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Convex Optimization in Robust Identification of Nonlinear Feedback

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Abstract— **A nonlinear system identification setup is formulated as a task of finding a stable feedback system of fixed complexity providing the best robust fit for a given set of input-output data. New techniques, based on incremental passivity, are proposed for casting such problems in a format which allows application of efficient convex optimization engines. Case studies of specific implementations of the approach are provided.**

I. INTRODUCTION

This paper is concerned with a specific aspect of system identification: finding stable discrete time dynamical nonlinear models of limited complexity to fit given sets of input-output data while minimizing certain robust macthing error criteria.

For example, in the case of a system with scalar input $v = v[t]$ and scalar output $y = y[t]$, the data may come in the form of a finite collection input-output pairs (v_i, y_i) of finite length $n(i)$, and the reduced complexity model may be sought in the implicit form

$$F(y[t], y[t-1], \dots, y[t-d], v[t-1], \dots, v[t-d]) = 0, \quad (1)$$

where F is to be selected from a given class $\mathcal{F} = \{F\}$ of functions $F : \mathbb{R}^{2d+1} \mapsto \mathbb{R}$ to minimize the i/o mismatch measure

$$E = E(F) = \sum_i \sum_{t=0}^{n(i)} |y_i[t] - \tilde{y}_i[t])|^2,$$

where $y = \tilde{y}_i[t]$ satisfy equations (1) for $t = 0, 1, \dots, n(i)$, with $v = v_i$ and with initial conditions $\tilde{y}_i[-t] = y_i[-t]$ for $t = 1, \dots, n$. The original data set $\{v_i, y_i)\}$ can represent either actual physical measurements (the classical system identification setup) or, alternatively, computer simulations (which makes it possible to apply the framework to the task of model reduction).

In the classical optimization-based approach to system identification (see, for example, [1]) it would be typical to seek $F \in \mathcal{F}$ minimizing the equation mismatch measure

$$J = J(F) = \lim_{T\to\infty} \frac{1}{T} \sum_i \sum_{t=1}^{T} |e_i[t])|^2,$$

where $e_i[t] =$

$$= F(y_i[t], \dots, y_i[t-m], v_i[t-1], \dots, v_i[t-m]).$$

In the most common setup, the class $\mathcal{F}$ of admissible models is linear (affine), in which case the minimization of $J(F)$ becomes a least squares optimization routine. However, unlike the i/o mismatch measure $E = E(F)$, the quantity $J(F)$ in general is not a valid measure of system identification performance, as it does not incorporate the degree of sensitivity of of the output y of the error model

$$F(y[t], \dots, y[t-m], v[t-1], \dots, v[t-m]) = e[t] \quad (2)$$

to its input e.

When the error model (2) has finite incremental L2 gain, smallness of $J(F)$ implies smallness of $E(F)$. However, incorporating robust stability of (2) as a constraint in the optimization of $J = J(F)$ can be tricky, unless

$$F(Y_0, Y_1, \dots, Y_m, V_1, \dots, V_m) = \tilde{F}(Y_0, V_1, \dots, V_m) \quad (3)$$

is not a function of all but one of its "y" arguments. Note that condition (3) means, essentially, that (1) is a (nonlinear) moving average model (for example, a Volterra series model). Since moving average models are remarkably inefficient in matching resonant behavior of dynamical systems, imposing the constraint (3) is not a viable option in many applications.

A major point of this paper is to enable optimization of a meaningful i/o mismatch measure by reducing the task of minimizing an upper bound of $E = E(F)$, subject to the condition of incremental stability of model (2), to an efficient convex optimization routine. The approach of this paper calls for limiting attention to *passive* feedback loops, and using robustess analysis

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in bounding the error. It is a natural extension of the relaxation-based LTI model reduction techniques [4] and [3], but also shares the positivity and non-parametric identification aspects with [2].

