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Attitude Consensus with Fixed Rotational Axis
via Energy Dissipation

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Abstract

A consensus control framework for the rotational dynamics of multiple spacecraft is developed. The approach is energy based and guarantees asymptotic convergence of relative states between the spacecraft. Since the proposed control law emulates internal forces between the spacecraft, the overall angular momentum is kept constant. Furthermore, it is shown that the angular velocities of each spacecraft can be made to asymptotically converge towards a constant value, i.e. each spacecraft can be made to rotate around a fixed common rotational axis.

1. Introduction

One of the primary reasons of recent huge interest in consensus control is due to its simplicity of the framework. Specifically, the control laws given in [1] constitute emulation of a mass-spring-damper system so that the agents eventually converge to each other. This entire behavior can be explained by employing the energy function defined in terms of the inertial frame and by showing that it decreases until the relative velocities between the agents coincide. Furthermore, the Krasovskii-LaSalle's invariance principle is invoked to show convergence of relative positions as well.

In contrast to the case of translational motion of particles, consensus of attitudes during the rotational motion of rigid bodies is not trivial although they have various applications such as a coordinated cluster of satellites carrying telescopes for astronomical interferometry and enhancing resolution compared to a single satellite. The application of attitude synchronization also extends to a fleet of sensor-equipped underwater vehicles that move together in an organized pattern to identify and track features in ocean [2].

The difficulty in attitude synchronization is the fact that if the mass moment of inertia is not spatially isotropic, the Euler equation describing rotational motion becomes inherently nonlinear. Synchronization/consensus on non-Euclidean manifolds (e.g., circle, SO(3)-a subgroup of orthogonal matrices with determinant +1) gives birth to various interesting phenomena and are discussed in [3–6]. Specifically, there have been a number of researches (e.g., [6–12]) addressing distributed synchronization problems on SO(3). Most of these works on SO(3), using the absolute states with respect to the inertial frame of each agent, either make the agents follow an externally given trajectory or take a leader-follower approach. In [10] the authors showed that attitude consensus can be achieved using only the relative states between the rigid bodies. However, they consider only the kinematic equation and the Euler equation is disregarded. In contrast, [11, 12] explicitly consider the Euler equation while aiming for attitude synchronization on SO(3).

Our research not only considers the Euler equation but also, to the best of our knowledge, produces the first result in attitude synchronization that uses a simple and intuitive control framework which emulates torques due to springs and dampers between the agents. We also show that the angular velocities of each agent can be made to asymptotically converge towards a constant value, i.e., the agents can be made to rotate around a fixed rotational axis in the inertial frame, by applying two types of the virtual (springs and) dampers in different timings.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, and $\mathbb{N}_0$ denotes the set of nonnegative integers. Furthermore, we write $(\cdot)^{\mathrm{T}}$ for transpose, $A_{(i,j)}$ for the (i,j)th (block) element of the matrix A, 1_n for the ones vector of dimension n, mspec(A) for the spectrum of the matrix A, and $|\mathcal{N}|$ for the cardinal number of the set $\mathcal{N}$.

2. Kinetic and Kinematic Equations of Space-craft

Consider the nonlinear dynamical system representing controlled n rigid spacecraft given by

$$I_{\mathrm{b}}\dot{\omega}_i(t) = -\omega_i^{\times}(t)I_{\mathrm{b}}\omega_i(t) + u_i(t), \quad \omega_i(0) = \omega_{i0},$$
$$t \geq 0, \quad i = 1, \ldots, n, \quad (1)$$

where $\omega_i(t) \in \mathbb{R}^3$ represents the angular velocity of the spacecraft i with respect to the body-fixed frame, $I_{\mathrm{b}} \triangleq \mathrm{diag}[I_{\mathrm{b}1}, I_{\mathrm{b}2}, I_{\mathrm{b}3}]$ is a positive-definite inertia matrix common to all the spacecraft, $u_i(t) \in \mathbb{R}^3$ is the input vector with control inputs providing body-fixed torques about three mutually perpendicular axes defining the body-fixed frame of the spacecraft i, and the notation $w^{\times}$ denotes the skew-symmetric matrix

$$w^{\times} \triangleq \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix},$$

representing cross-product. Furthermore, the kinematic equations of the rigid spacecraft i, $i = 1, \ldots, n$, are given by

$$\dot{\hat{\eta}}_i(t) = \tfrac{1}{2}(\bar{\eta}_i(t)\omega_i(t) - \omega_i^{\times}(t)\hat{\eta}_i(t)), \quad (2)$$
$$\dot{\bar{\eta}}_i(t) = -\tfrac{1}{2}\omega_i^{\mathrm{T}}(t)\hat{\eta}_i(t), \quad (3)$$

where $\eta_i \triangleq [\hat{\eta}_i^{\mathrm{T}}, \bar{\eta}_i]^{\mathrm{T}} \in \mathbb{R}^3 \times \mathbb{R}$ denotes Euler parameters representing the orientation of the spacecraft i with respect to the inertial frame $\mathcal{I}$. The corresponding rotation matrix $C(\eta_i) \in \mathrm{SO}(3)$, is given by

$$C(\eta_i) = (\bar{\eta}_i^2 - \hat{\eta}_i^{\mathrm{T}}\hat{\eta}_i)I_3 + 2\hat{\eta}_i\hat{\eta}_i^{\mathrm{T}} - 2\bar{\eta}_i\hat{\eta}_i^{\times}, \quad i = 1, \ldots, n. \quad (4)$$

