now presented (for the sake of notational simplicity, the dependency of J°_{FHD} on X_f will be omitted).

Theorem 6.2 ($\hat{\mathcal{C}}_i(X_f)$): Given a positive integer $i \in \mathbb{Z}_{>0}$ and a compact set X_f set such that Assumption 3 holds, consider the recursion (10) with ϑ defined as follows

$$\vartheta(\hat{\mathcal{C}}_i(X_f)_j,X_f) \triangleq \hat{\mathcal{C}}_i(X_f)_j \oplus \mathcal{B}\left(L_{f_x}^{-i} \min_{x\in\partial\hat{\mathcal{C}}_i(X_f)_j}\left\{J^\circ_{FHD}(x,\mathbf{u}^\circ_{0,i})\right\}\right). \tag{12}$$

Then the sequence of sets $\{\hat{\mathcal{C}}_i(X_f)_j, j\in\mathbb{Z}_{\geq 1}\}$ satisfies Points i)-iii) of Problem 6.1. □

The proof of Theorem 6.2 is given in Appendix III.

Remark 6.1: Notice that Theorem 6.2 assumes that the optimal value $J^\circ_{FHD}(x,\mathbf{u}^\circ_{0,i})$ for each $x \in \partial\hat{\mathcal{C}}_i(X_f)_j$ as well as the global minimum $\min_{x\in\partial\hat{\mathcal{C}}_i(X_f)_j}\left\{J^\circ_{FHD}(x,\mathbf{u}^\circ_{0,i})\right\}$ can be actually obtained. For a generic nonlinear system this is not always the case, therefore a numerical method to approximate the set-valued function ϑ is described. Notably, in this case the constraints imposed by Points i)-iii) of Problem 6.1 cannot be strictly fulfilled, but can be violated with an arbitrarily small tolerance specified by the designer, as described in the following section. □

VII. Numerical implementation of the set-iterative scheme

In order to derive a numerically affordable implementation of the set iterations (10)-(12), some properties of the optimal value function $J^\circ_{FHD}(x_0, \mathbf{u}^\circ_{0,i}(x_0))$ are going to be analyzed.

Lemma 7.1: Under Assumptions 1-3, given a vector $x_0' \in \partial\hat{\mathcal{C}}_i(X_f)_j$, the optimal cost $J^\circ_{FHD}(x_0', \mathbf{u}^\circ_{0,i}(x_0'))$, the optimal control sequence $\mathbf{u}^\circ_{0,i}(x_0')$, and the optimal state prediction $\hat{x}(x_0', \mathbf{u}^\circ_{0,i}(x_0'), i)$, then the optimal value of the function $J^\circ_{FHD}(x_0'', \mathbf{u}^\circ_{0,i}(x_0''))$ is lower bounded by

$$J^\circ_{FHD}(x_0'',\mathbf{u}^\circ_{0,i}(x_0''))\geq J^\circ_{FHD}(x_0',\mathbf{u}^\circ_{0,i}(x_0'))-\alpha L_{f_x}{}^i \tag{13}$$

for any vector $x_0'' \in \mathbb{R}^n : |x_0' - x_0''| \leq \alpha$, with $\alpha \in \mathbb{R}_{>0}$. □

Proof: In view of Assumption 1, it follows that

$$|\hat{x}(x_0', \mathbf{u}^\circ_{0,i}(x_0'), i) - \hat{x}(x_0'', \mathbf{u}^\circ_{0,i}(x_0'), i)| \leq L_{f_x}{}^i\alpha. \tag{14}$$

Let $\eta \in \mathbb{R}_{>0}$ be such that $\alpha = \eta L_{f_x}^{-i} J^\circ_{FHD}(x_0', \mathbf{u}^\circ_{0,i}(x_0'))$, then it follows that the state vector $\hat{x}(x_0'', \mathbf{u}^\circ_{0,i}(x_0'), i) \in \mathcal{B}\left(\hat{x}(x_0', \mathbf{u}^\circ_{0,i}(x_0'), i), \eta J^\circ_{FHD}(x_0', \mathbf{u}^\circ_{0,i}(x_0'))\right)$. Considering that $J^\circ_{FHD}(x_0'', \mathbf{u}^\circ_{0,i}(x_0'')) \geq J^\circ_{FHD}(x_0'', \mathbf{u}^\circ_{0,i}(x_0'))$, then

$$J^\circ_{FHD}(x_0'', \mathbf{u}^\circ_{0,i}(x_0''))\geq(1-\eta)J^\circ_{FHD}(x_0', \mathbf{u}^\circ_{0,i}(x_0')).$$

Finally, substituting the expression for η, the statement of the lemma trivially follows. ■

In the sequel, an algorithm for numerically approximating the set-valued function ϑ in (12) is discussed.

