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Analysis of Sampled-data Interconnected Systems

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Abstract— **In this paper, we consider the sampled-data problem of interconnected systems, specifically, time- and space-invariant systems. Our main contribution is to provide sufficient conditions on well-posedness, stability, and contractiveness of sampled-data interconnected systems in the form of a group of Linear Operator Inequalities (LOIs); And despite their infinite dimensionality, further reduce them to Linear Matrix Inequalities (LMIs). The technique is also applicable when dynamics are spatially continuous, and measurement and actuation take place in spatially localized patches; namely, when a spatial, rather than temporal, sampled-data arrangement is present.**

I. INTRODUCTION

Many systems are composed of multiple similar units that are interconnected to create a larger composite system, where the connection between units may for instance be simply neighboring units directly interacting. If the number of units is large, a centralized control approach to design can become prohibitively expensive from both computational and engineering perspectives due to fast increasing measurement costs and system complexity. In such cases distributed or decentralized control becomes natural choice to consider.

Frequently, interconnected systems can be approximated by *time- and space-invariant systems*, for which the continuous $\mathcal{H}_\infty$-alike distributed controllers with exact interconnection structure of the plant are discussed in [1] and [2]. A different controller design approach using Fourier transformation on both temporal and spatial dimensions is presented in [3].

However, these controllers are not immediate for digital implementation due to their continuous dynamics. Digital controllers are highly desired for any realistic system, due to the flexibility in design and the convenience in implementation, [4]. In order to bridge the gap, we adopt the *sampled-data*, in particular, *lifting* technique originated from [5] and [6] in our development. Although we focus here on temporal sampling, the approach presented has applicability to the spatially sampled case, which could for instance arise in micro- or nano-applications.

In this paper, we consider time- and space-invariant interconnected systems built from identical basic building blocks that only interact with their direct neighbors. For the sake of simplicity, we assume the dynamics of the system is of discrete-space but continuous-time. Also, we assume the system is controlled by a distributed digital controller with the same interconnection structure as the plant. The controller is assumed to be of discrete-time and -space.

In order to connect the continuous plant to the discrete controller, measurements must be sampled and controls must be held. However, in order to preserve the exact continuous interconnection between units, the interconnecting signals must be *lifted* instead of sampled. This leads to the major difficulty in this paper - the infinite dimensionality of the state space of the lifted interconnected system.

In this paper, we study analysis problems of the lifted open-loop system, which can be shown to have the same state space representation as the closed-loop system. Controller synthesis will be discussed in a future paper. Our main contribution is to provide sufficient conditions for well-posedness, stability and contractiveness of the sampled-data interconnected system in the form of a group of Linear Operator Inequalities and further reduce them to a group of Linear Matrix Inequalities despite the infinite dimensionality of the state space and the input/output spaces.

The rest of the paper is organized as follows. In Section II, we introduce the notation. The problem is mathematically formulated in Section III, followed by analysis of the lifted system in Section IV, where sufficient conditions are developed. Section V deals with the finite reduction. Conclusions are given in Section VI. Due to space limitation, most proofs and detailed derivations are omitted, interested readers can check [7] for details.

II. PRELIMINARIES

The sets of integers, non-negative integers, real numbers, and non-negative real numbers are denoted by $\mathbb{Z}$, $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{R}^+$ respectively. The notation $\mathbb{R}^\bullet$ denotes real-valued vectors whose size are either clear from context or irrelevant.

The space of n by m matrices is denoted by $\mathbb{R}^{n\times m}$, the space of symmetric n by n matrices is denoted by $\mathbb{S}^n$.

The space of square integrable functions mapping $[a,b] \mapsto \mathbb{R}^n$ is denoted as $L_2^n[a,b]$; The space of square summable sequences mapping $\mathbb{N}_0 \mapsto \mathbb{R}^n$ is denoted as ℓ_2^n. When n is clear or irrelevant, we simply denote them as $L_2[a,b]$ or ℓ_2.

