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Lyapunov Analysis of Quadratically Symmetric Neighborhood Consensus Algorithms

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Abstract— **We consider a class of neighborhood consensus algorithms for multi-agent systems. Within this class, the agents move along the gradients of a particular function, which can be represented as the sum of the minimums of several quadratically symmetric nonnegative functions. For these systems, we provide generic Lyapunov functions that are non-increasing along the trajectories. Under some mild technical assumptions, the Lyapunov functions prove convergence of the algorithms when the number of agents is finite. We show that a well-known model of multi-agent systems, namely the *opinion dynamics* model, is a special case of this class. The opinion dynamics model was first introduced by Krause and consists of a distribution of agents on the real line, where the agents simultaneously update their positions by moving to the average of the positions of their neighbors including themselves. We show that a specific Lyapunov function that was previously proposed for the opinion dynamics model by Blondel et. al. can be recovered from our generic Lyapunov function. In addition to providing intuition about the dynamics of neighborhood consensus algorithms, our Lyapunov analysis is particularly useful for analysis of the infinite-dimensional case, where extensions of the combinatorial approaches may not be convenient or possible.**

I. Introduction

Our research is motivated by the study of a simple model of a distributed averaging algorithm for multi-agent systems, known as the opinion dynamics model. The opinion dynamics model was first introduced by Krause [4], [5] and is defined as follows: Consider a systems of n agents where at time $k \in \mathbb{Z}$, every agent $i \in \{1, 2..., n\}$ has a position (an opinion) represented by a real number x_k^i. At each time step k, the agents update their positions by moving to the average of the positions of all the agents that are within a fixed distance of R from themselves. This system can be modeled in the following way:

$$x_{k+1}^i = \frac{\sum\limits_j e_k^{ij} x_k^j}{\sum\limits_j e_k^{ij}} \tag{1}$$

where

$$e_k^{ij} = \begin{cases} 1, & \left|x_k^i - x_k^j\right| \le R \\ 0, & \left|x_k^i - x_k^j\right| > R \end{cases}$$

The opinion dynamics model has been studied by many researchers. To name a few, the papers [1] and [5], contain various results on stability, convergence properties, and/or qualitative behavior of the system. For instance, it is shown in [5] that if the number of agents is finite, the state of every agents x^i converges to a limit value $\overline{x}^i$ in finite time, and that the absolute difference between the limit values of every two agents is either zero or greater than R. In [1], the authors introduce the notion of equilibrium stability and provide a lower bound on the inter-cluster distance between stable equilibria. They also consider the infinite-dimensional case where there is a continuum of agents on the real line. They provide a Lyapunov function which proves that the infinite-dimensional case does not produce cycles and that the variation rate decays to zero. Based on these observations, it is then conjectured that the continuum of agents converges to an equilibrium. See [6], [2] for more on the infinite-dimensional case.

In this paper, we first introduce a class of distributed multi-agent systems. In this class, the agents update their positions by moving along the gradient of a function which can be represented as the sum of the minimums of several (possibly infinite) quadratically symmetric nonnegative functions. That is:

$$x_{k+1}^i = \arg\min_u \sum_j \sigma_{\phi\left(x_k^i, x_k^j\right)}\left(u, x_k^j\right) \tag{2}$$

$$\phi\left(x^i, x^j\right) = \arg\min_{\lambda \in \Lambda} \sigma_\lambda\left(x^i, x^j\right) \tag{3}$$

where, Λ is an index set which can be finite or infinite. In addition, we require that for all $\lambda \in \Lambda$, the quadratic functions $\sigma_\lambda(.,.) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be symmetric, and nonnegative (The nonnegativity constraint can be relaxed to being bounded from below as we will discuss in more detail in Section II). Whenever (3) does not define $\phi\left(x^i, x^j\right)$ uniquely, we pick the smallest index (assuming for simplicity that it exists), and whenever (2) does not define x_{k+1}^i uniquely, we define $x_{k+1}^i = x_k^i$. It is easy to verify that the opinion dynamics model (1) can be recovered from (2) by choosing:

$$\Lambda = \{1, 2\}, \ \sigma_1(p, q) = (p - q)^2, \ \sigma_2(p, q) = R.$$

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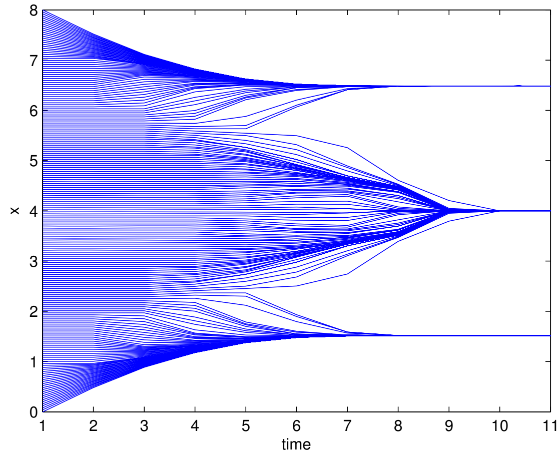

Fig. 1. Krause's opinion dynamics model (System (1)): Evolution of 200 agents initially distributed uniformly in the interval [0,8]. $\Lambda = \{1, 2\}$, $\sigma_1(x, y) = (x - y)^2$, $\sigma_2(x, y) = 1$.

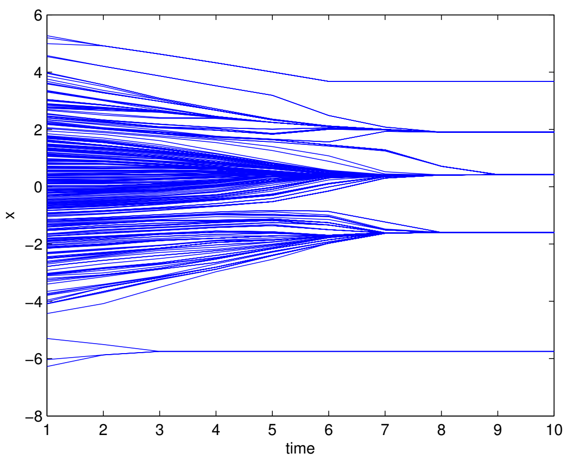

Fig. 2. Krause's opinion dynamics model (System (1)): Evolution of 200 agents with an initial normal distribution of zero mean and standard deviation equal to two. $\Lambda = \{1, 2\}$, $\sigma_1(x, y) = (x - y)^2$, $\sigma_2(x, y) = 1$.

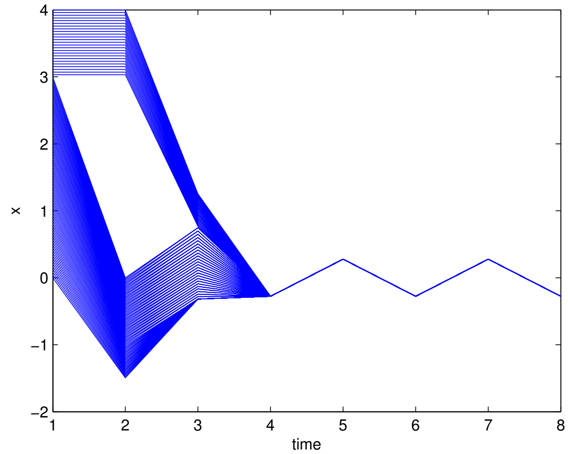

Fig. 3. Evolution of System (2) with 100 agents initially distributed unformly in the interval [0,4]. $\Lambda = \{1, 2\}$, $\sigma_1(x, y) = (x + y)^2$, $\sigma_2(x, y) = 3$.