Procedure 7.1 (Numerical recipe for $\vartheta(\hat{\mathcal{C}}_i(X_f)_j, X_f)$): First, notice that, given a lower bound

$$\underline{J}_j \leq \min_{x\in\partial\hat{\mathcal{C}}_i(x_f)_j}\{J^\circ_{FHD}(x, \bar{u}^\circ_{0,i})\},\ \ \underline{J}_j\in\mathbb{R}_{>0}$$

then the following inequality holds

$$\mathcal{C}_i(x_f)_j \subseteq \left(\hat{\mathcal{C}}_i(x_f)_j \oplus (L_{f_x}^{-1}\underline{J}_j)\right) \subseteq \vartheta(\hat{\mathcal{C}}_i(X_f)_j, X_f)$$

Thanks to Lemma 7.1, $\underline{J}_j$ can be obtained by performing a series of FHDOCP's in suitably chosen vectors belonging to $\partial\hat{\mathcal{C}}_i(X_f)_j$. In order to ensure the termination of the procedure in a finite number of steps, let us fix an arbitrary tolerance $\delta \in \mathbb{R}_{>0}$, whose significance will be cleared later on. At

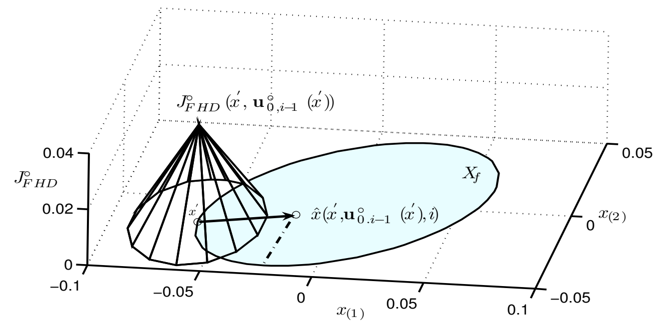

Fig. 2. Graphical representation of the conic lower bound on $J^{\circ}_{FHD}(x_0^{''})$, for $|x_0^{''} - x_0^{'}| < L_{f_x}^{-i} J^{\circ}_{FHD}(x^{'}, \mathbf{u}^{\circ}_{0,i-1}(x_0^{'}))$.

this point, let us consider a grid-like subset $X_H \subset \partial \hat{\mathcal{C}}_i(X_f)_j$ such that $\mathrm{d}(x, X_H) \leq \delta$, $\forall x \in \partial \hat{\mathcal{C}}_i(X_f)_j$ and $\exists \epsilon \in \mathbb{R}_{>0}$: $\mathrm{d}(x, X_H \backslash \{x\}) \geq \epsilon$, $\forall x \in X_H$. Being $\hat{\mathcal{C}}_i(X_f)_j$ compact, X_H is numerable. Then, performing a finite number of FHDOCP's on all the vectors of X_H, we can compute

$$\underline{J}_j = (1 - 0.5 L_{f_x} \delta) \min_{x \in X_H} \{J^{\circ}_{FHD}(x)\}$$

If $\underline{J}_j < 0$, the recursion is terminated. Notice that δ can be reduced to allow for a finer gridding, which permits to compute a possibly tighter bound, at the cost of an increase in the computational load. Conversely, if $\underline{J}_j > 0$, the set $\hat{\mathcal{C}}_i(X_f)_{j+1} = \hat{\mathcal{C}}_i(x_f)_j \oplus (L_{f_x}^{-1} \underline{J}_j)$ is computed and the recursion is continued.