II. SETUP AND OBJECTIVES

In this paper, a system identification task is defined by integers $d \geq 0$, $m > 0$, $k > 0$, a convex set $\mathcal{F} = \{F\}$ of continuous functions

$$F: \mathbb{R}^{dk+k+m} \mapsto \mathbb{R}^k,$$

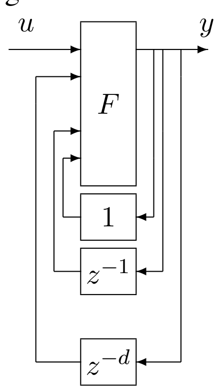
and a collection $Xi = \{\xi_1, \ldots, \xi_N\}$ of pairs $\xi_i = (y_i, u_i)$ of finite sequences

$$y_i = \{y_i[t]\}_{t=-d}^{n(i)}, \quad u_i = \{u_i[t]\}_{t=0}^{n(i)}$$

(where $n(i) \geq d$) of vectors $y_i[t] \in \mathbb{R}^k$, $u_i[t] \in \mathbb{R}^m$. The set $\mathcal{F} = \{F\}$ defines a family of low complexity dynamical models governed by implicit recursive equations

$$F(y[t], y[t-1], \ldots, y[t-d], u[t]) = 0, \tag{4}$$

Ξ represents the available discrete time input-output data, and a system identification algorithm is to find $F \in \mathcal{F}$ such that (4) is a *well-posed* and *stable* model minimizing a *fitting error* functional.



Specifically, model (4) is called *well - posed* when for every $v_1, \ldots, v_d \in \mathbb{R}^k$ and $w \in \mathbb{R}^m$ the equation $F(y, v_1, \ldots, v_d, w) = 0$ has a solution $y \in \mathbb{R}^k$ (not necessarily a unique one). This condition guarantees that for every set of *initial conditions*

$$y[-1] = v_1, \ \ldots, \ y[-d] = v_d$$

and every *input sequence* $u = \{u[t]\}_{t=0}^n$, equations (4) with $t = 0, 1, \ldots, n$ have at least one solution $y = \{y[t]\}_{t=0}^n$.

Furthermore, system (4) is called (incrementally) *stable* if it is well-posed, and

$$\sum_{t=0}^{\infty} |\bar{y}[t] - \hat{y}[t]|^2 < \infty \tag{5}$$

for every two solutions $y = \{\bar{y}[t]\}$, $y = \{\hat{y}[t]\}$ of (4) with the same $u = \{u[t]\}$. The stability condition guarantees *fading memory* for model (4).

Finally, for a given data set $\Xi = \{(y_i, u_i)\}_{i=1}^N$, the error of the fit produced by (4) is defined as

$$E = E(F) = E(\Xi, F)$$
$$= \sum_{i=1}^{N} \sum_{t=0}^{n(i)} |y_i[t] - \tilde{y}_i[t]|^2,$$

where the sequences $\tilde{y}_i = \{\tilde{y}_i[t]\}_{t=0}^{n(i)}$ satisfy

$$F(\tilde{y}_i[t], \ldots, \tilde{y}_i[t-d], u_i[t]) = 0 \quad (0 \leq t \leq n(i)) \tag{6}$$

with initial conditions

$$\tilde{y}_i[-t] = y_i[-t] \quad \text{for} \quad t \in \{1, \ldots, n\}. \tag{7}$$

Note that the evolution equation (4) is written in terms of several samples ($y[t]$, $y[t-1]$, …, $y[t-d]$) of the output signal y, but only one sample ($u[t]$) of the input signal u. This is done for the sake of convenience, but does not really limit the applicability of the framework in other situations, since a dynamical equation

$$F_0(y[t], \ldots, y[t-d], v[t], \ldots, v[t-r]) = 0$$

can always be re-written in the form (4) with

$$F(v_0, \ldots, v_d, x) = F_0(v_0, \ldots, v_d, w_0, \ldots, w_r), \tag{8}$$

$$u[t] = \begin{bmatrix} v[t] \\ \vdots \\ v[t-r] \end{bmatrix}, \quad x = \begin{bmatrix} w_0 \\ \vdots \\ w_r \end{bmatrix}.$$

This paper does not offer a solution to the task of finding the exact minimizer of the fit error over all stable models (4). Instead, it restricts F to satisfy a *generalized passivity* condition (which implies stability), and introduces an *upper bound* of E which can be minimized via convex optimization.