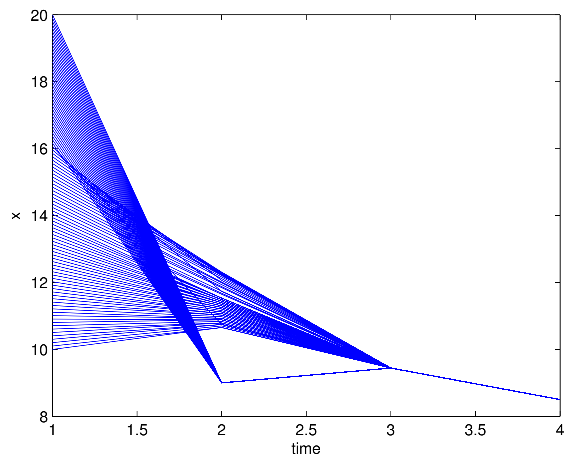

Fig. 4. Evolution of System (2) with 100 agents initially distributed unformly in the interval [10,20]. $\Lambda = \{1, 2\}$, $\sigma_1(x, y) = (x - y)^2 + 0.2xy$, $\sigma_2(x, y) = 0.1(x + y)^2 - 0.08xy + 30$.

Then we have:

$$\begin{aligned} x_{k+1}^{i} &= \arg\min_{u} \sum_{j} e_k^{ij} \left(u - x_k^j\right)^2 + \left(1 - e_k^{ij}\right) \cdot R \\ &= \frac{\sum_j e_k^{ij} x_k^j}{\sum_j e_k^{ij}}, \end{aligned}$$

where

$$e_k^{ij} = \begin{cases} 1 & \text{if} \quad \phi(x_i, x_j) = 1 \\ 0 & \text{if} \quad \phi(x_i, x_j) = 2 \end{cases}$$

Figures 1 and 2 show simulation results for system (1). Interested readers are referred to [1], [2] for more detailed discussions about the qualitative behavior of the opinion dynamics model, including the $2R$ conjecture. Figures 3 and 4 show simulation results for system (2). For the system in Figure 3, all the agents converge to the same value by time $k = 4$. Yet the system does not reach an equilibrium and produces a limit cycle. For the system in Figure 4, by time $k = 3$, all the agents converge to the same value, yet they keep updating their values until they all converge to zero asymptotically (Not shown in the picture).

In this paper we provide Lyapunov functions that are valid along the trajectories of (2). We prove that under some mild technical assumptions system (2) converges when the number of agents is finite. We also show that the discrete analog of a specific Lyapunov function that was previously proposed for the opinion dynamics model by Blondel *et. al.*

[1] can be recovered from our generic Lyapunov function. Finally, we will discuss the relation between our results and already existing results, mostly in [2]. In that aspect, we argue that Lyapunov analysis of neighborhood consensus algorithms provides a measure for their convergence rate. In addition, our Lyapunov analysis can be readily extended to draw conclusions about the behavior of these algorithms in the infinite-dimensional case, where combinatorial analysis may not be convenient or applicable.

II. Main Results

A. Lyapunov Analysis

We consider a system of n agents. To every agent $i \in \{1,2,..,n\}$, we associate a real vector $x^i \in \mathbb{R}^m$. We will refer to this vector as the *position vector* or the *value vector* of agent i. At each time step, the agents update their positions according to (2). In this section, we discuss convergence of such algorithms and present Lyapunov functions that support our results. The following Lemma is central to the development of our results.

Lemma 1: Let $\sigma(.,.) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a quadratic form which satisfies the following properties:

$$\begin{array}{llll} 1. & \sigma(p,q) & = & \sigma(q,p), \quad \forall p,q \in \mathbb{R}^n. \\ 2. & \sigma(p,-p) & \geq & 0, \qquad \forall p \in \mathbb{R}^n. \\ 3. & \sigma(p,0) & > & 0, \qquad \forall p \in \mathbb{R}^n/\{0\}. \end{array}$$

Define:

$$\gamma(p) := \arg\min_q \sigma(p,q)$$

Then

$$\sigma(\gamma(p),\gamma(p)) \leq \sigma(p,p), \ \forall p \in \mathbb{R}^n.$$

Proof: The first property implies that $\sigma(.,.)$ can be defined as $\sigma(p,q) = p^TQp + q^TQq - 2p^TRq$, where Q and R are symmetric matrices, and the third property implies that Q is positive definite. By definition, $\gamma(p) = Q^{-1}Rp$. Then, we have:

$$\begin{aligned} & \sigma(p,p) - \sigma(\gamma(p),\gamma(p)) \\ = \ & 2p^T\left[Q - R - RQ^{-1}(Q-R)Q^{-1}R\right]p \\ = \ & 2p^T\left[Q - R - RQ^{-1}R + RQ^{-1}RQ^{-1}R\right]p \\ = \ & 2p^T\left[\left(I - RQ^{-1}\right)(Q+R)\left(I - Q^{-1}R\right)\right]p \\ = \ & 2(p-\gamma(p))^T(Q+R)(p-\gamma(p)) \\ = \ & \sigma(p-\gamma(p), -p+\gamma(p)) \geq 0 \end{aligned}$$

where the last inequality follows from property 2. ■

Remark 1: The Lemma remains valid if σ is a quadratic function which contains linear and/or constant terms. I.e.