For implementation purposes, it is convenient that X_f is given as a polytope [1], such that the sets $\hat{\mathcal{C}}_i(X_f)_{j+1}$, $j \in \{1, \ldots, N_c\}$ can be obtained by performing the Minkowski addition between polytopes, having previously inner approximated the ball addendum in (12) by a parallelotope. $\square$

VIII. Application Example

Consider the following discrete-time model of an undamped nonlinear oscillator

$$\begin{cases} x_{(1)_{t+1}} = x_{(1)_t} + 0.05 \left[-x_{(2)_t} + 0.5 \left(1 + x_{(1)_t}\right) u_t \right] \\ x_{(2)_{t+1}} = x_{(2)_t} + 0.05 \left[x_{(1)_t} + 0.5 \left(1 - 4x_{(1)_t}\right) u_t \right], \end{cases} \tag{15}$$

where the subscript (i) denotes the i-th component of the vector x_t. System (15) is subject to the constraints (2) and (3). The set X is given by $\{x \in \mathbb{R}^n : x = [x_{(1)}\ x_{(2)}]^T, |x_{(1)}| \leq 0.2,\ |x_{(2)}| \leq 0.2\}$, while $U \triangleq \{u \in \mathbb{R} : |u| \leq 2\}$. Given X and U, the Lipschitz constant of the system is $L_{f_x} = 1.1390$. A linear state feedback control law $u_t = \kappa_f(x_t) = k^T x_t$, with $k \in \mathbb{R}^2$, stabilizing (15) in a neighborhood of the origin, can be designed as described in [13]. Choosing $k = [0.74\ \ 1.80]^T$, the following ellipsoidal set, $X_f \in X$, is RPI under the nominal dynamics (15) in c-l with $\kappa_f(x_t)$

$$X_f \triangleq \left\{ x_t \in \mathbb{R}^n : x_t^T \begin{bmatrix} 309.21 & -162.53 \\ -162.53 & 602.72 \end{bmatrix} x_t \leq 1 \right\}.$$

In X_f, the auxiliary control law satisfies Assumption 3. Assuming that the system is controlled by a constraint tightening RH scheme having X_f as terminal set and $N_c = 9$, in order to obtain the upper norm bound on admissible uncertainties, the set-iterative procedure (10) has been applied to X_f, obtaining a sequence of sets $\hat{\mathcal{C}}_i(X_f)_j$, $j \in \mathbb{Z}_{\geq 1}$, $i \in \{1, \ldots, 8\}$, as depicted in Figure 1. The following bounds can be inferred from the computed sets by using (9)

$$\bar{d}_1 = 4.02 \cdot 10^{-4} \quad \bar{d}_2 = 4.92 \cdot 10^{-4} \quad \bar{d}_3 = 3.31 \cdot 10^{-4}$$

Finally, the upper norm bound on admissible uncertainty is $\bar{d} = 4.92 \cdot 10^{-4}$, which is larger than $\bar{d}_1$, that coincides with the upper limit computable by the tools provided in [14].

Conclusions

This paper has presented a novel method, based on set-invariance theoretic arguments, to evaluate the maximal admissible uncertainty under which a nonlinear discrete-time system can be stabilized by a constrained Receding Horizon (RH) state-feedback control scheme. Remarkably, the norm bound on the admissible uncertainties has been shown to depend on the robust invariance properties of the terminal set X_f, which is a free design parameter of the FHOCP.

Finally, a set-iterative procedure has been proposed to approximate by numerical computations, with arbitrary accuracy, the bound on admissible uncertainty.

Appendix I

Proof: [Lemma 5.1] According to [12], given the state vector x_t at time t, and a feasible control sequence $\bar{\mathbf{u}}_{t,t+N_c-1|t}$, the prediction error $\hat{e}_{t+i|t} \triangleq x_{t+i} - \hat{x}_{t+i|t}$, with $i \in \{1, \ldots, N_c\}$, and x_{t+i} obtained applying $\bar{\mathbf{u}}_{t,t+N_c-1|t}$ in open loop to the uncertain system (1), is upper bounded by

$$|\hat{e}_{t+i|t}| \leq \frac{L_{f_x}^i - 1}{L_{f_x} - 1} \bar{d}, \ \ \forall i \in \{1, \ldots, N_c\}$$

where $\bar{d}$ is defined as in Assumption 2. Then it follows that $x_{t+i} = \hat{x}_{t+i|t} + \hat{e}_{t+i|t} \in X$, $\forall i \in \{1, \ldots, N_c\}$. ■

Appendix II

Proof: [Theorem 5.1] The proof consists in showing that if the FHOCP is feasible at time t for some x_t (i.e $x_t \in X_{RH}$), then, applying the state feedback control law $u_t = \kappa_{RH}(x_t)$, $t \in \mathbb{Z}_{>0}$, the FHOCP admits a feasible solution for all x_{t+i}, with $i \in \mathbb{Z}_{>0}$, subject to the c-l dynamics $x_{t+1} = \hat{f}(x_t, \kappa_{RH}(x_t)) + d_t$, with $d_t \in \mathcal{B}(\bar{d})$. Let us define $\bar{\delta}_i \triangleq (L_{f_x} - 1)/(L_{f_x}^i - 1) \mathrm{dist}(\mathbb{R}^n \backslash \mathcal{C}_i(X_f), X_f)$.