III. FITTING WITH PASSIVE MODELS

Let us call a functon $F: \mathbb{R}^{dk+k+m} \mapsto \mathbb{R}^k$ (as well as the correspoinding system (4)) *strictly passive* if there exist $\epsilon > 0$ and a function $V: \mathbb{R}^{2kd} \mapsto [0, \infty)$ such that

$$(z_0 - q_0)'[F(z_0, \ldots, z_d, v) - F(q_0, \ldots, q_d, v)]$$
$$\geq \epsilon |z_0 - q_0|^2 - V(z_1, \ldots, z_d, q_1, \ldots, q_d)$$
$$+ V(z_0, \ldots, z_{d-1}, q_0, \ldots, q_{d-1}), \tag{9}$$

and $V(z_1,\dots,z_d,z_1,\dots,z_d)=0$ for all $z_i,q_i\in\mathbb{R}^k$, $v\in\mathbb{R}^m$. Note that condition (9), as a family of linear inequalities, defines a convex set of pairs of functions (F,V).

One reason to impose the passivity requirement in (9) is that it guarantees well posedness and stability of system (4), and guarantees that

$$\tilde{E}_\epsilon(F)=\epsilon^{-2}\sum_{i=1}^{N}\sum_{t=0}^{n(i)}|F(y_i[t],\dots,y_i[t-d],u[t])|^2 \tag{10}$$

is an upper bound of $E(F)$.

Theorem 1. *A strictly passive system (4) is well-posed and stable. Moreover, the inequality $E(F)\le\tilde{E}_\epsilon(F)$, is satisfied for the coefficient $\epsilon>0$ from condition (9) in the definition of passivity.*

In addition, the passivity condition from (9) guarantees finiteness of a tighter (though usually more difficult to calculate) upper bound of $E(F)$. More specifically for a given data set $\Xi=\{(y_i,u_i)\}$, a real number $\epsilon>0$, and a function $F\in\mathcal{F}$ let $\hat{E}=\hat{E}[F]=\hat{E}_\epsilon(\Xi,F)$ be defined as the minimal upper bound of the sum

$$\sum_{i=1}^{N}\sum_{t=0}^{n(i)}\delta_i[t]'[\delta_i[t]+2\epsilon^{-1}F(z_i[t],\dots,z_i[t-d],u_i[t])], \tag{11}$$

where $z_i[t]=y_i[t]-\delta_i[t]$, over all possible $\delta_i[t]\in\mathbb{R}^k$ such that $\delta[t]=0$ for $t<0$. Note that $\hat{E}$, as a minimal upper bound of a family of linear functionals, is a convex function of its argument F.

Theorem 2. *Assume that system (4) is passive, i.e. condition (9) holds for some $\epsilon>0$. Then*

$$E(F)\le\hat{E}_\epsilon(F)\le\tilde{E}_\epsilon(F).$$

Essentially, Theorem 1 provides a rather general set of conditions under which stability of a feedback system is linked to a convex constraint imposed on the set of its coefficients. It also establishes a nicely optimizable upper bound for the i/o mismatch. In turn, Theorem 2 establishes a more accurate upper bound for the i/o mismatch, which is still amenable to efficient minimization.

To use Theorems 1,2 in a practical application, one has to select a workable description of a convex set $\Omega=\{(F,V)\}$, to contain all model/certificate pairs of potential interest, complete with efficient algorithms for verifying, given a pair $F,V\in\Omega$, the validity of (9), and computing an upper bound of $E(F)$.

A proof of Theorems 1 and 2 is given in the Appendix.

IV. CASE STUDIES

This section presents examples in which the general framework is applied to relatively simple sets $\mathcal{F}=\{F\}$ and $\Omega=\{(F,V)\}$.