Note that

$$\dot{C}(\eta_i(t)) = -\omega_i^{\times}(t)C(\eta_i(t)), \quad t \geq 0, \quad i = 1, \ldots, n. \quad (5)$$

3. Attitude Consensus Control for Multiple Spacecraft

In the preceding work [13, 14], Tanner *et al.* considered an energy-based controller that emulates forces due to springs and dampers for translational motion of particle systems. Adopting a similar idea, in this paper we consider the control law given by

$$u_i(t) = -\frac{1}{|\mathcal{N}_i|}\sum_{j \in \mathcal{N}_i}\hat{\eta}_j^{-1}(t)\hat{\eta}_i(t)$$
$$- \frac{1}{|\mathcal{N}_i|}\sum_{j \in \mathcal{N}_i}(\omega_i(t) - C(\eta_i(t))C^{\mathrm{T}}(\eta_j(t))\omega_j(t)), \quad (6)$$

for the rotation dynamics (1)–(3), where $\mathcal{N}_i \subset \{1, \ldots, n\}\backslash\{i\}$ represents the set of agents which the

agent i can communicate with and $\hat{\eta}_j^{-1}\hat{\eta}_i \in \mathbb{R}^3$ is the quaternion error between quarternions η_1, η_2 given by

$$\hat{\eta}_j^{-1}\hat{\eta}_i \triangleq \bar{\eta}_j\hat{\eta}_i - \bar{\eta}_i\hat{\eta}_j - \hat{\eta}_j^{\times}\hat{\eta}_i. \quad (7)$$

Note that $\hat{\eta}_j^{-1}\hat{\eta}_i$ is a torque along the rotational axis of the rotation matrix $C(\eta_i)C^{\mathrm{T}}(\eta_j)$. It is also assumed that if the agent i can communicate with the agent j, then the agent j can also communicate with the agent i.

The control law (6) states that the torque input $u_i(t)$ of the agent i is computed with the relative state measurements with respect to the other agents specified by $\mathcal{N}_i$. Note that this control law (6) can also be written as

$$u(t) = -[(A_{\mathrm{adj}} \otimes I_3)\hat{\eta}^{-1}(t)]\hat{\eta}(t)$$
$$- \tilde{C}(\eta(t))(L \otimes I_3)\tilde{C}^{\mathrm{T}}(\eta(t))\omega(t), \quad (8)$$

where $u = [u_1^{\mathrm{T}}, \ldots, u_n^{\mathrm{T}}]^{\mathrm{T}}$, $\eta \triangleq (\hat{\eta}, \bar{\eta}) \in \mathbb{R}^{3n} \times \mathbb{R}^n$, $\hat{\eta} \triangleq [\hat{\eta}_1^{\mathrm{T}}, \ldots, \hat{\eta}_n^{\mathrm{T}}]^{\mathrm{T}}$, $\bar{\eta} \triangleq [\bar{\eta}_1, \ldots, \bar{\eta}_n]^{\mathrm{T}}$, $\hat{\eta}^{-1} \triangleq [(\hat{\eta}_1^{-1})^{\mathrm{T}}, \ldots, (\hat{\eta}_n^{-1})^{\mathrm{T}}]^{\mathrm{T}}$, $\omega \triangleq [\omega_1^{\mathrm{T}}, \ldots, \omega_n^{\mathrm{T}}]^{\mathrm{T}}$, $\tilde{C}(\eta) \triangleq$ block-diag$[C(\eta_1), \ldots, C(\eta_n)]$, $L \in \mathbb{R}^{n \times n}$ is the normalized Laplacian matrix defined as

$$L_{(i,j)} \triangleq \begin{cases} 1, & i = j, \\ -\frac{1}{|\mathcal{N}_i|}, & j \in \mathcal{N}_i, \\ 0, & \text{otherwise}, \end{cases} \quad (9)$$

and $A_{\mathrm{adj}} \triangleq D - L$ is the adjacency matrix associated with the Laplacian L and the associated degree matrix D. Note that if the graph that a Laplacian matrix L corresponds to is connected, then L possesses a simple zero eigenvalue and the corresponding eigenvector can be 1_n.