The proof will be carried out in four steps.

i) $x_0 \in \mathcal{C}_i(X_f) \Rightarrow \exists \mathbf{u}^*_{0,i-1|0} \in U^i | x(x_0, \mathbf{u}^*_{0,i-1|0}, \mathbf{d}_{0,i-1}, i) \in \mathcal{C}_i(X_f), \forall \mathbf{d}_{0,i-1} \in \mathcal{B}(\bar{\delta}_i)^i$: Given $x_0 \in \mathcal{C}_i(X_f)$, let us denote as $\mathbf{u}^*_{0,i-1|0}$ a feasible control sequence (implicitly function of x_0) such that $\hat{x}_{i|0} = \hat{x}(x_0, \mathbf{u}^*_{0,i-1|0}, i) \in X_f$. It is straightforward to prove that the norm difference between the prediction $\hat{x}_{i|0}$ and $x_i = x(x_0, \mathbf{u}^*_{0,i-1|0}, \mathbf{d}_{0,i-1}, i)$, with $\mathbf{d}_{0,i-1} \in \mathcal{B}(\bar{\delta}_i)^i$, can be upper bounded by

$$|\hat{x}_{i|0} - x_i| \leq \frac{L_{f_x}^i - 1}{L_{f_x} - 1} \bar{\delta}_i \tag{16}$$

Substituting the value of $\bar{\delta}_i$ in (16), it is trivial to show that, since $\hat{x}_{i|0} \in X_f$, then $x_i \in \mathcal{C}_i(X_f)$.

ii) Uncertainty bound $\bar{d}$: First, let us consider $i = 2$. Thanks to Point i), $\forall x^{'} \in \mathcal{C}_1(X_f) \oplus \mathcal{B}(\bar{\delta}_2)\ \exists u^{'} : \hat{f}(x^{'}, u^{'}) \in \mathcal{C}_2(X_f) \backsim \mathcal{B}(\bar{\delta}_2)$; then it follows that

$$\begin{aligned} &(\mathcal{C}_1(X_f) \oplus \mathcal{B}(\bar{\delta}_2)) \cup \mathcal{C}_2(X_f) \\ &\quad \subseteq \mathcal{C}_1\Big(\mathcal{C}_2(X_f) \backsim \mathcal{B}(\bar{\delta}_2)\Big) \cup \mathcal{C}_1\left(\mathcal{C}_1(X_f)\right) \\ &\quad = \mathcal{C}_1\Big(\mathcal{C}_1(X_f) \cup \Big(\mathcal{C}_2(X_f) \backsim \mathcal{B}(\bar{\delta}_2)\Big)\Big) \\ &\quad \subseteq \mathcal{C}_1\Big(\Big[(\mathcal{C}_1(X_f) \oplus \mathcal{B}(\bar{\delta}_2)) \cup \mathcal{C}_2(X_f)\Big] \backsim \mathcal{B}(\bar{\delta}_2)\Big). \end{aligned} \tag{17}$$

By noting that $\mathcal{C}_1(X_f) \cup \big(\mathcal{C}_2(X_f) \backsim \mathcal{B}(\bar{\delta}_2)\big) \subseteq \mathcal{C}_2(X_f) \subseteq (\mathcal{C}_1(X_f) \oplus \mathcal{B}(\bar{\delta}_2)) \cup \mathcal{C}_2(X_f)$, then from (17) it follows that $\mathrm{dist}(\mathbb{R}^n \backslash \mathcal{C}_3(X_f), \mathcal{C}_2(X_f)) \geq L_{f_x}^{-1} \bar{\delta}_2$.