A. The SISO LTI Case

Consider the situation when the input-output data is given in the form of harmonic response of an LTI system with scalar input w and scalar output y, i.e. N samples $g_i=G(\theta_i)$ of a stable transfer function G (possibly non-rational) are given at $\theta_i=\exp(j\omega_i)$, to be fitted by a stable rational transfer function

$$\hat{G}(z)=\frac{b(z)}{a(z)}=\frac{b_0+b_1z^{-1}+\cdots+b_dz^{-d}}{a_0+a_1z^{-1}+\cdots+a_dz^{-d}}$$

of order not larger than d. The desired quantity to minimize in this case is the cumulative transfer function fitting error

$$E_0=\sum_{i=1}^{N}\left|g_i-\frac{b(\theta_i)}{a(\theta_i)}\right|^2. \tag{12}$$

However, there is no known polynomial time algorithms for minimizing E_0.

In time domain terms, the setup means that the input-output data contains long sequences ($n(i)\to+\infty$) of pure sinusoids

$$u_i[t]=\begin{bmatrix}\cos(\omega_i t)\\ \vdots\\ \cos(\omega_i t-\omega_i d)\end{bmatrix},\quad y_i[t]=r_i\cos(\omega_i t+\phi_i),$$

where $g_i=r_ie^{j\phi_i}$, and the set $\mathcal{F}=\{F\}$ consists of *linear* functions

$$F(q_0,\dots,q_d,v)=a_0q_0+\cdots+a_dq_d-\tilde{b}v, \tag{13}$$

where $\tilde{b}=[b_0,\dots,b_d]$. Consider a class $\mathcal{V}=\{V\}$ of quadratic storage functions of the form

$$V(z_1,\dots,z_d,q_1,\dots,q_d)=\sigma(z_1-q_1,\dots,z_d-q_d),$$

where σ ranges over the set of all positive semidefinite quadratic forms. Note that the convex set $\Omega=\{(F,V)\}$ has a finite dimension in this case, which reflects the fact that the models under consideration are defined by a finite set of scalar parameters (the coefficients of the polynomials a,b).

According to the Kalman - Yakubovich - Popov Lemma, strict passivity of system (4) defined by a function F from (13) is equivalent to the positive real condition:

$$\text{Re}(a(\theta)) > 0 \quad \text{for all} \quad |\theta| = 1, \tag{14}$$

and can always be certified by a quadratic storage function $V \in \mathcal{V}$. Moreover, the maximal lower bound for the coefficient $\epsilon > 0$ from (9) is

$$\epsilon_0 = \min\{\text{Re}(a(\theta)) : \quad |\theta| = 1\}. \tag{15}$$

For large values of $n(i) \to \infty$, the normalized upper bound $\tilde{E}(F)$ of $E(F)$ approaches

$$\tilde{E}_0 = \frac{\sum_{i=1}^{N} |a(\theta_i)g_i - b(\theta_i)|^2}{\min\{|\text{Re}(a(\theta))|^2 : \quad |\theta| = 1\}}.$$

Minimization of $\tilde{E}_0$ over all b and all positive real a is equivalent to minimization of

$$\tilde{E}_* = \sum_{i=1}^{N} |a(\theta_i)g_i - b(\theta_i)|^2 \tag{16}$$

subject to the normalization inequality

$$\text{Re}(a(\theta)) \geq 1 \quad \text{for all} \quad |\theta| = 1. \tag{17}$$

Similarly, subject to (17), a normalized version of $\hat{E}(F)$ can be shown to approach

$$\hat{E}_0 = \sum_{i=1}^{N} \frac{|a(\theta_i)g_i - b(\theta_i)|^2}{\text{Re}(a(\theta_i))}. \tag{18}$$

The functionals (16) and (18) (which, subject to (17), serve as upper bounds for (12), and are jointly convex with respect to the coefficients of a and b) demonstrate the application of the general approach of this paper to frequency domain data fitting for LTI SISO systems. While the gap between (18) and (12) can be large, minimization of (18) subject to $\text{Re}(a(\theta_i)) \geq 1$ offers a valuable compromise between accuracy and computational complexity.

B. Nonlinear SISO Feedback Model

Consider the model shown on Figure 1, where B is a Volterra series model of depth r (i.e. the output $w = w(t)$ of B is defined by

$$w[t] = b(u[t]), \quad u[t] = (v[t]; \ldots ; v[t-r]),$$

where b is a multivariate polynomial), A stands for the linear FIR transformation with input y and output

$$e[t] = a_0 y[t] + \cdots + a_d t[t-d],$$

and $\phi : \mathbb{R} \mapsto \mathbb{R}$ is a strictly monotonic function.