We use s to denote the *L-tuple* $(s_1, s_2, \cdots, s_L)$. The variable s_i can either be in $\mathbb{Z}$ or in a subset $\{1, 2, \cdots, N_i\}$ of $\mathbb{Z}$. We use $\mathbb{D}_i$ to denote either of them, therefore, $s_i \in \mathbb{D}_i$.

Definition 1: The space l_2 is the set of sequences u mapping $\mathbb{D} := \mathbb{D}_1 \times \mathbb{D}_2 \times \cdots \times \mathbb{D}_L \mapsto \mathbb{R}^\bullet$, where $\sum_{\mathbf{s}\in\mathbb{D}} u^*(\mathbf{s})u(\mathbf{s}) < \infty$. It is a Hilbert space with inner product defined as $\langle u, w\rangle_{l_2} := \sum_{\mathbf{s}\in\mathbb{D}} u^*(\mathbf{s})w(\mathbf{s})$, The associated norm is defined as $\|u\|_{l_2} := \sqrt{\langle u, u\rangle_{l_2}}$.

Apparently, when $\mathbb{D} = \mathbb{Z}$, this is the standard ℓ_2 space.

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Definition 2: The space $\mathcal{K}_2$ is defined as the space $L_2[0,h)$. And $\mathcal{K}_2^n$ denotes the space $L_2^n[0,h)$.

Definition 3: The space $\tilde{\mathcal{K}}_2(\mathbb{D})$ (or simply $\tilde{\mathcal{K}}_2$ when $\mathbb{D}$ is clear from context or irrelevant) is the set of functions $\tilde{u}$ mapping $\mathbb{D} := \mathbb{D}_1 \times \mathbb{D}_2 \times \cdots \times \mathbb{D}_L \mapsto \mathcal{K}_2$ for which $\sum_{\mathbf{s}\in\mathbb{D}} \|\tilde{u}(\mathbf{s})\|_{\mathcal{K}_2} < \infty$ The inner product on this space is defined as

$$\begin{aligned}\langle \tilde{u}, \tilde{w}\rangle_{\tilde{\mathcal{K}}_2} &:= \sum_{\mathbf{s}\in\mathbb{D}} \langle \tilde{u}(\mathbf{s}), \tilde{w}(\mathbf{s})\rangle_{\mathcal{K}_2} \\ &= \sum_{\mathbf{s}\in\mathbb{D}} \int_0^h [(\tilde{u}^*(\mathbf{s}))(t)][(\tilde{w}(\mathbf{s}))(t)]dt\end{aligned}$$

The norm on this space is defined as $\|\tilde{u}\|_{\tilde{\mathcal{K}}_2} := \sqrt{\langle \tilde{u}, \tilde{u}\rangle_{\tilde{\mathcal{K}}_2}}$.

Definition 4: The space $\tilde{l}_2$ is the set of sequences $\tilde{u}$ mapping $\mathbb{N}_0 \mapsto \tilde{\mathcal{K}}_2$ for which $\sum_{k\in\mathbb{N}_0} \|\tilde{u}(k)\|_{\tilde{\mathcal{K}}_2}$ is finite. The inner product on this space is defined as

$$\begin{aligned}\langle \tilde{u}, \tilde{w}\rangle_{\tilde{l}_2} &:= \textstyle\sum_{k\in\mathbb{N}_0} \langle \tilde{u}(k), \tilde{w}(k)\rangle_{\tilde{\mathcal{K}}_2} \\ &= \textstyle\sum_{k\in\mathbb{N}_0} \sum_{\mathbf{s}\in\mathbb{D}} \langle \tilde{u}(k,\mathbf{s}), \tilde{w}(k,\mathbf{s})\rangle_{\mathcal{K}_2} \\ &= \textstyle\sum_{k\in\mathbb{N}_0} \sum_{\mathbf{s}\in\mathbb{D}} \int_0^h [(\tilde{u}^*(k,\mathbf{s}))(t)][(\tilde{w}(k,\mathbf{s}))(t)]dt\end{aligned}$$

The norm on this space is defined as $\|\tilde{u}\|_{\tilde{l}_2} := \sqrt{\langle \tilde{u}, \tilde{u}\rangle_{\tilde{l}_2}}$.