Theorem 3.1. Consider the rotational dynamics of n rigid spacecraft given by (1) with kinematic equations given by (2), (3). Let $\mathcal{N}_1, \ldots, \mathcal{N}_n$ be such that resulting Laplacian represents a connected graph. Then the feedback control law given by (6) guarantees

$$\lim_{t \to \infty} C(\eta_i(t))C^{\mathrm{T}}(\eta_j(t)) = I_3, \quad i, j = 1, \ldots, n. \quad (10)$$

Proof. First, note that with $u_i(t)$ given by (6) it follows that

$$I_{\mathrm{b}}\dot{\omega}_i(t) = -\omega_i^{\times}(t)I_{\mathrm{b}}\omega_i(t) - \frac{1}{|\mathcal{N}_i|}\sum_{j \in \mathcal{N}_i}\hat{\eta}_j^{-1}(t)\hat{\eta}_i(t)$$
$$- \frac{1}{|\mathcal{N}_i|}\sum_{j \in \mathcal{N}_i}(\omega_i(t) - C(\eta_i(t))C^{\mathrm{T}}(\eta_j(t))\omega_j(t)),$$
$$\omega_i(0) = \omega_{i0}, \quad t \geq 0, \quad i = 1, \ldots, n, \quad (11)$$

or, equivalently,

$$(I_n \otimes I_{\mathrm{b}})\dot{\omega}(t)$$

$$
\begin{aligned}
= \ & -\text{block-diag}[\omega_1^\times(t), \dots, \omega_n^\times(t)](I_n \otimes I_b)\omega(t) \\
& - [(A_{\text{adj}} \otimes I_3)\hat{\eta}^{-1}(t)]\hat{\eta}(t) \\
& - \tilde{C}(\eta(t))(L \otimes I_3)\tilde{C}^{\mathrm{T}}(\eta(t))\omega(t), \\
& \qquad \omega(0) = [\omega_{10}^{\mathrm{T}}, \dots, \omega_{n0}^{\mathrm{T}}]^{\mathrm{T}}, \quad t \ge 0. \quad (12)
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{i=1}^n & C^{\mathrm{T}}(\eta_i(t))u_i(t) \\
&= (1_n^{\mathrm{T}} \otimes I_3)\tilde{C}^{\mathrm{T}}(\eta)u(t) \\
&= -(1_n^{\mathrm{T}} \otimes I_3)[(A_{\text{adj}} \otimes I_3)\hat{\eta}^{-1}(t)]\hat{\eta}(t) \\
& \quad - (1_n^{\mathrm{T}} \otimes I_3)(L \otimes I_3)\tilde{C}^{\mathrm{T}}(\eta(t))\omega(t) \\
&= 0, \quad t \ge 0, \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (13)
\end{aligned}
$$

it follows that the angular momentum of the overall system conserves so that the angular momentum of the system in the inertial frame is given by

$$
\begin{aligned}
L_0 \ &= \ \sum_{k=1}^n C^{\mathrm{T}}(\eta_{k0})I_b\omega_{k0} \\
&\equiv \ \sum_{k=1}^n C^{\mathrm{T}}(\eta_k(t))I_b\omega_k(t). \quad (14)
\end{aligned}
$$

Next, consider the energy-like function

$$
\begin{aligned}
\mathcal{H}(\omega, \eta) \ = \ & \frac{1}{2}\sum_{i=1}^n \omega_i^{\mathrm{T}}I_b\omega_i \\
& + \frac{1}{2}\sum_{i=1}^n \sum_{j \in \mathcal{N}_i}(\hat{\eta}_i - \hat{\eta}_j)^{\mathrm{T}}(\hat{\eta}_i - \hat{\eta}_j) \\
& + \frac{1}{2}\sum_{i=1}^n \sum_{j \in \mathcal{N}_i}(\bar{\eta}_i - \bar{\eta}_j)^2 \\
= \ & \frac{1}{2}\omega^{\mathrm{T}}(I_n \otimes I_b)\omega + \frac{1}{2}\hat{\eta}^{\mathrm{T}}(t)(L \otimes I_3)\hat{\eta}(t) \\
& + \frac{1}{2}\bar{\eta}^{\mathrm{T}}(t)L\bar{\eta}(t). \quad (15)
\end{aligned}
$$

Note that since I_b is positive definite and L is nonnegative definite, $\mathcal{H}(\omega, \eta) \ge 0$ for all $(\omega, \eta) \in \mathbb{R}^{3n} \times \mathbb{R}^{4n}$. Now, letting $\eta(t)$, $\omega(t)$, denote the solution to (2), (3), and (12), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$
\begin{aligned}
\dot{\mathcal{H}}&(\omega(t), \eta(t)) \\
= \ & \sum_{i=1}^n \omega_i^{\mathrm{T}}(t)I_b\dot{\omega}_i(t) \\
& + \sum_{i=1}^n \sum_{j \in \mathcal{N}_i}(\hat{\eta}_i(t) - \hat{\eta}_j(t))^{\mathrm{T}}(\dot{\hat{\eta}}_i(t) - \dot{\hat{\eta}}_j(t)) \\
& + \sum_{i=1}^n \sum_{j \in \mathcal{N}_i}(\bar{\eta}_i(t) - \bar{\eta}_j(t))(\dot{\bar{\eta}}_i(t) - \dot{\bar{\eta}}_j(t))
\end{aligned}
$$