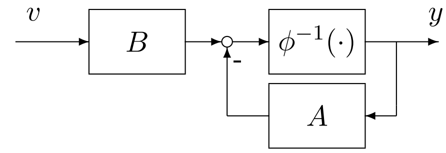


Fig. 1. Nonlinear System with Dynamical Feedback

The setup can be viewed as a special case of the general framework, with $u = u(t)$ defined according to (8), and $\mathcal{F} = \{F\}$ consisting of functions

$$F(q_0, \ldots, q_d, v) = \phi(q_0) + a_0 q_0 + \cdots + a_d q_d - b(v).$$

Note the non-parametric character (and hence infinite dimensionality) of the model class $\mathcal{F}$: no specific finite basis representation of ϕ is assumed, and the low complexity of system (4) is assured by the numbers d and r, a bound on the degree of the polynomial b, and the monotonicity constraint for ϕ. The passivity of system (4) is guaranteed by the frequency domain inequality $\text{Re}(a(\theta)) > 0$ for $|\theta| = 1$. The exact calculation of the upper bound $\hat{E}$ presents a challenge, but $\tilde{E}(F)$ is easy to optimize, as the maximal lower bound for $\epsilon > 0$ given by (15) is still valid. The identified model is obtained by minimizing (subject to the normalization inequality (17)) the sum

$$\tilde{E}_0(a, b, \phi) =$$

$$= \sum_{i=1}^{n(i)} \sum_{t=0}^{n(i)} |\phi(y_i[t]) + a_0 y_i[t] + \cdots + a_d y_i[t-d] - b(u[t])|^2$$

over the coefficients of a,b, and all monotonic functions $\phi(\cdot)$. Since only the samples of $p_{it} = \phi(y_i[t])$ enter the expression for $\tilde{E}_0$, the actual number of parameters in the resulting convex optimization is finite.

V. APPENDIX

This section contains some technical details and formal proofs of mathematical statements.

A. A Feasibility Lemma

The following simple observation, to be used in the proof of Theorem 1, is a corollary of the Brouwer's fixed point theorem.

Lemma 1. *If $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a continuous function such that $|u|^2 + u'f(u)$ is bounded over $u \in \mathbb{R}^n$ then $f(u_0) = 0$ fr some $u_0 \in \mathbb{R}^n$.*

PROOF. Take $R > 0$ such that R^2 is strictly larger than the upper bound for $|u|^2 + u'f(u)$. Let $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ be defined by

$$g(u) = (u + f(u))/\max\{1, |u + f(u)|^2/R^2\}.$$

Since g is continuous and $|g(u)| \leq R$ for all u, g maps the disc $|u| \leq R$ to itself, it has a fixed point u_0, i.e. $g(u_0) = u_0$. Multiplying the equality by $(u_0 + f(u_0))'$ on the left and using the inequality $u_0'(u_0 + f(u_0)) < R^2$ yields $|u + f(u)| < R$, hence $g(u_0) = u_0 + f(u_0)$ and $f(u_0) = 0$. ■

B. Proof of Theorem 1

Applying (9) with $q_i = 0$, $z_0 = 0$, and using the fact that V is non-negative shows that the function $f(y) = -F(y, z_1, \ldots, z_n, v)/2\epsilon$ satisfies the conditions of Lemma 1 for all z_i, v. Hence the equation $F(y, z_1, \ldots, z_n, v) = 0$ has a solution for all z_i, v, i.e. system (4) is well posed.