$$\sigma(p,q) := \widehat{\sigma}(p,q) - 2L(p,q) + c \tag{4}$$

where $\widehat{\sigma}$ satisfies properties $1,2,3$ and L is a symmetric linear function, i.e. $L(p,q) = L(q,p) = L^T(q+p)$. Since the proof idea for the general case is the same as the one presented above, for the sake of brevity, we do not present the proof in the presence of linear terms. For the rest of this paper we assume that $\sigma(.)$'s that satisfy conditions of Lemma 1, are allowed to have linear and constant terms as in (4).

Theorem 1: Let $x^i \in \mathbb{R}^m$, $i \in \{1,2,..,n\}$ represent a multi-agent system of size n with m-dimensional position vectors. Let Λ be a set of indices and $\sigma_\lambda(.,.)$, $\lambda \in \Lambda$ be a family of convex quadratic functions, where each σ_λ satisfies properties 1, 2, 3 of Lemma 1, or, is possibly a constant. Let $\Lambda_c \subset \Lambda$ be the set of indices of constant functions:

$$\forall \lambda \in \Lambda_c, \ \exists c_\lambda \in \mathbb{R}, \text{ s.t. } \sigma_\lambda(.,.) = c_\lambda.$$

Let the index set Λ^* and the map $\phi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be defined by:

$$\begin{aligned} \Lambda^*(x,y) &= \{\lambda^* \mid \sigma_{\lambda^*}(x,y) \leq \sigma_\lambda(x,y), \ \forall \lambda \in \Lambda\} \\ \phi &: (x,y) \to \min(\Lambda^*(x,y)) \end{aligned} \tag{5}$$

Assume that at each time step k, agents update their positions according to:

$$x^i_{k+1} = \begin{cases} x^i_k, & \text{if } \phi\left(x^i_k, x^j_k\right) \in \Lambda_c, \ \forall j. \\ \arg\min_u \sum_j \sigma_{\phi\left(x^i_k,x^j_k\right)}\left(u, x^j_k\right), & \text{otherwise.} \end{cases} \tag{6}$$

Then, the following function is non-increasing along the trajectories of (6):

$$V(x_k) = \sum_{i,j} \sigma_{\phi\left(x^i_k,x^j_k\right)}\left(x^i_k, x^j_k\right) \tag{7}$$

Proof: If there exists i such that $\phi(x^i_k, x^j_k) \in \Lambda_c$ for all j, then at time k, agent i does not get influenced by any of the other agents. Since $\phi(.,.)$ is symmetric, none of the other agents gets influenced by i. Thus, for time step k, the movement of all other agents can be isolated from that of i. Moreover, since $\sigma_{\phi(x^i_k,x^j_k)}$ is a constant for all j, and σ_λ is convex for all λ, the minimum at the next step cannot increase and we necessarily have:

$$\sigma_{\phi\left(x^i_{k+1},x^j_{k+1}\right)} \leq \sigma_{\phi\left(x^i_k,x^j_k\right)}, \ \forall j$$

Therefore, the overall contribution of agent i in $V(x_k) - V(x_{k+1})$ is nonnegative as desired. We can repeat this argument for all such agents and isolate them from other agents which influence (and get influenced by) at least one other agent. Therefore, without loss of generality, for the rest of the proof we assume that at every time step k, and for every agent i, there exists an agent j such that $\phi\left(x^i_k, x^j_k\right) \in \Lambda \backslash \Lambda_c$. Notice that x^α_{k+1}, $\alpha \in \{1,2,..,n\}$ can be written as

$$x^\alpha_{k+1} = \arg\min_{u^\alpha} \sum_{ij} \sigma_{\phi\left(x^i_k,x^j_k\right)}\left(u^i, x^j_k\right)$$