$$
\begin{aligned}
= \ & \sum_{i=1}^n \omega_i^{\mathrm{T}}(t)(-\omega_i^\times(t)I_b\omega_i(t) - \hat{\eta}_j^{-1}(t)\hat{\eta}_i(t) \\
& - (\omega_i(t) - C(\hat{\eta}_i(t))C^{\mathrm{T}}(\eta_j(t))\omega_j(t))) \\
& + \sum_{i=1}^n \sum_{j \in \mathcal{N}_i}(\hat{\eta}_i(t) - \hat{\eta}_j(t))^{\mathrm{T}}((\bar{\eta}_i(t)\omega_i(t) - \omega_i^\times(t)\hat{\eta}_i(t)) \\
& - (\bar{\eta}_j(t)\omega_j(t) - \omega_j^\times(t)\hat{\eta}_j(t))) \\
& - \sum_{i=1}^n \sum_{j \in \mathcal{N}_i}(\bar{\eta}_i(t) - \bar{\eta}_j(t))(\omega_i^{\mathrm{T}}(t)\hat{\eta}_i(t) - \omega_j^{\mathrm{T}}(t)\hat{\eta}_j(t)) \\
= \ & \sum_{i=1}^n \sum_{j \in \mathcal{N}_i}(-\omega_i^{\mathrm{T}}(t)\hat{\eta}_j^{-1}(t)\hat{\eta}_i(t) - \omega_i^{\mathrm{T}}(t)\omega_i(t) \\
& - \omega_i^{\mathrm{T}}(t)C(\hat{\eta}_i(t))C^{\mathrm{T}}(\eta_j(t))\omega_j(t) \\
& + \omega_i^{\mathrm{T}}(t)\hat{\eta}_j^{-1}(t)\hat{\eta}_i(t)) \\
= \ & -\sum_{i=1}^n \sum_{j \in \mathcal{N}_i}[(C^{\mathrm{T}}(\eta_i(t))\omega_i(t) - C^{\mathrm{T}}(\eta_j(t))\omega_j(t))^{\mathrm{T}} \\
& \cdot (C^{\mathrm{T}}(\eta_i(t))\omega_i(t) - C^{\mathrm{T}}(\eta_j(t))\omega_j(t))] \\
\le \ & 0, \quad t \ge 0. \quad\quad\quad\quad\quad\quad\quad\quad (16)
\end{aligned}
$$

Hence, it follows from Theorem 4.4 of [15] that

$$
\begin{aligned}
\lim_{t \to \infty}(C^{\mathrm{T}}(\eta_i(t))\omega_i(t) - C^{\mathrm{T}}(\eta_j(t))\omega_j(t)) \ = \ & 0, \\
& i, j = 1, \dots, n. \quad (17)
\end{aligned}
$$

To consider the attitude consensus, consider

$$
\begin{aligned}
\mathcal{R} \ \triangleq \ & \{(\omega, \eta) \in \mathbb{R}^{3n} \times \mathbb{R}^{4n} : \\
& C^{\mathrm{T}}(\eta_i(t))\omega_i(t) - C^{\mathrm{T}}(\eta_j(t))\omega_j(t) = 0, \\
& i, j = 1, \dots, n\}, \quad (18)
\end{aligned}
$$

and and let $\mathcal{M}$ be the largest invariant set contained in $\mathcal{R}$. Note that for the system to guarantee the condition $\dot{\mathcal{H}}(\omega(t), \eta(t)) = 0$, the trajectory of the system must lie on the set $\mathcal{R}$.

In this case, since

$$
C^{\mathrm{T}}(\eta_i(t))\dot{\omega}_i(t) - C^{\mathrm{T}}(\eta_j(t))\dot{\omega}_j(t) \equiv 0, \quad i, j = 1, \dots, n, \quad (19)
$$

holds, it follows from

$$
\begin{aligned}
(I_n &\otimes I_b)\dot{\omega}(t) \\
= \ & -\text{block-diag}[\omega_1^\times(t), \dots, \omega_n^\times(t)](I_n \otimes I_b)\omega(t) \\
& - \tilde{C}(\eta(t))(L \otimes I_3)\tilde{C}^{\mathrm{T}}(\eta(t))\omega(t), \\
& \qquad \omega(0) = [\omega_{10}^{\mathrm{T}}, \dots, \omega_{n0}^{\mathrm{T}}]^{\mathrm{T}}, \quad t \ge 0, \quad (20)
\end{aligned}
$$

that

$$
\begin{aligned}
0 \ = \ & I_b^{-1}\omega_i^\times I_b\omega_i - C(\eta_i)C^{\mathrm{T}}(\eta_j)I_b^{-1}\omega_j^\times I_b\omega_j \\
& + (I_3 + C(\eta_i)C^{\mathrm{T}}(\eta_j))I_b^{-1}\hat{\eta}_j^{-1}\hat{\eta}_i, \\
= \ & I_b^{-1}[I_3 - I_bC(\eta_i)C^{\mathrm{T}}(\eta_j)I_b^{-1}C(\eta_j)C^{\mathrm{T}}(\eta_i)]\omega_i^\times I_b\omega_i \\
& + (I_3 + C(\eta_i)C^{\mathrm{T}}(\eta_j))I_b^{-1}\hat{\eta}_j^{-1}\hat{\eta}_i, \\
& \qquad\qquad\qquad\qquad\qquad i, j = 1, \dots, n. \quad (21)
\end{aligned}
$$

