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Dynamic Output Feedback Control with Finite Frequency Specifications for Time-Delay Systems

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Abstract— **This paper considers the problem of control synthesis via dynamic output feedback for linear time-delay systems with finite frequency specifications. A finite frequency performance analysis condition for time-delay systems is presented. Then, a procedure of control synthesis via dynamic output feedback such that finite frequency specifications are captured, is given in the framework of linear matrix inequality (LMI) approach. Finally, the design procedure and the effectiveness of the proposed method are illustrated via a numerical design example.**

I. Introduction

One of the most fundamental results in the field of dynamical systems analysis, feedback control, and signal processing, is the Kalman-Yakubovic-Popov (KYP) lemma. Various properties of dynamical systems can be characterized by a set of inequality constraints in the frequency domain. The KYP lemma establishes equivalence between such frequency domain inequality (FDI) for a transfer function and a linear matrix inequality (LMI) for its state space realization. While the KYP lemma has been a major machinery for developing systems theory, it is not completely compatible with practical requirements. In particular, design specifications are often given for a certain frequency range of relevance. The generalized KYP (GKYP) lemma ([1]-[3]) provides a unified LMI characterization of FDIs in finite frequency ranges.

The phenomena of time delay are often encountered in many practical systems, such as chemical engineering systems, inferred grinding model, manual control, neural network, nuclear reactor, population dynamic model, and ship stabilization. Time delay often causes instability and generation of oscillation. Hence, many stabilization approaches have been proposed to deal with the problem of specification requirements. H_∞ control is proposed to reduce the effect of the disturbance input on the regulated output to within a prescribed level [4]-[9]. Positive real control is proposed to guarantee that the resulting closed-loop system is stable, and its transfer function is positive real [10]-[13]. Guaranteed cost control is proposed to stabilize the system and provides an upper bound on the performance index [14]-[18].

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The main objective of this paper is to derive a delay-independent finite frequency performance analysis condition and propose a design procedure for dynamic output feedback controllers such that the desired specifications in finite frequency ranges for linear time-delay systems are satisfied. The synthesis condition via dynamic output is given in terms of solutions to a set of LMIs by using full multiplier expansion approach based on the analysis result. A numerical design example will illustrate the procedures and usefulness of the proposed method.

The paper is organized as follows. Section 2 presents a finite frequency performance analysis condition for linear time-delay systems, and Section 3 provides a design procedure of dynamic output feedback controller. In Section 4, a numerical example is proposed to illustrate the design procedure and demonstrate their effectiveness. Some concluding remarks are shown in Section 5.

The following notations are used throughout this paper. For a matrix A, its transpose and complex conjugate transpose are denoted by A^T and A^*, respectively. The Hermitian part of a square matrix A is denoted by $\mathbf{He}(A) := A + A^*$. The symbol $\mathbf{H}_n$ stands for the set of $n\times n$ Hermitian matrices. I denotes the identity matrix with an appropriate dimension. The set of matrices $N = N^* \le 0$ is denoted by $\mathbf{N}$. For matrices Φ and P, $\Phi \otimes P$ means the Kronecker product. For matrices $G \in \mathbf{C}^{n\times m}$ and $\Pi \in \mathbf{H}_{n+m}$, a function $\sigma : \mathbf{C}^{n\times m} \times \mathbf{H}_{n+m} \to \mathbf{H}_m$ is defined by

$$\sigma(G,\Pi) := \begin{bmatrix} G \\ I_m \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I_m \end{bmatrix}.$$

II. Finite frequency performance analysis for linear time-delay systems

Consider a linear time-delay system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-d) + B\varpi(t) \\ z(t) &= Cx(t) + D\varpi(t) \end{aligned} \qquad (1)$$

for the continuous-time case or

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k-d) + B\varpi(k) \\ z(k) &= Cx(k) + D\varpi(k) \end{aligned} \qquad (2)$$

for the discrete-time case, where $x \in \mathbf{R}^n$ is the state vector, $\varpi \in \mathbf{R}^{n_\varpi}$ is the disturbance input and $z \in \mathbf{R}^{n_z}$ is the regulated output, respectively. A, A_d, B, C and D are known constant matrices of appropriate dimensions. $d > 0$ is an unknown state delay.

The transfer function matrix $G(\lambda)$ from ϖ to z is denoted by

$$G(s) = C(sI - A - e^{-ds}A_d)^{-1}B + D \tag{3}$$

for the continuous-time case or

$$G(z) = C(zI - A - z^{-d}A_d)^{-1}B + D \tag{4}$$

for the discrete-time case. Given a Hermitian matrix Π, the specification can be described by

$$\sigma(G(\lambda),\Pi) < 0 \qquad \forall\lambda \in \overline{\Lambda}(\Phi,\Psi) \tag{5}$$

where

$$\Lambda(\Phi,\Psi) := \{\lambda \in \mathbf{C} | \sigma(\lambda,\Phi) = 0, \sigma(\lambda,\Psi) \geq 0\} \tag{6}$$

and $\overline{\Lambda} := \Lambda$ if Λ is bounded and $\overline{\Lambda} := \Lambda \cup \{\infty\}$ if unbounded. Subsequently, a finite frequency performance analysis condition is given by the following theorem.