III. PROBLEM SETUP

A. Interconnected Systems

For simplicity, we only consider systems with one spatial dimension (with both forward and backward channels) in our derivation. Results here can be generalized to systems with multiple spatial dimensions.

Fig. 1 depicts the basic building block of the plant, which is assumed to be a finite-dimensional Linear Time-invariant (FD-LTI) system with the following state space representation

$$\begin{aligned}\begin{bmatrix}\dot{x}(t,s)\\ w(t,s)\\ z(t,s)\end{bmatrix} &= \begin{bmatrix}A_{TT} & A_{TS} & B_T\\ A_{ST} & A_{SS} & B_S\\ C_T & C_T & D\end{bmatrix}\begin{bmatrix}x(t,s)\\ v(t,s)\\ z(t,s)\end{bmatrix}\\ x(0,s) &= x_0 \in \mathbb{R}^\bullet\end{aligned} \quad (1)$$

where $v(t,\mathbf{s}) = \begin{bmatrix}v_+(t,\mathbf{s})\\ v_-(t,\mathbf{s})\end{bmatrix}, \quad w(t,\mathbf{s}) = \begin{bmatrix}w_+(t,\mathbf{s})\\ w_-(t,\mathbf{s})\end{bmatrix}$.

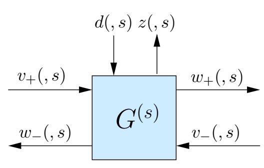

Fig. 1. Basic Building Block

We consider an infinite interconnection in this paper. This approximation is sufficient for a large number of systems; In particular, when the scale of the influence of localized effects is much less than that of the whole system, [1] and [3].

The first order infinite connection is shown in Fig. 2, where we assume

$$\begin{aligned}v_+(t,s+1) &= w_+(t,s) \in \mathbb{R}^{m+}\\ v_-(t,s-1) &= w_-(t,s) \in \mathbb{R}^{m-}, \quad \forall s \in \mathbb{Z}\end{aligned} \quad (2)$$

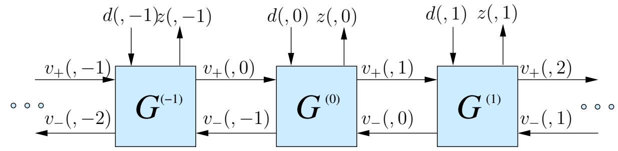

Fig. 2. Infinite Interconnected System

Define the following bi-directional shift operator:

$$\mathbf{\Delta}_{S,\mathbf{m}} = \mathbf{diag}(\mathbf{S}I_{m_+}, \mathbf{S}^{-1}I_{m_-})$$

where $\mathbf{S}$ is the shift operator on the forward spatial dimension. Then the interconnected system can be written as

$$\begin{aligned}\begin{bmatrix}\dot{x}(t,s)\\ (\mathbf{\Delta}_{S,\mathbf{m}}v)(t,s)\\ z(t,s)\end{bmatrix} &= \begin{bmatrix}A_{TT} & A_{TS} & B_T\\ A_{ST} & A_{SS} & B_S\\ C_T & C_T & D\end{bmatrix}\begin{bmatrix}x(t,s)\\ v(t,s)\\ z(t,s)\end{bmatrix}\\ x(0,s) &= x_0 \in \mathbb{R}^\bullet\end{aligned} \quad (3)$$

It is also possible to form a period connections or other higher order connections, see [1], [7] and references therein.