Thus:

$$x_{k+1} = \arg\min_u \sum_{ij} \sigma_{\phi\left(x_k^i, x_k^j\right)} \left(u^i, x_k^j\right)$$

For fixed vectors $p, q \in \mathbb{R}^n$ define $f_{pq} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ according to:

$$f_{pq}(x, y) = \sum_{i,j} \sigma_{\phi(p^i, q^j)} \left(x^i, y^j\right)$$

Then, $f_{pq}(x, y)$ satisfies properties $1, 2, 3$ of Lemma 1. Define

$$z := \arg\min_u f_{pq}(u, x_k)$$

Lemma 1 then implies that

$$\sum_{i,j} \sigma_{\phi(p^i, q^j)} \left(z^i, z^j\right) \leq \sum_{i,j} \sigma_{\phi(p^i, q^j)} \left(x_k^i, x_k^j\right)$$

Since this is true for arbitrary p and q, for $p = q = x_k$ we have

$$\sum_{i,j} \sigma_{\phi\left(x_k^i, x_k^j\right)} \left(z^i, z^j\right) \leq \sum_{i,j} \sigma_{\phi\left(x_k^i, x_k^j\right)} \left(x_k^i, x_k^j\right)$$

However, $p = q = x_k$ implies that $z = x_{k+1}$, which immediately implies that:

$$\sum_{i,j} \sigma_{\phi\left(x_k^i, x_k^j\right)} \left(x_{k+1}^i, x_{k+1}^j\right) \leq \sum_{i,j} \sigma_{\phi\left(x_k^i, x_k^j\right)} \left(x_k^i, x_k^j\right) \quad (8)$$

Next, note that by definition:

$$\sigma_{\phi\left(x_{k+1}^i, x_{k+1}^j\right)} \left(x_{k+1}^i, x_{k+1}^j\right) \leq \sigma_{\phi\left(x_k^i, x_k^j\right)} \left(x_{k+1}^i, x_{k+1}^j\right)$$

Therefore,

$$\sum_{i,j} \sigma_{\phi\left(x_{k+1}^i, x_{k+1}^j\right)} \left(x_{k+1}^i, x_{k+1}^j\right) \leq \sum_{i,j} \sigma_{\phi\left(x_k^i, x_k^j\right)} \left(x_{k+1}^i, x_{k+1}^j\right) \quad (9)$$

The result immediately follows from (8) and (9). ■

Remark 2: It can be shown in a similar fashion that

$$V(x_k) = \sum_{i,j} \sigma_{\phi\left(x_k^i, x_k^j\right)} \left(x_{k+1}^i, x_{k+1}^j\right)$$

is also a Lyapunov function for system (6).

Theorem 2: Consider a multi-agent system defined as in Theorem 1. Define $x_k = \left[x_k^1 \ x_k^2 \ \cdots \ x_k^n\right]$. Then, there exists a vector $x^* \in \mathbb{R}^{m \times n}$ such that

$$\lim_{k \to \infty} \|x_k - x^*\| = 0,$$

if the following condition holds:

$$\sigma_\lambda(p, -p) - \sigma_\lambda(0, 0) > 0, \ \forall \lambda \in \Lambda \backslash \Lambda_c, \ p \in \mathbb{R} \backslash \{0\}. \quad (10)$$

Proof: Condition (10) implies that the function

$$h(x) := \sum_{i,j} \sigma_{\phi(x^i, x^j)} \left(x^i, -x^j\right)$$

defines a norm in $\mathbb{R}^{n \times m}$. It follows from Theorem 1 and the proof of Lemma 1 that

$$\begin{aligned} & V(x_k) - V(x_{k+1}) \\ \geq \ & \sum_{i,j} \sigma_{\phi\left(x_k^i, x_k^j\right)} \left(x_{k+1}^i - x_k^i, x_k^j - x_{k+1}^j\right) \\ = \ & \|x_{k+1} - x_k\|_h \end{aligned}$$

Therefore,

$$\sum_{k=0}^{\infty} \|x_{k+1} - x_k\|_h < \infty. \quad (11)$$

The result then follows from (11). Note that without condition (10) convergence cannot be guaranteed and the system may indeed produce a limit cycle (See Figure 3). ■

Remark 3: It is well-known that for Krause's opinion dynamics model convergence happens in finite time when the number of agents is finite. This need not be true for the general case. For instance, for the system presented in Figure 4 of Section I, convergence is asymptotic.