Theorem 1: Let matrices $A \in \mathbf{C}^{n\times n}$, $A_d \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{n\times n_\varpi}$, $C \in \mathbf{C}^{n_z\times n}$, $D \in \mathbf{C}^{n_z\times n_\varpi}$, $\Pi \in \mathbf{H}_{n_\varpi+n_z}$, and Φ, $\Psi \in \mathbf{H}_2$ be given and define Λ by (6). Suppose Λ represents curves on the complex plane. Then, $\sigma(G(\lambda),\Pi) < 0$ holds for all $\lambda \in \overline{\Lambda}(\Phi,\Psi)$ if there exist $P = P^*$, $Q = Q^* > 0$ and $X = X^*$ such that

$$\begin{bmatrix} A & B & A_d \\ I & 0 & 0 \end{bmatrix}^* (\Phi\otimes P + \Psi\otimes Q) \begin{bmatrix} A & B & A_d \\ I & 0 & 0 \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & -X \end{bmatrix} < 0. \tag{7}$$

Proof. For the continuous-time setting, the specification (5) can be denoted by the FDI

$$J^* \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} J < 0, \quad \omega \in \Lambda_c \tag{8}$$

where $J := \begin{bmatrix} (j\omega I - A - e^{-dj\omega}A_d)^{-1}B \\ I \end{bmatrix}$. It is equivalent to

$$L^* \begin{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & -X \end{bmatrix} L < 0, \tag{9}$$

where $L := \begin{bmatrix} (j\omega I - A - e^{-dj\omega}A_d)^{-1}B \\ I \\ e^{-dj\omega}(j\omega I - A - e^{-dj\omega}A_d)^{-1}B \end{bmatrix}$ and $X = X^*$.

Similarly, for the discrete-time setting, we can get the following FDI from (5)

$$J^* \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} J < 0, \tag{10}$$

where $\omega \in \Lambda_d$. It follows that

$$L^* \begin{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & -X \end{bmatrix} L < 0. \tag{11}$$

Defining

$$\zeta := \begin{bmatrix} (j\omega I - A - e^{-dj\omega}A_d)^{-1}B \\ I \\ e^{-dj\omega}(j\omega I - A - e^{-dj\omega}A_d)^{-1}B \end{bmatrix} \eta \tag{12}$$

with $\eta \in \mathbf{C}^{n_\varpi}$, $\eta \neq 0$ and $\omega \in \Lambda_c$ for the FDI in (9) or

$$\zeta := \begin{bmatrix} (e^{j\omega} I - A - e^{-dj\omega}A_d)^{-1}B \\ I \\ e^{-dj\omega}(e^{j\omega} I - A - e^{-dj\omega}A_d)^{-1}B \end{bmatrix} \eta \tag{13}$$

with $\eta \in \mathbf{C}^{n_\varpi}$, $\eta \neq 0$ and $\omega \in \Lambda_d$ for the FDI in (11), we have

$$\Gamma_\lambda F \zeta = 0, \quad \lambda \in \overline{\Lambda} \tag{14}$$

where

$$\Gamma_\lambda := \begin{cases} [I \quad -\lambda I] & (\lambda \in \Lambda) \\ [0 \quad -I] & (\lambda = \infty) \end{cases}, \quad F := \begin{bmatrix} A & B & A_d \\ I & 0 & 0 \end{bmatrix}. \tag{15}$$

Now we consider the generalization of the strict S-procedure in [1] given by

$$\mathrm{tr}(\Theta\mathbf{S}) < 0 \quad \Leftrightarrow \quad (\Theta + \mathbf{M}) \cap \mathrm{int}(\mathbf{N}) \neq \phi \tag{16}$$

where $\mathbf{S}$ is a set specified by $\mathbf{M}$ as follows:

$$\mathbf{S} := \{S \in \mathbf{H}_q : S \neq 0, S \geq 0, \mathrm{tr}(\mathbf{M}S) \geq 0, \mathrm{rank}(S) = 1\}. \tag{17}$$

Correspondingly, the set $\mathbf{S}$ would be given by

$$\begin{aligned} \mathbf{S} &:= \{\xi\xi^* : \xi \in \mathbf{G}_\lambda, \lambda \in \overline{\Lambda}\} \\ \mathbf{G}_\lambda &:= \{\xi \in \mathbf{C}^{2n+n_\varpi} : \xi \neq 0, \Gamma_\lambda F \xi = 0\}. \end{aligned} \tag{18}$$

From Lemma 2, the set $\mathbf{S}$ can be characterized by (17) with $\mathbf{M}$ in (43). By Lemma 3, the set $\mathbf{M}$ is admissible and rank-one separable. Hence, from Lemma 1, $\mathrm{tr}(\Theta\mathbf{S}) < 0$ where $\Theta := \begin{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & -X \end{bmatrix}$ is equivalent to the existence of $M \in \mathbf{M}$ satisfying

$$\Theta + M < 0. \tag{19}$$

From (14), it is easy to see that $\zeta \in \mathbf{G}_\lambda$, so $\zeta^*\Theta\zeta < 0$ holds if (19) holds. Thus (9) and (11) hold.