To prove stability, for every two solutions $y = \{\bar{y}[t]\}$, $y = \{\hat{y}[t]\}$ of (4) with the same $u = \{u[t]\}$, substituting

$$z_i = \bar{y}[t-i],\ q_i = \hat{y}[t-i],\ v = u[t]$$

into (9) yields

$$\epsilon|\bar{y}[t] - \hat{y}[t]|^2 \leq$$

$$V(\bar{y}[t-1], \ldots, \bar{y}[t-d], \hat{y}[t-1], \ldots, \hat{y}[t-d])$$

$$-V(\bar{y}[t], \ldots, \bar{y}[t-d+1], \hat{y}[t], \ldots, \hat{y}[t-d+1]).$$

Summation from $t = 0$ to $t = \infty$ yields the inequality

$$\epsilon \sum_{t=0}^{\infty} |\bar{y}[t] - \hat{y}[t]|^2$$

$$\leq V(\bar{y}[-1], \ldots, \bar{y}[-d], \hat{y}[-1], \ldots, \hat{y}[-d]).$$

Evidently, in this definition of passivity, V plays the role of a distance-like *storage function*.

To prove the inequality $E[F] \leq \tilde{E}_\epsilon[F]$, substitute

$$z_k = \tilde{y}_i[t-k],\ q_k = y_i[t-k],\ v = u_i[t]$$

into (9) to get the inequalities

$$\epsilon|\delta_i[t]|^2 + \delta_i[t]'F(y_i[t], \ldots, y_i[t-d], u_i[t]) \leq$$

$$V(\tilde{y}_i[t-1], \ldots, \tilde{y}_i[t-d], y_i[t-1], \ldots, y_i[t-d])$$

$$-V(\tilde{y}_i[t], \ldots, \tilde{y}_i[t-d+1], y_i[t], \ldots, y_i[t-d+1]).$$

where $\delta_i = \tilde{y}_i[t] - y_i[t]$. Since $\tilde{y}_i[t] = y_i[t]$ for $t < 0$, it follows that

$$V(\tilde{y}_i[-1], \ldots, \tilde{y}_i[-d], y_i[-1], \ldots, y_i[-d]) = 0.$$

Hence, taking into account that V takes non-negative values, the summation of the last inequality from $t = 0$ to $t = n$ yields

$$\epsilon \sum_{t=0}^{\infty} |\delta_i[t]|^2 \leq -\sum_{t=0}^{\infty} \delta_i[t]'F(y_i[t], \ldots, y_i[t-d], u_i[t]),$$

which in turn implies $E[F] \leq \tilde{E}_\epsilon[F]$. ■

C. Proof of Theorem 2

Let the sequences $\hat{y}_i = \{\hat{y}_i(t)\}_{t=0}^{n(i)}$ be defined by equations (6) with initial conditions (7). Then $E(F)$ equals the value of the sum (11) for $\delta_i(t) = \hat{y}_i(t) - y_i(t)$. Since $\hat{E}(F)$ is an upper bound for the sum, $E(F) \leq \hat{E}(F)$.

To show that $\hat{E}(F) \leq \tilde{E}(F)$, note that, due to (9),

$$\begin{aligned}
&\delta_i[t]'F(z_i[t], \ldots, z_i[t-d], u_i[t]) \\
&= \delta_i[t]'\{F(z_i[t], \ldots, z_i[t-d], u_i[t]) \\
&\quad -F(y_i[t], \ldots, y_i[t-d], u_i[t])\} \\
&\quad +\delta_i[t]'F(y_i[t], \ldots, y_i[t-d], u_i[t]) \\
&\leq -\epsilon|\delta_i[t]|^2 \\
&\quad +V(z_i[t], \ldots, z_i[t-d+1], y_i[t], \ldots, y_i[t-d+1]) \\
&\quad -V(z_i[t-1], \ldots, z_i[t-d], y_i[t-1], \ldots, y_i[t-d]) \\
&\quad +\delta_i[t]'F(y_i[t], \ldots, y_i[t-d], u_i[t]).
\end{aligned}$$

Summation from $t = 0$ to $t = n(i)$ yields

$$\sum_{i=1}^{N}\sum_{t=0}^{n(i)} \delta_i[t]'[\delta_i[t] + 2\epsilon^{-1}F(z_i[t], \ldots, z_i[t-d], u_i[t])]$$

$$\leq \sum_{i=1}^{N}\sum_{t=0}^{n(i)} \{2\epsilon^{-1}\delta_i[t]'F(y_i[t], \ldots, y_i[t-d] - |\delta_i[t]|^2\}$$

$$\leq \tilde{E}_\epsilon(F).$$

■

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