We showed earlier that the one-dimensional Krause's opinion dynamics model is a special case of System (6) with

$$\Lambda = \{1, 2\}, \ \sigma_1(p, q) = (p - q)^2, \ \sigma_2(p, q) = R.$$

Theorem 1 then implies that the function

$$V(x_k) := \sum_{i,j} e_{ij} \left(x_k^i - x_k^j\right)^2 + \left(1 - e_k^{ij}\right) R \quad (12)$$

is non-increasing along the trajectories of (1). Since $\sigma_2(p - p) = 4p^2 > 0$, Theorem 2 then implies that the opinions converge to real values. The function (12) is the discrete analog of the Lyapunov function that Blondel *et. al.* obtain in [1] for the infinite-dimensional version of (1), where there is a continuum of agents on the real line. This Lyapunov function proves that the infinite-dimensional system does not produce cycles and that the variation rate decays to zero [1].

B. Averaging with Arbitrary Weights

In this section we consider a generalized version of Krause's opinion dynamics model. In model (1), the agents put a weight of either 0 or 1 on the opinions of other agents based on their distance, and use this weighted opinion in the averaging process. Here, we consider a case where the weights can be arbitrary as long as they are symmetric and monotonically non-increasing as a function of the relative distance.

Corollary 1: Let $f : \mathbb{R} \to \mathbb{R}^+$ be an even function which is radially non-increasing. Then, the multi-agent system defined by

$$x_{k+1}^i := \frac{\sum_j f\left(x_k^i - x_k^j\right) x_j}{\sum_j f\left(x_k^i - x_k^j\right)} \quad (13)$$

converges.

To prove this corollary we use the following lemma:

Lemma 2: Let $\Lambda \subset \mathbb{R}$, and let $f : \Lambda \to \mathbb{R}^+$ be an even function which is lower semi-continuous and radially non-increasing. Then there exists a family of convex quadratic functions θ_λ, $\lambda \in \Lambda$, that satisfy the following properties:

$$(i)\;\; \theta_x(x) \;\leq\; \theta_\lambda(x)\,,\;\forall x, \lambda \in \Lambda$$
$$(ii)\;\; f(x) \;=\; \frac{d^2}{dx^2}\min_\lambda \theta_\lambda(x)$$

Proof: Since f is radially monotonic it is integrable. Define

$$\theta_\lambda(x) = f(\lambda)\,x^2 - \int_{f(0)}^{f(\lambda)} t^2 df(t) \qquad (14)$$

Since f is lower semi-continuous $\theta_\lambda(x)$ is well defined. Then, we have

$$\begin{aligned}\theta_\lambda(x) - \theta_x(x) &= x^2\left(f(\lambda) - f(x)\right) - \int_{f(x)}^{f(\lambda)} t^2 df(t)\\ &\geq x^2\left(f(\lambda) - f(x)\right) - \int_{f(x)}^{f(\lambda)} x^2 df(t)\\ &= 0\end{aligned}$$

where the last inequality follows from the fact that $f(.)$ is radially non-increasing and even. The first property is thus satisfied. The second property follows from the first property and the definition of $\theta_\lambda(.)$. ■

Remark 4: If f is constant over an interval $[a, b)$, then $\theta_\lambda(x) = \theta_a(x)\,, \forall \lambda, x \in [a, b)$. Therefore, if f is piecewise constant with finitely many points of discontinuity, then a finite family of quadratic functions θ_λ, $\lambda \in \mathcal{D} \cup \{0\}$ satisfies the conditions of Lemma 2, where $\mathcal{D} \subset \Lambda$ is the set of all discontinuities of f over the positive real line.

We can now present a proof of Corollary 1.