Remark 2: Theorem 1 gives a sufficient condition for finite frequency performance analysis of linear time-delay systems. It should be emphasized that the condition of Theorem 1 is only sufficient, not necessary. Correspondingly, it will lead to some conservatism.

The following result provides a dual version of Theorem 1.

Theorem 2: Let matrices $A \in \mathbf{C}^{n\times n}$, $A_d \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{n\times n_\varpi}$, $C \in \mathbf{C}^{n_z\times n}$, $D \in \mathbf{C}^{n_z\times n_\varpi}$, $\Pi \in \mathbf{H}_{n_\varpi+n_z}$, and Φ, $\Psi \in \mathbf{H}_2$ be given and consider Λ defined by (6). Suppose Λ represents curves on the complex plane. Then $\sigma(G(\lambda)^*,\Pi) < 0$ holds for all

$\lambda \in \overline{\Lambda}(\Phi^*, \Psi^*)$ if there exist $P = P^*$, $Q = Q^* > 0$ and $X = X^*$ such that

$$\begin{bmatrix} A & I \\ C & 0 \\ A_d & 0 \end{bmatrix} (\Phi \otimes P + \Psi \otimes Q) \begin{bmatrix} A & I \\ C & 0 \\ A_d & 0 \end{bmatrix}^* +$$

$$\begin{bmatrix} \begin{bmatrix} B & 0 \\ D & I \end{bmatrix} \Pi \begin{bmatrix} B & 0 \\ D & I \end{bmatrix}^* + \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & -X \end{bmatrix} < 0. \quad (20)$$

By Theorem 1, we can obtain the following result in the entire frequency range.

Corollary 1: Let matrices $A \in \mathbf{C}^{n\times n}$, $A_d \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{n\times n_\varpi}$, $C \in \mathbf{C}^{n_z\times n}$, $D \in \mathbf{C}^{n_z\times n_\varpi}$ and $\Pi \in \mathbf{H}_{n_\varpi+n_z}$ be given. Then, $\sigma(G(\lambda), \Pi) < 0$ holds if there exist $P = P^* > 0$ and $X = X^* > 0$ such that

$$\begin{bmatrix} A & B & A_d \\ I & 0 & 0 \end{bmatrix}^* (\Phi \otimes P) \begin{bmatrix} A & B & A_d \\ I & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & -X \end{bmatrix} < 0, \quad (21)$$

where $\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for continuous-time case or $\Phi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ for discrete-time case.

Remark 3: Corollary 1 provides an exact solution to the FDI specification given for the entire frequency range. In particular, the parameters P and X are required to be positive definite to enforce a stability constraint.

III. DYNAMIC OUTPUT FEEDBACK CONTROL SYNTHESIS

Consider the plant $G(\lambda)$ described by

$$\begin{aligned} \lambda x &= Ax + A_d x_d + B_1 \varpi + B_2 u \\ z &= C_1 x + D_{11} \varpi + D_{12} u \\ y &= C_2 x + D_{21} \varpi \end{aligned} \quad (22)$$

with a dynamic output feedback controller $K(\lambda)$ of the following form:

$$\begin{aligned} \lambda x_K &= A_K x_K + B_K y \\ u &= C_K x_K + D_K y \end{aligned} \quad (23)$$

where λ is the frequency variable (s for continuous-time and z for discrete-time cases), and $x \in \mathbf{R}^n$ is the state vector, $x_d \in \mathbf{R}^n$ is the delayed state with a time delay $d > 0$, $\varpi \in \mathbf{R}^{n_\varpi}$ is the disturbance input, $u \in \mathbf{R}^{n_u}$ is the control input, $z \in \mathbf{R}^{n_z}$ is the regulated output, $y \in \mathbf{R}^{n_y}$ is the measured output and $x_K \in \mathbf{R}^n$ is the state of the controller, respectively. A, A_d, B_1, B_2, C_1, C_2, D_{11}, D_{12} and D_{21} are known constant matrices of appropriate dimensions. A_K, B_K, C_K and D_K are the controller parameter matrices. Then the resulting closed-loop system is

$$\begin{aligned} \lambda x &= (A + B_2 D_K C_2)x + B_2 C_K x_K + A_d x_d + (B_1 + B_2 D_K D_{21})\varpi \\ \lambda x_K &= B_K C_2 x + A_K x_K + B_K D_{21} \varpi \\ z &= (C_1 + D_{12} D_K C_2)x + D_{12} C_K x_K + (D_{11} + D_{12} D_K D_{21})\varpi. \end{aligned} \quad (24)$$