Note that (21) is rewritten as

$$0 = I_{\mathrm{b}}^{-1}[I_3 - I_{\mathrm{b}}C(\eta_j^{-1}\eta_i)I_{\mathrm{b}}^{-1}C^{\mathrm{T}}(\eta_j^{-1}\eta_i)]\omega_i^\times I_{\mathrm{b}}\omega_i$$
$$+ (I_3 + C(\eta_j^{-1}\eta_i))I_{\mathrm{b}}^{-1}\hat{\eta}_j^{-1}\hat{\eta}_i, \quad i,j = 1,\dots,n,$$
$$(22)$$

where $\eta_j^{-1}\eta_i \triangleq [(\hat{\eta}_j^{-1}\hat{\eta}_i)^{\mathrm{T}}, \hat{\eta}_j^{\mathrm{T}}\hat{\eta}_i + \bar{\eta}_j\bar{\eta}_j]^{\mathrm{T}}$. This equality holds only if

$$\hat{\eta}_j^{-1}(t)\hat{\eta}_i(t) \equiv 0, \tag{23}$$

and hence it further follows that

$$\lim_{t\to\infty}\hat{\eta}_i(t) = \lim_{t\to\infty}\hat{\eta}_j(t), \quad i,j = 1,\dots,n, \tag{24}$$

which implies (10). $\qquad\square$

Note that (22) holds in $\mathcal{M}$ irrespective of whether the term $\omega_i^\times I_{\mathrm{b}}\omega_i$ is zero or not, which implies that $\dot{\omega}_i(t)$ does not necessarily goes to 0, i.e., the angular velocities of the spacecraft does not asymptotically converge towards a constant value.

It is known that the angular velocity of a single rigid body rotating with energy dissipation and conserved angular momentum asymptotically aligns with the largest principal axis of the rigid body. This interesting phenomenon, called the major axis rule, seems extendable to our multiple rigid body problem since our proposed controller conserves the overall angular momentum with its spring damper emulations and dissipates energy with dampers. This phenomenon is adopted in the next section which provides a control framework to bring the angular velocities of each spacecraft to a constant value, i.e., to make each spacecraft rotate around a fixed common axis in the inertial frame.

4. Synchronized Rotation around a Fixed Axis

In this section we characterize the control law that resembles the case where there is no 'stiffness' term in our controller. Specifically, we remove the first term in the right-hand side of (6) so that the control law in the following theorem is given by

$$u_i(t) = -\frac{1}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i}(\omega_i(t) - C(\eta_i(t))C^{\mathrm{T}}(\eta_j(t))\omega_j(t)). \tag{25}$$

Theorem 4.1. Consider the rotation dynamics of n rigid spacecraft given by (1) with kinematic equations given by (2), (3). Let $\mathcal{N}_1,\dots,\mathcal{N}_n$ be such that resulting Laplacian represents a connected graph. Then the feedback control law given by (25) guarantees conservation of angular momentum and makes either of the following hold:

i) $\lim_{t\to\infty}\omega_i^\times(t)I_{\mathrm{b}}\omega_i(t) = 0$, $i = 1,\dots,n$;

ii) $\lim_{t\to\infty}C(\eta_i(t))C^{\mathrm{T}}(\eta_j(t)) = I_3$ or $-I_3$, $i,j = 1,\dots,n$.

Proof. First, note that with $u_i(t)$ given by (25) it follows that

$$I_{\mathrm{b}}\dot{\omega}_i(t) = -\omega_i^\times(t)I_{\mathrm{b}}\omega_i(t)$$
$$-\frac{1}{|\mathcal{N}_i|}\sum_{j\in\mathcal{N}_i}(\omega_i(t) - C(\eta_i(t))C^{\mathrm{T}}(\eta_j(t))\omega_j(t)),$$
$$\omega_i(0) = \omega_{i0}, \quad t \geq 0, \quad i = 1,\dots,n, \tag{26}$$

or, equivalently (20). Since

$$\sum_{i=1}^{n}C^{\mathrm{T}}(\eta_i(t))u_i(t)$$
$$= (1_n^{\mathrm{T}}\otimes I_3)\tilde{C}^{\mathrm{T}}(\eta)u(t)$$
$$= -(1_n^{\mathrm{T}}\otimes I_3)(L\otimes I_3)\tilde{C}^{\mathrm{T}}(\eta(t))\omega(t)$$
$$= 0, \tag{27}$$

it follows that the angular momentum of the overall system conserves so that the angular momentum of the system in the inertial frame is given by (14). Next, consider the Lyapunov-like function

$$V(\omega) = \frac{1}{2}\sum_{i=1}^{n}\omega_i^{\mathrm{T}}I_{\mathrm{b}}\omega_i$$
$$= \frac{1}{2}\omega^{\mathrm{T}}(I_n\otimes I_{\mathrm{b}})\omega. \tag{28}$$