Proof: [of Corollary 1] With θ defined as in (14), define:

$$\sigma_\lambda(p, q) = \theta_\lambda(p - q)^2\,.$$

The update law (13) can then be written as:

$$x^i_{k+1} = \arg\min_u \sum_j \sigma_{\phi\left(x^i_k, x^j_k\right)}\left(u, x^j_k\right)$$

By construction, the family of functions σ_λ satisfy (10) for all λ. Theorem 2 then yields the claimed result. ■

Remark 5: Recent results by Hendrickx [2] (Theorem 9.4) can in principle be used to prove the statement of Corollary 1. The advantage of our result is in providing a specific Lyapunov function which can be used for quantifying the convergence rate of the algorithm and in addition, can be extended to the infinite-dimensional case.

III. Multi-Dimensional Case

We now investigate the dynamics of system (6) in higher dimensions ($m > 1$). For simplicity, consider the case where $|\Lambda| = 3$ with two quadratic functions and a constant. The

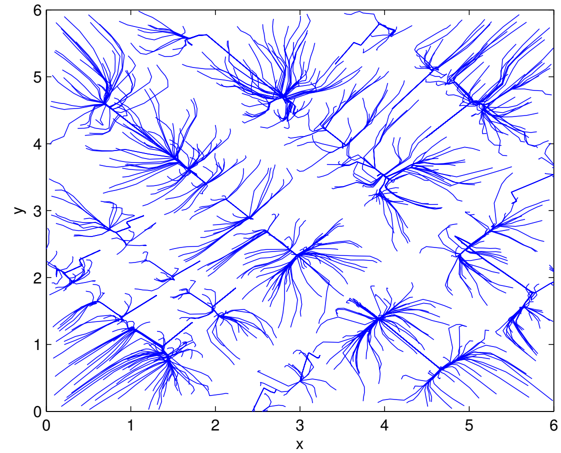

Fig. 5. Evolution of system (2) with 800 agents initially distributed randomly in the box $[0, 6]^2$. Simulation time: $k = 0, ..60$. The system data is given in (15).

derivations will be similar when $|\Lambda| > 3$. An interesting case to consider would be when the agents update their positions based on their relative distance. Consider:

$$\Lambda = \{1, 2, 3\}\,,\;\sigma_1(p, q) = (p - q)^T S_1 (p - q)\,,$$
$$\sigma_2(p, q) = (p - q)^T S_2 (p - q)\,,\;\sigma_3(p, q) = 1.$$

where $p, q \in \mathbb{R}^m$, and S_1 and S_2 are symmetric positive-definite matrices. Specifying the position of each agent i at time k with an m-dimensional coordinate vector c^i_k we obtain:

$$\begin{aligned}c^i_{k+1} &= \Gamma^{i^{-1}}_k \sum_j \left(e^{ij}_k S_1 + v^{ij}_k S_2\right) c^j_k,\\ \Gamma^i_k &= \sum_j e^{ij}_k S_1 + v^{ij}_k S_2\end{aligned}$$

where

$$e^{ij}_k = \begin{cases} 1, & \phi\left(c^i_k, c^j_k\right) = 1\\ 0, & \text{otherwise}\end{cases}\;,\quad v^{ij}_k = \begin{cases} 1, & \phi\left(c^i_k, c^j_k\right) = 2\\ 0, & \text{otherwise}\end{cases}$$

Figure 5 shows simulation results in $\mathbb{R}^2$ with

$$S_1 = \begin{bmatrix} 1 & -0.996\\ -0.996 & 1\end{bmatrix},\; S_2 = \begin{bmatrix} 1 & 0.996\\ 0.996 & 1\end{bmatrix}. \qquad (15)$$

Figure 6 shows simulation results in $\mathbb{R}^2$ with

$$S_1 = \begin{bmatrix} 1 & -0.996\\ -0.996 & 1\end{bmatrix},\; S_2 = \begin{bmatrix} 0.004 & 0\\ 0 & 1.996\end{bmatrix}. \qquad (16)$$

Another interesting case to consider in $\mathbb{R}^2$ is:

$$\Lambda = \{1, 2\}\,,\;\sigma_1(p, q) = (p - q)^T S_1 (p - q)\,,\;\sigma_2(p, q) = 1.$$