Denote by $H(\lambda)$ the closed-loop transfer function from ϖ to z. The control synthesis problem under consideration is to design a dynamic output feedback controller $K(\lambda)$ such that

$$\sigma(H(\lambda)^*, \Pi) < 0 \qquad \forall \lambda \in \overline{\Lambda}(\Phi^T, \Psi^T). \quad (25)$$

For the later development, the following preliminaries are required.

Lemma 4 (Finsler's Lemma): Let $x \in \mathbf{R}^n$, symmetric matrix $\Xi \in \mathbf{R}^{n\times n}$, and $\Omega \in \mathbf{R}^{m\times n}$ such that $\mathrm{rank}(\Omega) = r < n$. Then the following statements are equivalent:

i) $x^* \Xi x < 0$, for any $x \neq 0$ and $\Omega x = 0$.

ii) $\exists \Delta \in \mathbf{R}^{n\times m}$: $\Xi + \Delta\Omega + \Omega^* \Delta^* < 0$.

The following result provides a feasible method to solve the control synthesis problem via dynamic output feedback.

Theorem 3: Consider the linear time-delay system (22) with a dynamic output feedback controller (23). Let $\Phi, \Psi \in \mathbf{H}_2$ and $\Pi \in \mathbf{H}_{n_\varpi+n_z}$ be given. Suppose $R \in \mathbf{C}^{(n+n_y)\times(4n+n_\varpi+n_z)}$ satisfies

$$YT \begin{bmatrix} \Phi \otimes P + \Psi \otimes Q & 0 & 0 & 0 \\ 0 & \Pi & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & -X \end{bmatrix} T^* Y^* - \mu Y R^* R Y^* < 0, \quad (26)$$

$$Y := \begin{bmatrix} \hat{A} + \hat{B}_0 K \hat{C}_0 & I & \hat{B} + \hat{B}_0 K \hat{D}_2 & I & 0 & 0 \\ \hat{C} + \hat{D}_1 K \hat{C}_0 & 0 & \hat{D} + \hat{D}_1 K \hat{D}_2 & 0 & I & 0 \\ \hat{A}_d & 0 & 0 & 0 & 0 & I \\ \hat{C}_0 & 0 & \hat{D}_2 & 0 & 0 & 0 \end{bmatrix}, \quad (27)$$

$$K := \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \quad (28)$$

where

$$\hat{A} := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{A}_d := \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{B} := \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

$$\hat{B}_0 := \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}, \quad \hat{C} := \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad \hat{C}_0 := \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix},$$

$$\hat{D} := D_{11}, \quad \hat{D}_1 := \begin{bmatrix} 0 & D_{12} \end{bmatrix}, \quad \hat{D}_2 := \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}, \quad (29)$$

$\mu > 0$ is a real scalar and T is the permutation matrix such that

$$\begin{bmatrix} M_1 & M_2 & M_3 & M_4 & M_5 & M_6 \end{bmatrix} T = \begin{bmatrix} M_1 & M_2 & M_3 & M_5 & M_4 & M_6 \end{bmatrix} \quad (30)$$

for arbitrary matrices M_1, M_2, M_3, M_4, M_5 and M_6 with column dimensions $2n$, $2n$, n_ϖ, n_z, $2n$ and $2n$, respectively. If there exist matrices $P = P^*$, $Q = Q^* > 0$, $X = X^*$, W, V_1, V_2, V_3 and $\mathscr{K}$ such that the following inequality

$$TZT^* < \mathbf{He} \begin{bmatrix} -\hat{C}_0^* \Sigma \Lambda + V_1 \\ V_2 \\ -\hat{D}_2^* \Sigma \Lambda + V_3 \\ (\hat{A}\hat{C}_0^* + \hat{B}\hat{D}_2^*)\Sigma\Lambda - \hat{A}V_1 - V_2 - \hat{B}V_3 - \hat{B}_0 \mathscr{K} R \\ (\hat{C}\hat{C}_0^* + \hat{D}\hat{D}_2^*)\Sigma\Lambda - \hat{C}V_1 - \hat{D}V_3 - \hat{D}_1 \mathscr{K} R \\ \hat{A}_d \hat{C}_0^* \Sigma\Lambda - \hat{A}_d V_1 \end{bmatrix}, \quad (31)$$

$$\Lambda := \hat{C}_0 V_1 + \hat{D}_2 V_3 - WR,$$
$$Z := \begin{bmatrix} \Phi\otimes P+\Psi\otimes Q & 0 & 0 & 0 \\ 0 & \Pi & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & -X \end{bmatrix}$$

holds where

$$\Sigma = \begin{bmatrix} I & 0 \\ 0 & (C_2C_2^* + D_{21}D_{21}^*)^{-1} \end{bmatrix}, \tag{32}$$

then the resulting closed-loop system (24) captures the finite frequency specification (25). In this case, the parameter matrices of the controller (23) is given by

$$\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} := \mathscr{K} W^{-1}. \tag{33}$$