Note that since I_{b} is positive definite, $V(\omega) > 0$ for all $\omega \in \mathbb{R}^{3n}\backslash\{0\}$. Now, letting $\omega(t)$ denote the solution to (20), it follows that the time derivative of (28) along the closed-loop system trajectories is given by

$$\dot{V}(\omega(t),\eta(t))$$
$$= \sum_{i=1}^{n}\omega_i^{\mathrm{T}}(t)I_{\mathrm{b}}\dot{\omega}_i(t)$$
$$= \sum_{i=1}^{n}\omega_i^{\mathrm{T}}(t)(-\omega_i^\times(t)I_{\mathrm{b}}\omega_i(t)$$
$$- (\omega_i(t) - C(\hat{\eta}_i(t))C^{\mathrm{T}}(\eta_j(t))\omega_j(t)))$$
$$= -\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i}[(C^{\mathrm{T}}(\eta_i(t))\omega_i(t) - C^{\mathrm{T}}(\eta_j(t))\omega_j(t))^{\mathrm{T}}$$
$$\cdot (C^{\mathrm{T}}(\eta_i(t))\omega_i(t) - C^{\mathrm{T}}(\eta_j(t))\omega_j(t))]$$
$$\leq 0, \quad t \geq 0. \tag{29}$$

Hence, it follows from Theorem 4.1 of [16] that the trajectory $(\omega(t),\eta(t)) \equiv (0,\cdot)$ of the closed-loop system given by (2), (3), and (20) is Lyapunov stable with respect to ω and hence ω is bounded. Furthermore, it follows from Theorem 4.4 of [15] that

$$\lim_{t\to\infty}(C^{\mathrm{T}}(\eta_i(t))\omega_i(t) - C^{\mathrm{T}}(\eta_j(t))\omega_j(t)) = 0,$$
$$i,j = 1,\dots,n. \tag{30}$$

Now, consider $\mathcal{R}$ given by (18) and let $\mathcal{M}$ be the largest invariant set contained in $\mathcal{R}$. Note that for the

system to guarantee the condition $\dot{V}(\omega(t), \eta(t)) = 0$, the trajectory of the system must lie on the set $\mathcal{R}$. Hence, in this case, using (5), it follows that (19) holds in $\mathcal{M}$ and hence it follows from (20) that

$$0 = C^{\mathrm{T}}(\eta_i)I_{\mathrm{b}}^{-1}\omega_i^{\times}I_{\mathrm{b}}\omega_i - C^{\mathrm{T}}(\eta_j)I_{\mathrm{b}}^{-1}\omega_j^{\times}I_{\mathrm{b}}\omega_j,$$
$$i, j = 1, \ldots, n, \quad (31)$$

holds in $\mathcal{M}$. Note that $\omega_i^{\times}I_{\mathrm{b}}\omega_i = 0$, $i = 1, \ldots, n$, satisfy (31) and hence $\{(\omega, \eta) \in \mathbb{R}^{3n} \times \mathbb{R}^{4n} : \omega_i^{\times}I_{\mathrm{b}}\omega_i = 0, \quad i, j = 1, \ldots, n\}$ is an invarient set in $\mathcal{M}$.

Now, suppose $\omega_i^{\times}I_{\mathrm{b}}\omega_i \neq 0$, $i = 1, \ldots, n$. In this case, using (18), it follows that

$$\begin{aligned}
0 &= C^{\mathrm{T}}(\eta_i)I_{\mathrm{b}}^{-1}\omega_i^{\times}I_{\mathrm{b}}\omega_i - C^{\mathrm{T}}(\eta_j)I_{\mathrm{b}}^{-1}\omega_j^{\times}I_{\mathrm{b}}\omega_j \\
&= C^{\mathrm{T}}(\eta_i)[I_{\mathrm{b}}^{-1}\omega_i^{\times}I_{\mathrm{b}}\omega_i - C(\eta_i)C^{\mathrm{T}}(\eta_j)I_{\mathrm{b}}^{-1} \\
&\quad \cdot (C(\eta_j)C^{\mathrm{T}}(\eta_i)\omega_i)^{\times}I_{\mathrm{b}}C(\eta_j)C^{\mathrm{T}}(\eta_i)\omega_i] \\
&= C^{\mathrm{T}}(\eta_i)I_{\mathrm{b}}^{-1}[\omega_i^{\times}I_{\mathrm{b}}\omega_i - I_{\mathrm{b}}C(\eta_i)C^{\mathrm{T}}(\eta_j)I_{\mathrm{b}}^{-1} \\
&\quad \cdot C(\eta_j)C^{\mathrm{T}}(\eta_i)\omega_i^{\times}I_{\mathrm{b}}\omega_i] \\
&= C^{\mathrm{T}}(\eta_i)I_{\mathrm{b}}^{-1}[I_3 - I_{\mathrm{b}}C(\eta_i)C^{\mathrm{T}}(\eta_j)I_{\mathrm{b}}^{-1} \\
&\quad \cdot C(\eta_j)C^{\mathrm{T}}(\eta_i)]\omega_i^{\times}I_{\mathrm{b}}\omega_i, \quad i = 1, \ldots, n, \quad (32)
\end{aligned}$$