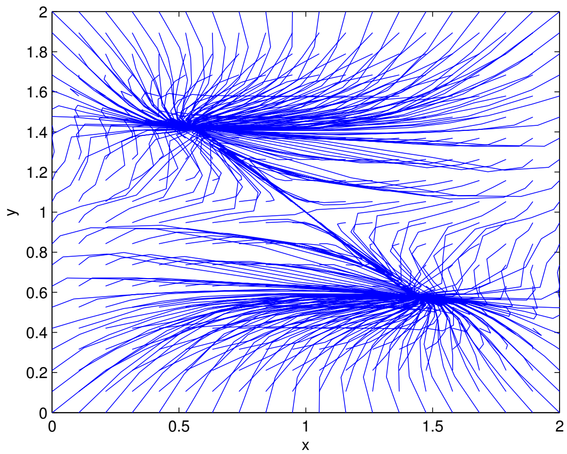

Fig. 6. Evolution of system (2) with 400 agents initially distributed uniformly in the box $[0,2]^2$. Simulation time: $k = 0,..60$. The system data is given in (16).

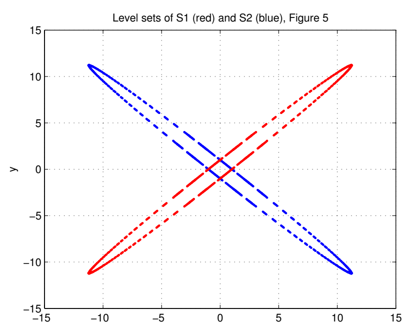

Fig. 7. The level sets of the quadratic functions of the system in Figure 5. (S1 in red and S2 in blue) The shape and the direction of the level sets determines the movement of agents.

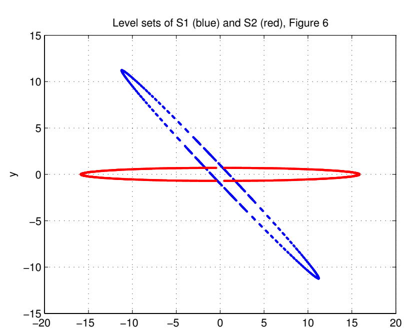

Fig. 8. The level sets of the quadratic functions of the system in Figure 6. (S1 in blue and S2 in red) The shape and the direction of the level sets determines the movement of agents.

It can be verified that this case leads to a simple averaging of the x and y coordinates of the neighbors independently. The measure of the distance would be quadratic norm in $\mathbb{R}^2$ defined by the positive-definite matrix S_1. The function

$$V(c_k) := \sum_{i,j} e_k^{ij} \left\| c_k^i - c_k^j \right\|_{S_1}^2 + 1 - e_k^{ij}$$

is a Lyapunov function, where:

$$\|v\|_{S_1}^2 \quad : \quad = v^T S_1 v, \;\; c_k := [x_k; y_k]$$
$$e_k^{ij} \quad = \quad 1, \text{ if } \left\| c_k^i - c_k^j \right\|_{S_1}^2 \leq 1, \text{ and } e_k^{ij} = 0 \text{ otherwise.}$$

IV. Conclusions

We showed that a well-studied model of multi-agent dynamical systems, known as the opinion dynamics model, is a special case of a larger class of neighborhood consensus algorithms. Within this class, the agents move along the gradients of a particular function, which can be represented as the sum of the minimums of several quadratically symmetric convex functions. Simulation results for one and two dimensions show that such algorithms exhibit very interesting dynamics. We presented a generic Lyapunov function which is non-increasing along the trajectories of systems within the class, and we showed that under some mild technical assumptions, such systems converge when the number of agents in the system is finite. We used Lyapunov function analysis to prove that the one-dimensional opinion dynamics model remains convergent under an arbitrary weight function which is symmetric and radially non-increasing. Recent results by Hendrickx [2] (Theorem 9.4) can be used to prove the same statement via combinatorial analysis of the system behavior. However, our Lyapunov function can be used to provide a quantified measure of the system behavior which is particularly convenient when extensions of the analysis to the infinite-dimensional case is considered.

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