Proof. By virtue of Theorem 2, the specification (25) is captured if the following inequality for the closed-loop system (24)

$$\begin{bmatrix} \mathscr{M} & I \end{bmatrix} T \begin{bmatrix} \Phi\otimes P+\Psi\otimes Q & 0 & 0 & 0 \\ 0 & \Pi & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & -X \end{bmatrix} T^* \begin{bmatrix} \mathscr{M} & I \end{bmatrix}^* < 0, \tag{34}$$

$$\mathscr{M} := \begin{bmatrix} \mathbf{A} & I & \mathbf{B} \\ \mathbf{C} & 0 & \mathbf{D} \\ \mathbf{A}_d & 0 & 0 \end{bmatrix}$$

holds, where T is defined by (30) and

$$\mathbf{A} := \begin{bmatrix} A+B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{bmatrix}, \quad \mathbf{A}_d := \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{B} := \begin{bmatrix} B_1+B_2D_KD_{21} \\ B_KD_{21} \end{bmatrix},$$

$$\mathbf{C} := \begin{bmatrix} C_1+D_{12}D_KC_2 & D_{12}C_K \end{bmatrix}, \quad \mathbf{D} := \begin{bmatrix} D_{11}+D_{12}D_KD_{21} \end{bmatrix}. \tag{35}$$

Defining a parameter matrix K of the controller (23) by (28) and matrices by (29), the closed-loop system matrices can be denoted by

$$\mathbf{A} = \hat{A} + \hat{B}_0 K \hat{C}_0, \qquad \mathbf{A}_d = \hat{A}_d, \qquad \mathbf{B} = \hat{B} + \hat{B}_0 K \hat{D}_2,$$

$$\mathbf{C} = \hat{C} + \hat{D}_1 K \hat{C}_0, \qquad \mathbf{D} = \hat{D} + \hat{D}_1 K \hat{D}_2. \tag{36}$$

So $\mathscr{M}$ can be rewritten as follows:

$$\begin{aligned} \mathscr{M} &:= \mathscr{A} + \mathscr{B}K\mathscr{C} \\ &= \begin{bmatrix} \hat{A} & I & \hat{B} \\ \hat{C} & 0 & \hat{D} \\ \hat{A}_d & 0 & 0 \end{bmatrix} + \begin{bmatrix} \hat{B}_0 \\ \hat{D}_1 \\ 0 \end{bmatrix} K \begin{bmatrix} \hat{C}_0 & 0 & \hat{D}_2 \end{bmatrix}. \end{aligned} \tag{37}$$

We assume that $\mathscr{C}$ has full row rank without loss of generality. According to Lemma 4, it follows that

$$T \begin{bmatrix} \Phi\otimes P+\Psi\otimes Q & 0 & 0 & 0 \\ 0 & \Pi & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & -X \end{bmatrix} T^* < \mathbf{He}\left(\begin{bmatrix} I \\ -\mathscr{M} \end{bmatrix} \mathscr{W} \right). \tag{38}$$

To make the problem tractable, the multiplier $\mathscr{W}$ is given to be

$$\mathscr{W} := \mathscr{C}^+ WR + (I - \mathscr{C}^+\mathscr{C})V, \qquad \det(W) \neq 0 \tag{39}$$

where $W \in \mathbf{C}^{(n+n_y)\times(n+n_y)}$, $V \in \mathbf{C}^{(4n+n_\varpi)\times(4n+n_\varpi+n_z)}$ and $R \in \mathbf{C}^{(n+n_y)\times(4n+n_\varpi+n_z)}$ is a matrix to be specified. If there exists a real scalar $\mu > 0$ such that R is chosen to satisfy (26), then (38) is equivalent to

$$T \begin{bmatrix} \Phi\otimes P+\Psi\otimes Q & 0 & 0 & 0 \\ 0 & \Pi & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & -X \end{bmatrix} T^* <$$
$$\mathbf{He}\left(\begin{bmatrix} \mathscr{C}^+ & I - \mathscr{C}^+\mathscr{C} \\ -\mathscr{A}\mathscr{C}^+ - \mathscr{B}K & \mathscr{A}(\mathscr{C}^+\mathscr{C} - I) \end{bmatrix} \begin{bmatrix} WR \\ V \end{bmatrix} \right) \tag{40}$$

without introducing conservatism. Defining $\mathscr{K} := KW$, it follows that

$$T \begin{bmatrix} \Phi\otimes P+\Psi\otimes Q & 0 & 0 & 0 \\ 0 & \Pi & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & -X \end{bmatrix} T^* <$$
$$\mathbf{He}\left(\begin{bmatrix} \mathscr{C}^+ & I - \mathscr{C}^+\mathscr{C} & 0 \\ -\mathscr{A}\mathscr{C}^+ & \mathscr{A}(\mathscr{C}^+\mathscr{C} - I) & -\mathscr{B} \end{bmatrix} \begin{bmatrix} WR \\ V \\ \mathscr{K}R \end{bmatrix} \right). \tag{41}$$