B. Lifting the Interconnected System

Lifting is now a standard technique for lumped-parameter sampled-data systems, see [4], [5], [8] and [9]. In this section, we generalize this technique to the interconnected system shown in Fig. 2.

The *lifting operator* $\mathcal{L} : L_2[0,\infty) \mapsto \tilde{\mathcal{K}}_2(\mathbb{N}_0)$ is defined such that for a signal $u \in L_2[0,\infty)$,

$$\tilde{u} = \mathcal{L}u, \quad \tilde{u}_k(t) = u(kh+t), \quad \text{for } 0 \le t < h$$

The *inverse lifting operator* $\mathcal{L}^{-1} : \tilde{\mathcal{K}}_2(\mathbb{N}_0) \mapsto L_2[0,\infty)$ is defined such that, for a signal $\tilde{u} \in \tilde{\mathcal{K}}_2(\mathbb{N}_0)$,

$$u = \mathcal{L}^{-1}\tilde{u}, \quad u(t) = \tilde{u}_k(t-kh), \quad \text{for } kh \le t < (k+1)h.$$

Remark 1: The key property of the lifting operator is that it defines an *isometry* between the two spaces $L_2[0,\infty)$ and $\tilde{\mathcal{K}}_2(\mathbb{N}_0)$ [5], in the way that

$$\|\tilde{u}\|^2_{\tilde{\mathcal{K}}_2(\mathbb{N}_0)} = \|u\|^2_{L_2[0,\infty)}$$

Let us consider the interconnected system (3). A lifted basic building block with measurements and controls included is shown in Fig. 3.

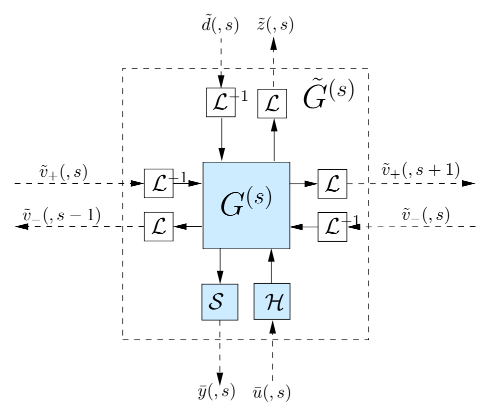

Fig. 3. Lifted Basic Building Block

The sampler $\mathcal{S}$ and the zero-order hold $\mathcal{H}$ are assumed to have perfect synchronization. The sampling period is denoted as h. The sampler and the hold work as follows,

$$\bar{y}(k,s) = \mathcal{S}y(t,s) = y(kh,s) \qquad \text{for } kh \le t < (k+1)h$$
$$u(t,s) = \mathcal{H}\bar{u}(k,s) \equiv u(k,s) \qquad \text{for } kh \le t < (k+1)h$$

The setup in Fig. 3 preserves the exact continuous interconnection between units by lifting instead of sampling the interconnection signal v. However, this comes at the cost of infinite-dimensional state-space.

It can be shown that the closed-loop system has the same state space representation as the lifted open-loop system without measurements y and control signals $\bar{u}$, if we assume the controller has the same interconnection structure as the plant [7]. The lifted open-loop interconnected system is shown in Fig. 4.

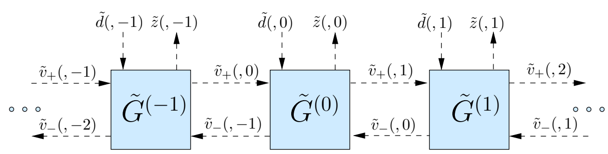

Fig. 4. Lifted Interconnected System

Thus in order to consider analysis problems of the closed-loop system, we can equivalently study the lifted open-loop interconnected system,