and thus, it follows that either $C(\eta_i)C^{\mathrm{T}}(\eta_j) = I_3$ satisfies (31). $\qquad\square$

Finally, note from (14) that the system of n spacecraft has the constant angular moment L_0 depending on the initial condition of the system. Now, together with (14), it follows that

$$\begin{aligned}
L_0 &\equiv \sum_{k=1}^{n} C^{\mathrm{T}}(\eta_k(t))I_{\mathrm{b}}\omega_k(t) \\
&= \sum_{k=1}^{n} C^{\mathrm{T}}(\eta_k(t))I_{\mathrm{b}}C(\eta_k(t))C^{\mathrm{T}}(\eta_k(t))\omega_k(t) \\
&= \lim_{t \to \infty} \left(\sum_{k=1}^{n} C^{\mathrm{T}}(\eta_k(t))I_{\mathrm{b}}C(\eta_k(t)) \right)C^{\mathrm{T}}(\eta_i(t))\omega_i(t), \\
&\qquad\qquad i = 1, \ldots, n. \quad (33)
\end{aligned}$$

It is important to note that $i)$ in Theorem 4.1 indicates that each of the spacecraft rotates about one of its principal axes in the case where $I_{\mathrm{b}1} \leq I_{\mathrm{b}2} < I_{\mathrm{b}3}$. Specifically, even when rotating axes of the spacecraft are aligned, they rotate with constant offset maintained. On the other hand, when $ii)$ in Theorem 4.1 holds, orientation of all the spacecraft coincide and each of the spacecraft rotates as if it is independent of the other spacecraft.

In the case of $i)$, the constant offset can be removed by applying the 'spring' term, i.e., changing the control input from (25) to (6). Making this switching only when the angular velocities are almost aligned to the common fixed axis, the effect of the torque due to spring on the direction of the angular velocities can be made minimal and thus attitude consensus with rotation around the fixed axis is achieved.
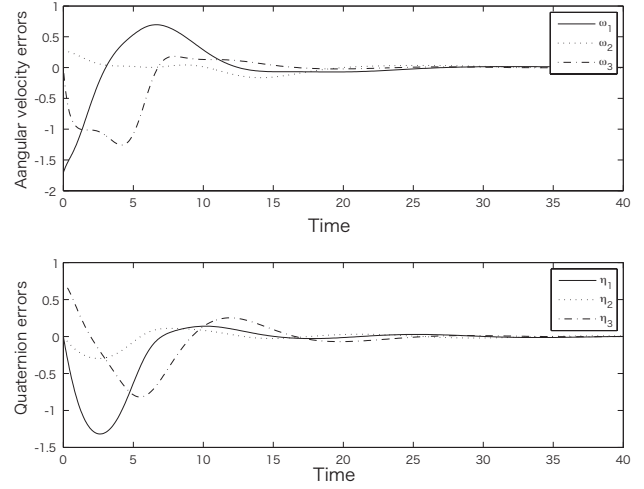

Figure 5.1: Angular velocity errors $\omega_1(t) - \omega_2(t)$ and the error of three quaternion elements $\hat{\eta}_1(t) - \hat{\eta}_2(t)$ versus time

5. Illustrative Numerical Example

Consider the two spacecraft given by (1) ($n = 2$). It follows from Theorem 3.1 that the torque inputs (6) achieve attitude consensus.

With $I_{\mathrm{b}} = \mathrm{diag}[5, 4, 1]$, and the initial conditions $\omega_1(0) = [-0.7071, 1, 0]^{\mathrm{T}}$, $\eta_1(0) = [0, 0, 0.7071, 0.7071]^{\mathrm{T}}$, $\omega_2(0) = [1, 0.7071, 0]^{\mathrm{T}}$, $\eta_2(0) = [0, 0, 0, 0]^{\mathrm{T}}$, Figure 5.1 shows the relative angular velocity $\omega_1(t) - \omega_2(t)$, and the error of the three quarternion elements $\hat{\eta}_1(t) - \hat{\eta}_2(t)$ versus time and Figure 5.2 shows the control signals versus time.