In view of $\mathscr{C}^+ = \mathscr{C}^*(\mathscr{C}\mathscr{C}^*)^{-1}$, we have

$$\begin{aligned} \mathscr{C}^+ &= \begin{bmatrix} \hat{C}_0^* \\ 0 \\ \hat{D}_2^* \end{bmatrix} \left(\begin{bmatrix} \hat{C}_0 & 0 & \hat{D}_2 \end{bmatrix} \begin{bmatrix} \hat{C}_0^* \\ 0 \\ \hat{D}_2^* \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} \hat{C}_0^* \\ 0 \\ \hat{D}_2^* \end{bmatrix} \Sigma = \begin{bmatrix} \hat{C}_0^*\Sigma \\ 0 \\ \hat{D}_2^*\Sigma \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} I - \mathscr{C}^+\mathscr{C} &= I - \begin{bmatrix} \hat{C}_0^*\Sigma \\ 0 \\ \hat{D}_2^*\Sigma \end{bmatrix} \begin{bmatrix} \hat{C}_0 & 0 & \hat{D}_2 \end{bmatrix} \\ &= \begin{bmatrix} I - \hat{C}_0^*\Sigma\hat{C}_0 & 0 & -\hat{C}_0^*\Sigma\hat{D}_2 \\ 0 & I & 0 \\ -\hat{D}_2^*\Sigma\hat{C}_0 & 0 & I - \hat{D}_2^*\Sigma\hat{D}_2 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} -\mathscr{A}\mathscr{C}^+ &= -\begin{bmatrix} \hat{A} & I & \hat{B} \\ \hat{C} & 0 & \hat{D} \\ \hat{A}_d & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{C}_0^*\Sigma \\ 0 \\ \hat{D}_2^*\Sigma \end{bmatrix} \\ &= \begin{bmatrix} -(\hat{A}\hat{C}_0^* + \hat{B}\hat{D}_2^*)\Sigma \\ -(\hat{C}\hat{C}_0^* + \hat{D}\hat{D}_2^*)\Sigma \\ -\hat{A}_d\hat{C}_0^*\Sigma \end{bmatrix}, \end{aligned}$$

$$\mathscr{A}(\mathscr{C}^+\mathscr{C}-I)=\begin{bmatrix}(\hat{A}\hat{C}_0^*+\hat{B}\hat{D}_2^*)\Sigma\hat{C}_0-\hat{A} & -I & (\hat{A}\hat{C}_0^*+\hat{B}\hat{D}_2^*)\Sigma\hat{D}_2-\hat{B}\\ (\hat{C}\hat{C}_0^*+\hat{D}\hat{D}_2^*)\Sigma\hat{C}_0-\hat{C} & 0 & (\hat{C}\hat{C}_0^*+\hat{D}\hat{D}_2^*)\Sigma\hat{D}_2-\hat{D}\\ \hat{A}_d\hat{C}_0^*\Sigma\hat{C}_0-\hat{A}_d & 0 & \hat{A}_d\hat{C}_0^*\Sigma\hat{D}_2\end{bmatrix},$$

$$\Sigma=(\hat{C}_0\hat{C}_0^*+\hat{D}_2\hat{D}_2^*)^{-1}=\begin{bmatrix}I & 0\\ 0 & (C_2C_2^*+D_{21}D_{21}^*)^{-1}\end{bmatrix}. \quad (42)$$

By defining $V:=\begin{bmatrix}V_1\\V_2\\V_3\end{bmatrix}$, we can obtain (31). Namely, if the inequality (31) holds, the closed-loop system (24) meets the requirement of the finite frequency specification (25).

Remark 4: The result of Theorem 3 is given in the framework of LMIs which can be solved numerically. The condition in Theorem 3 is sufficient for the existence of feasible dynamic output feedback controllers such that the resulting closed-loop time-delay systems achieve finite frequency design specifications. The choice of R will affect the associated degree of conservatism. If R is chosen to satisfy (26) with Y defined by (27), the introduced conservatism is minimal.

Remark 5: In the single-objective setting, a sufficient condition is provided for the multiplier basis to yield design procedures. The full multiplier expansion method is used to render the synthesis conditions convex.