With similar arguments as above, it is possible to prove that $\mathrm{dist}(\mathbb{R}^n \backslash \mathcal{C}_{i+i-1}(X_f), \mathcal{C}_i(X_f)) \geq L_{f_x}^{1-i} \bar{\delta}_i, \ \forall i \geq 2$. Hence, one can apply recursively this result to obtain

$$\mathrm{dist}(\mathbb{R}^n \backslash \mathcal{C}_{(\sum_{j=1}^{i} j)}(X_f), \mathcal{C}_{(-1+\sum_{j=1}^{i} j)}(X_f)) \geq L_{f_x}^{i-(\sum_{j=1}^{i} j)} (\textstyle\prod_{k=1}^{i} (L_{f_x}-1)/(L_{f_x}^{k}-1)) \mathrm{dist}(\mathbb{R}^n \backslash \mathcal{C}_i(X_f), (X_f)).$$

Therefore, since $N_c \geq \sum_{j=1}^{i} j$, the statement of the theorem trivially follows.

iii) $\hat{x}_{t+j|t} \in X_{t+j|t} \Rightarrow \hat{x}_{t+j|t+1} \in X_{t+j|t+1}, \forall j \in \{1, \ldots, N_c\}$: Consider the predictions $\hat{x}_{t+j|t}$ and $\hat{x}_{t+j|t+1}$, initialized by x_t and x_{t+1}, and obtained respectively using the input sequences $\mathbf{u}^{\circ}_{t,t+N_c-1|t}$ and a feasible $\bar{\mathbf{u}}_{t+1,t+N_c-1|t+1}$, chosen equal to the subvector $\mathbf{u}^{\circ}_{t+1,t+N_c-1|t}$. Assuming that $\hat{x}_{t+j|t} \in X \backsim \mathcal{B}(\,(L_{f_x}^{j}-1)/(L_{f_x}-1)\bar{d})$, let us introduce $\eta \in \mathcal{B}(\,(L_{f_x}^{j-1}-1)/(L_{f_x}-1)\bar{d})$. Let $\xi \triangleq \hat{x}_{t+j|t+1} - \hat{x}_{t+j|t} + \eta$, then, in view of Assumption 1 and thanks to (16), it follows that

$$|\xi| \leq |\hat{x}_{t+j|t+1} - \hat{x}_{t+j|t}| + |\eta| \leq \frac{L_{f_x}^{j} - 1}{L_{f_x} - 1} \bar{d}, \qquad (18)$$

and hence, $\xi \in \mathcal{B}(\,(L_{f_x}^{j} - 1)/(L_{f_x} - 1)\bar{d})$. Since $\hat{x}_{t+j|t} \in X_{t+j|t}$, it follows that $\hat{x}_{t+j|t} + \xi = \hat{x}_{t+j|t+1} + \eta \in X$, $\forall \eta \in \mathcal{B}(\,(L_{f_x}^{j-1}-1)/(L_{f_x}-1)\bar{d})$, yielding to $\hat{x}_{t+j|t+1} \in X_{t+j|t+1}$.

iv) $\hat{x}_{t+N_c|t} \in X_f \Rightarrow \hat{x}_{t+N_c+1|t+1} \in X_{t+N_c+1|t+1}$: Since $\bar{d}$ is assumed to fulfill the following inequality

$$\bar{d} \leq \frac{L_{f_x} - 1}{L_{f_x}^{N_c} - 1} \mathrm{dist}(\mathbb{R}^n \backslash X, X_f),$$

then, in view of (8), $X_{t+N_c|t} = X_{t+N_c+1|t+1} \supseteq X_f$. In this regard, Point ii) ensures that $x_{t+1} \in X_{RH}$, and hence $\hat{x}_{t+N_c+1|t+1} \in X_f \subseteq X_{t+N_c+1|t+1}$.

Finally, in view of Points i)–iv), it follows that X_{RH} is RPI for bounded additive uncertainties $d \in \mathcal{B}(\bar{d})$, with $\bar{d}$ defined as in the statement of the theorem. ■

APPENDIX III

First, let us introduce the following

Theorem 3.1 (Geometric Condition for Invariance,[6]): A set $\Xi \in \mathbb{R}^n$ is a control/positively invariant set if and only if $\Xi \subseteq \mathcal{C}_1(\Xi)$. □

In a numerical framework, the geometric condition for set invariance stated in Theorem 3.1 must be replaced by its robust counterpart, i.e. $\Xi \subseteq \mathcal{RC}_1(\Xi)$.