With initial conditions $\omega_1(0) = [0.5, 0, 0]^{\mathrm{T}}$, $\eta_1(0) = [0, 0, 0.7071, 0.7071]^{\mathrm{T}}$, $\omega_2(0) = [0, 0.3, 0]^{\mathrm{T}}$, $\eta_2(0) = [0, 0.7071, 0, 0.7071]^{\mathrm{T}}$, and torque input (25), Figure 5.3 shows the angular velocities $\omega_1(t)$, $\omega_2(t)$, and the error of the three quaternion elements $\hat{\eta}_1(t) - \hat{\eta}_2(t)$ versus time. Note that a single pair of elements of the angular velocity reaches a constant value and the other two pairs reaches 0. This shows that the angular velocities asymptotically align with a principle axis. Furthermore, note that the quarternion error does not converge to 0.

Now, with the same initial conditions as above Figure 5.4 shows the angular velocities $\omega_1(t)$, $\omega_2(t)$, and the error of the three quaternion elements $\hat{\eta}_1(t) - \hat{\eta}_2(t)$ versus time, when the control input is switched from (25) to (6) at time $t = 30$. This shows that not only attitude consensus but also a synchronized rotation around a fixed axis is achieved.

6. Conclusion

In this paper we considered an attitude consensus problem of multiple spacecraft. A controller that emulates torque inputs which constitute internal forces of the overall system was proposed and attitude consensus
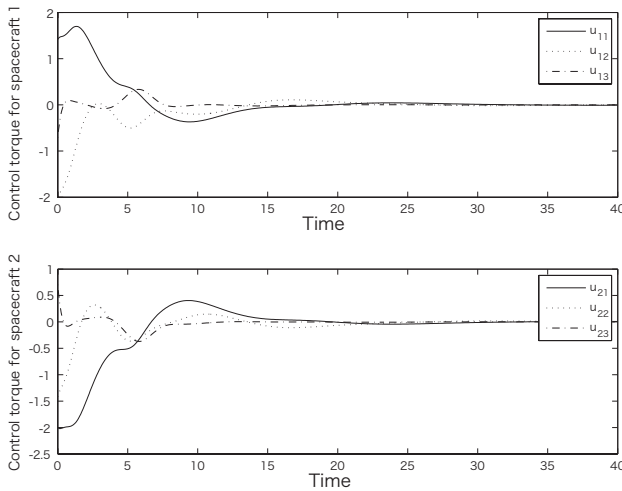
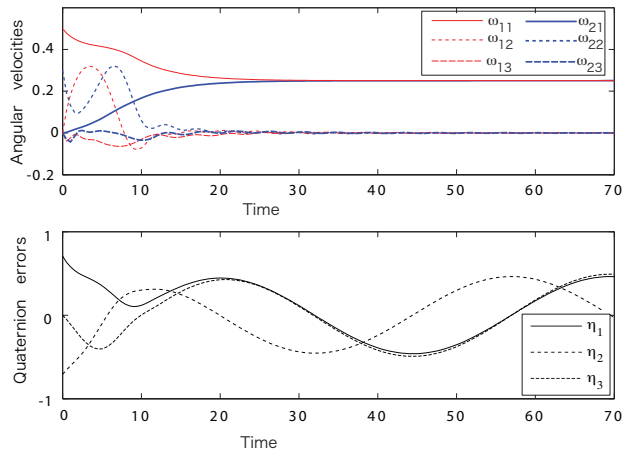**Figure 5.2**: Control torques $u_1(t)$, $u_2(t)$

Figure 5.3: Angular velocities $\omega_1(t)$, $\omega_2(t)$ and the error of three quaternion elements $\hat{\eta}_1(t) - \hat{\eta}_2(t)$ versus time

under the proposed controller was showed. Furthermore, a method to achieve common fixed rotational axis was proposed. Finally, we provided numerical examples to show the properties of the closed-loop system.

Figure 5.4: Angular velocities $\omega_1(t)$, $\omega_2(t)$ and the error of three quaternion elements $\hat{\eta}_1(t) - \hat{\eta}_2(t)$ versus time

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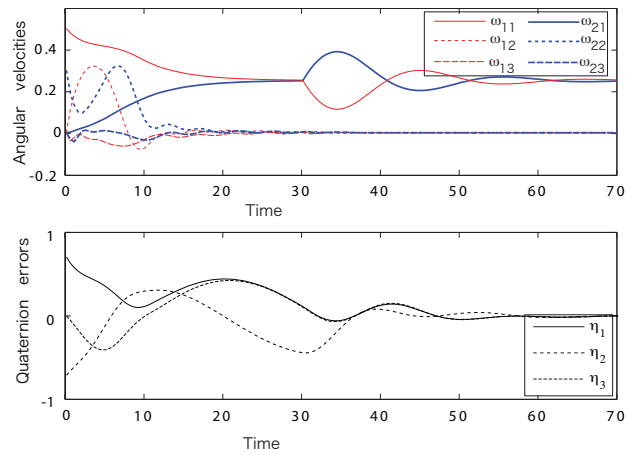
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