The design specification in (25) can guarantee that the closed-loop system meets the requirement of performance in a certain finite frequency range of rlevance. However, the closed-loop stability has not been captured, and hence one may wish to include a stability constraint as an additional design specification. The following lemma gives a basic result for the closed-loop stability.

Lemma 5: Let $\hat{A}$, $\hat{A}_d$, $\hat{B}_0$ and $\hat{C}_0$ in (29), R and $\Phi\in\mathbf{H}_2$ be given. Then the following statements are equivalent:

i) There exist a parameter matrix $K:=\begin{bmatrix}A_K & B_K\\ C_K & D_K\end{bmatrix}$ and matrices $P=P^*>0$ and $X=X^*>0$ such that

$$\begin{bmatrix}\hat{A}+\hat{B}_0K\hat{C}_0 & I\\ \hat{A}_d & 0\end{bmatrix}(\Phi\otimes P)\begin{bmatrix}\hat{A}+\hat{B}_0K\hat{C}_0 & I\\ \hat{A}_d & 0\end{bmatrix}^*+\begin{bmatrix}X & 0\\ 0 & -X\end{bmatrix}<0,$$

and

$$R^{*\perp}\left(\begin{bmatrix}I & 0\\ 0 & I\\ 0 & 0\end{bmatrix}(\Phi\otimes P)\begin{bmatrix}I & 0\\ 0 & I\\ 0 & 0\end{bmatrix}^*+\begin{bmatrix}0&0&0\\0&X&0\\0&0&-X\end{bmatrix}\right)R^{*\perp *}<0.$$

ii) There exist matrices W, $\mathscr{K}$, $P=P^*>0$ and $X=X^*>0$ such that

$$\begin{bmatrix}I & 0\\ 0 & I\\ 0 & 0\end{bmatrix}(\Phi\otimes P)\begin{bmatrix}I & 0\\ 0 & I\\ 0 & 0\end{bmatrix}^*+\begin{bmatrix}0&0&0\\0&X&0\\0&0&-X\end{bmatrix}<\mathbf{He}\left(\begin{bmatrix}\hat{C}_0^+W\\ -\hat{A}\hat{C}_0^+W-\hat{B}_0\mathscr{K}\\ -\hat{A}_d\hat{C}_0^+W\end{bmatrix}R\right),$$

where

$$\hat{C}_0^+:=\begin{bmatrix}0 & C_2^*(C_2C_2^*)^{-1}\\ I & 0\end{bmatrix}.$$

IV. EXAMPLE

In this section, the proposed design method of a stabilizing dynamic output feedback controller, which makes the resulting closed-loop time-delay system meet specifications in finite frequency ranges, is illustrated via an numerical example.

Example 1: Consider the linear continuous time-delay system given by

$$\begin{aligned}\dot{x}(t)&=Ax(t)+A_dx(t-d)+B_1\varpi(t)+B_2u(t)\\ z(t)&=C_1x(t)+D_{11}\varpi(t)+D_{12}u(t)\\ y(t)&=C_2x(t)+D_{21}\varpi(t)\end{aligned}$$

with the following parameters

$$A=\begin{bmatrix}-2 & 1\\ -1 & 1\end{bmatrix},\quad A_d=\begin{bmatrix}0.2 & 0.1\\ 0.3 & 0.1\end{bmatrix},\quad B_1=\begin{bmatrix}1\\ -1\end{bmatrix},$$

$$B_2=\begin{bmatrix}-2\\ 1\end{bmatrix},\quad C_1=\begin{bmatrix}-1 & 1\end{bmatrix},\quad D_{11}=0,$$

$$D_{12}=0,\quad C_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad D_{21}=0.$$

and d is an unknown positive scalar.

Our objective is to design a stabilizing dynamic output feedback controller (23) such that

$$|G_{z\varpi}(j\omega)|<\gamma,\quad \forall|\omega|\le\omega_l$$

holds, where $G_{z\varpi}$is the closed-loop transfer functions from ϖ to z. From Lemma 5 and Theorem 3, the synthesis conditions are given by

$$\begin{bmatrix}0 & P_s & 0\\ P_s & X_s & 0\\ 0 & 0 & -X_s\end{bmatrix}<\mathbf{He}\left(\begin{bmatrix}\hat{C}_0^+W\\ -\hat{A}\hat{C}_0^+W-\hat{B}_0\mathscr{K}\\ -\hat{A}_d\hat{C}_0^+W\end{bmatrix}R_s\right),$$

$$\begin{bmatrix}-Q_l & P_l & 0 & 0 & 0 & 0\\ P_l & \omega_l^2Q_l & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & X_l & 0 & 0\\ 0 & 0 & 0 & 0 & -\gamma^2 & 0\\ 0 & 0 & 0 & 0 & 0 & -X_l\end{bmatrix}<$$