$$\begin{bmatrix} \bar{x}(k+1,s) \\ (\tilde{\mathbf{\Delta}}_{S,\mathbf{m}}\tilde{v})(k,s) \\ \tilde{z}(k,s) \end{bmatrix} = \begin{bmatrix} A_{TTd} & \tilde{A}_{TS} & \tilde{B}_T \\ \tilde{A}_{ST} & \tilde{A}_{SS} & \tilde{B}_S \\ \tilde{C}_T & \tilde{C}_S & \tilde{D} \end{bmatrix} \begin{bmatrix} \bar{x}(k,s) \\ \tilde{v}(k,s) \\ \tilde{d}(k,s) \end{bmatrix}$$
$$\bar{x}(0,s) = x_0 \in \mathbb{R}^{m_0}. \tag{4}$$

where $\bar{x}(t,s) \in \mathbb{R}^{m_0}$, $\tilde{v}(t,s) \in \mathcal{K}_2^{m_+} \oplus \mathcal{K}_2^{m_-}$, $\tilde{d}(t,s) \in \mathcal{K}_2^d$, and $\tilde{z}(t,s) \in \mathcal{K}_2^z$. The operator $\tilde{\mathbf{\Delta}}_{S,\mathbf{m}}$ on $\mathcal{K}_2$ is defined as

$$\tilde{\mathbf{\Delta}}_{S,\mathbf{m}} := \mathbf{diag}(\tilde{\mathbf{S}}\tilde{I}_{m_+}, \tilde{\mathbf{S}}^{-1}\tilde{I}_{m_-})$$

where $\tilde{I}_{m_+}$ and $\tilde{I}_{m_-}$ are unit operators on $\mathcal{K}_2^{m_+}$ and $\mathcal{K}_2^{m_-}$ respectively, and $\tilde{\mathbf{S}}$ is the *shift operator* on the forward channel.

The other operators are defined as follows, derivations can be found in [7].

$$\begin{aligned} A_{TT_d} &:= e^{A_{TT}h} \\ \tilde{A}_{TS}\tilde{v}(k,s) &:= \int_0^h e^{A_{TT}(h-\sigma)}A_{TS}\tilde{v}_k(\sigma,s)d\sigma \\ \tilde{B}_T\tilde{d}(k,s) &:= \int_0^h e^{A_{TT}(h-\sigma)}B_T\tilde{d}_k(\sigma,s)d\sigma. \end{aligned} \tag{5}$$

$$\begin{aligned} (\tilde{A}_{ST}x)(t,s) &:= A_{ST}e^{A_{TT}t}x(t,s) \\ (\tilde{A}_{SS}\tilde{v}_k)(t,s) &:= A_{SS}\tilde{v}_k(t,s) \\ &\quad + \int_0^t \{A_{ST}e^{A_{TT}(t-\sigma)}A_{TS}\}\tilde{v}_k(\sigma,s)d\sigma \\ (\tilde{B}_S\tilde{d}_k)(t,s) &:= B_S\tilde{d}_k(t,s) \\ &\quad + \int_0^t \{A_{ST}e^{A_{TT}(t-\sigma)}B_T\}\tilde{d}_k(\sigma,s)d\sigma \end{aligned} \tag{6}$$

$$\begin{aligned} \tilde{C}_T &:= C_T e^{A_{TT}t} \\ (\tilde{C}_S\tilde{v}_k)(t,s) &:= C_S\tilde{v}_k(t,s) \\ &\quad + \int_0^t \{C_T e^{A_{TT}(t-\sigma)}A_{TS}\}\tilde{v}_k(\sigma,s)d\sigma \\ (\tilde{D}\tilde{d}_k)(t,s) &:= D\tilde{d}_k(t,s) \\ &\quad + \int_0^t \{C_T e^{A_{TT}(t-\sigma)}B_T\}\tilde{d}_k(\sigma,s)d\sigma. \end{aligned} \tag{7}$$

Define the following operators for convenience

$$\tilde{A} = \begin{bmatrix} A_{TTd} & \tilde{A}_{TS} \\ \tilde{A}_{ST} & \tilde{A}_{SS} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_T \\ \tilde{B}_S \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} \tilde{C}_T & \tilde{C}_S \end{bmatrix} \tag{8}$$

Remark 2: After lifting, the system becomes a discrete-time and -space linear system which is shift-invariant in both temporal and spatial dimensions.