Now, it is possible to prove Theorem 6.2

Proof: [Theorem 6.2] Points i), ii) and iii) of Problem 6.1 are addressed separately in the following.

i) $\hat{\mathcal{C}}_i(X_f)_j \subset \mathcal{C}_i(X_f) \Rightarrow \hat{\mathcal{C}}_i(X_f)_{j+1} \subset \mathcal{C}_i(X_f), \forall j \in \mathbb{Z}_{\geq 1}$: Given a vector $x' \in \hat{\mathcal{C}}_i(X_f)_j \oplus \mathcal{B}(L_{f_x}^{-i} \underline{J}_j)$, with $\underline{J}_j \triangleq \min_{x \in \partial \hat{\mathcal{C}}_i(X_f)_j} J^{\circ}_{FHD}(x, \mathbf{u}^{\circ}_{0,i})$, then $\exists x'' \in \partial \hat{\mathcal{C}}_i(X_f)_j$ ($\subset \mathcal{C}_i(X_f)$) such that $|x' - x''| \leq L_{f_x}^{-i} \underline{J}_j$. Hence, there exists a feasible sequence of controls $\bar{\mathbf{u}}_{0,i-1}$ which yields to $\hat{x}(x'', \bar{\mathbf{u}}_{0,i-1}, i) \in X_f \backsim \mathcal{B}(\underline{J}_j)$, with $\underline{J}_j \in \mathbb{R}_{\geq 0}$. Then, under Assumption 1, the inequality $|\hat{x}(x', \bar{\mathbf{u}}_{0,i-1}, i) - \hat{x}(x'', \bar{\mathbf{u}}_{0,i-1}, i)| \leq L_{f_x}^{i} |x' - x''|$, yields to $\hat{x}(x', \mathbf{u}_{0,i-1}, i) \in X_f$, and hence $x' \in \mathcal{C}_i(X_f)$. These arguments also prove the second inequality at Point iii) of Problem 6.1.

ii) $\mathrm{dist}(\mathbb{R}^n \backslash \hat{\mathcal{C}}_i(X_f)_{j+1}, X_f) < \mathrm{dist}(\mathbb{R}^n \backslash \mathcal{RC}_i(X_f), X_f) \Rightarrow \hat{\mathcal{C}}_i(X_f)_{j+1} \supset \hat{\mathcal{C}}_i(X_f)_j$ ($\supset X_f$), $\forall j \in \mathbb{Z}_{\geq 1}$: Under the stated assumption, being $\underline{J}_j \in \mathbb{R}_{>0}$, the subsequent inclusion follows from the properties of Minkowski addition. If $\underline{J}_j = 0$ for some $j \in \mathbb{Z}_{\geq 1}$, then $\hat{\mathcal{C}}_i(X_f)_{j+1} = \hat{\mathcal{C}}_i(X_f)_j, \forall j \in \mathbb{Z}_{\geq} j$, and hence the limit is finitely determined.

iii) $\mathrm{dist}(\mathbb{R}^n \backslash \mathcal{RC}_i(X_f), X_f) \leq \lim_{j \to \infty} \mathrm{dist}(\mathbb{R}^n \backslash \hat{\mathcal{C}}_i(X_f)_j, X_f)$: The proof can be carried out by contradiction. First, by Point ii), notice that $\vartheta(\hat{\mathcal{C}}_i(X_f)_\infty) = \hat{\mathcal{C}}_i(X_f)_\infty$. Assume there exists $\epsilon' \in \mathbb{R}_{>0}$ such that $\lim_{j \to \infty} \mathrm{dist}(\mathbb{R}^n \backslash \hat{\mathcal{C}}_i(X_f)_j, X_f) = \mathrm{dist}(\mathbb{R}^n \backslash \mathcal{RC}_i(X_f), X_f) - \epsilon'$. Then, the set $\partial \hat{\mathcal{C}}_i(X_f)_\infty \subset \mathcal{C}_i(\mathrm{int}(X_f))$. Since there exists $\epsilon'' \in \mathbb{R}_{>0}$ such that $\min_{x \in \partial \hat{\mathcal{C}}_i(X_f)_\infty} J^{\circ}_{FHD}(x, \mathbf{u}^{\circ}_{0,i}) \geq \epsilon''$, then there exist a set $\vartheta(\hat{\mathcal{C}}_i(X_f)_\infty) \supset \hat{\mathcal{C}}_i(X_f)_\infty$, which invalidates the original assumption. ■

ACKNOWLEDGMENT

The authors would like to thank prof. Franco Blanchini for his valuable, constructive and thorough comments.

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