$$\mathbf{He}\begin{bmatrix}-\hat{C}_0^*\Sigma\Lambda+V_1\\ V_2\\ -\hat{D}_2^*\Sigma\Lambda+V_3\\ (\hat{A}\hat{C}_0^*+\hat{B}\hat{D}_2^*)\Sigma\Lambda-\hat{A}V_1-V_2-\hat{B}V_3-\hat{B}_0\mathscr{K}R_l\\ (\hat{C}\hat{C}_0^*+\hat{D}\hat{D}_2^*)\Sigma\Lambda-\hat{C}V_1-\hat{D}V_3-\hat{D}_1\mathscr{K}R_l\\ \hat{A}_d\hat{C}_0^*\Sigma\Lambda-\hat{A}_dV_1\end{bmatrix},$$

$$\Lambda:=\hat{C}_0V_1+\hat{D}_2V_3-WR_l,$$

where W, $\mathscr{K}$, $P_s = P_s^* > 0$, $X_s = X_s^* > 0$, $P_l = P_l^*$, $Q_l = Q_l^* > 0$, $X_l = X_l^* > 0$, V_1, V_2 and V_3 are the real variables and R_s and R_l are given by the followings, respectively

$$R_s = \left[\begin{array}{cc|cc|cc} -2I & 2I & -I & 2I & 0 & 0 \\ 0 & 6I & 8I & -I & 0 & 0 \end{array}\right],$$

$$R_l = \left[\begin{array}{c|c|c|cc|c|c} 0 & 0 & 0 & -5I & -3I & 0 & 0 \\ 0 & 0 & 0 & I & -6I & 0 & 0 \end{array}\right].$$

We fix the value of ω_l as

$$\omega_l = 2,$$

and then minimize γ. The optimal value of γ is found to be

$$\gamma_{min} = 0.3102,$$

and the controller parameters (A_K, B_K, C_K, D_K) are found to be:

$$A_K = \begin{bmatrix} -67.9253 & -136.4110 \\ 14.9212 & 30.7040 \end{bmatrix},$$

$$B_K = \begin{bmatrix} 123.3933 & 288.5334 \\ -46.0145 & -109.3057 \end{bmatrix},$$

$$C_K = \begin{bmatrix} 99.7888 & 203.1330 \end{bmatrix},$$

$$D_K = \begin{bmatrix} -133.7425 & -293.5104 \end{bmatrix}.$$

V. Conclusion

In this paper, the finite frequency performance analysis conditions and synthesis conditions via dynamic output feedback control for linear time-delay systems have been addressed. A method for synthesizing dynamic output feedback controllers is developed to capture the desired specifications in finite frequency ranges based on the resulting analysis conditions. Sufficient conditions for the existence of feasible controllers are given in terms of solutions to a set of LMIs. Finally, a numerical example is given to demonstrate the utility of the proposed design approach.

Appendix

Before presenting the proof for Theorem 1, some preliminaries are required.

Lemma 1[1]: Let an admissible set $\mathbf{M} \subset \mathbf{H}_q$ be given and define $\mathbf{S}$ by (17). Then, (16) holds for an arbitrary $\Theta \in \mathbf{H}_q$, if and only if $\mathbf{M}$ is rank-one separable.

Lemma 2: Let $F \in \mathbf{C}^{2n\times(2n+n_\varpi)}$ and Φ, $\Psi \in \mathbf{H}_2$ be given such that Λ in (6) represents curves. Then, the set $\mathbf{S}$ defined in (18) can be characterized by (17) with

$$\mathbf{M} := \{F^*(\Phi\otimes P + \Psi\otimes Q)F : P, Q \in \mathbf{H}_n, Q > 0\}. \quad (43)$$

Lemma 3: Let matrices Φ, $\Psi \in \mathbf{H}_2$ and $F \in \mathbf{C}^{2n\times(2n+n_\varpi)}$ be given such that Λ in (6) represents curves. Define the set $\mathbf{M}$ by (43) and the matrix-valued mapping Γ_λ by (15). The following statements hold true.

a) The set $\mathbf{M}$ is admissible and rank-one separable.

b) The set $\mathbf{M}$ is regular and the matrix F is a minimal realization of the set $\mathbf{M}$ in the sense that

$$\begin{aligned} & F^*(\Phi\otimes P + \Psi\otimes Q)F = 0, \quad P, Q \in \mathbf{H}_n, \quad Q > 0 \\ \Rightarrow \quad & \Phi\otimes P + \Psi\otimes Q = 0 \end{aligned} \quad (44)$$

if and only if

$$\mathrm{rank}(\Gamma_\lambda F) = n \quad \forall \lambda \in \overline{\Lambda}. \quad (45)$$

Remark 1: Lemma 2 and Lemma 3 correspondingly follow from Lemma 5 and Lemma 6 of [1].

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