Remark 3: Part of the state variable, $\tilde{v}$ is in $\tilde{\ell}_2$, which is of infinite-dimensional. This structure prohibits the direct application of the techniques used in [5], where the lifted system is reduced to an equivalent FD-LTI system but depending on the fact that for a lumped parameter system the state-space stays finite after lifting.

IV. ANALYSIS

In this section, we develop sufficient conditions on *well-posedness, stability and performance* properties of the lifted interconnected system (4) in the form of a group of LOIs. We assume the system before lifting has these properties satisfied. Finite computation will be discussed in the next section.

Similar to [1], by eliminating the interconnection signal $\tilde{v}$, the system (4) can be re-written as

$$\begin{aligned} \bar{x}(k+1) &= \mathbf{A}\bar{x}(k) + \mathbf{B}\tilde{d}(k) \\ \tilde{z}(k) &= \mathbf{C}\bar{x}(k) + \mathbf{D}\tilde{d}(k), \end{aligned} \tag{9}$$

where $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} := \begin{bmatrix} A_{TT_d} & \tilde{B}_T \\ \tilde{C}_T & \tilde{D} \end{bmatrix} + \begin{bmatrix} \tilde{A}_{TS} \\ \tilde{C}_S \end{bmatrix} \times (\tilde{\mathbf{\Delta}}_{S,\mathbf{m}} - \tilde{A}_{SS})^{-1} \times \begin{bmatrix} \tilde{A}_{ST} & \tilde{B}_S \end{bmatrix}.$

A. Well-posedness

A system is well-posed if it is physically realizable. In our setup, we define the well-posedness property as follows,

Definition 5: A system is well-posed if and only if $(\tilde{\mathbf{\Delta}}_{S,\mathbf{m}} - \tilde{A}_{SS})$ is invertible.

We have the following Lyapunov type test on it.

Lemma 1: The system (4) is well-posed if there exists an invertible self-adjoint operator $\tilde{X}_S$ on the space $\mathcal{K}_2^{m_+ + m_-}$ such that

$$\tilde{A}_{SS}\tilde{X}_S\tilde{A}_{SS} - \tilde{X}_S < 0 \tag{10}$$

Proof: The system is well-posed if there exists an invertible self-adjoint operator $\tilde{X}_S$ on the space $\mathcal{K}_2^{m_+ + m_-}$ such that $\tilde{A}_{SS}\tilde{\mathbf{\Delta}}_{S,\mathbf{m}}^*\tilde{X}_S\tilde{\mathbf{\Delta}}_{S,\mathbf{m}}\tilde{A}_{SS} - \tilde{X}_S < 0$, (10) follows by considering the structure of $\tilde{\mathbf{\Delta}}_{S,\mathbf{m}}$. ■

B. Stability

For a well-posed system, given an initial condition $\bar{x}_0 \in l_2$, the solution to (9) is

$$\bar{x}(k) = \mathbf{A}^k\bar{x}_0 + \sum_{p=0}^{k-1}\mathbf{A}^{k-1}B\tilde{d}(p).$$

A system is said to be *(uniformly exponential) stable* if there exists a finite positive constant α and a constant $0 \le \beta < 1$ such that for all k, we have

$$\|\mathbf{A}^k\|_{\tilde{l}_2} \le \alpha\beta^k.$$

Lemma 2: The lifted system (4) is uniformly exponential stable if there exists an operator $\tilde{X} = \mathbf{diag}(X_T, \tilde{X}_S)$, where $X_T \in \mathbb{S}^{m_0}$ is a positive definite matrix and $\tilde{X}_S$ is an invertible self-adjoint operator on the space $\mathcal{K}_2^{m_+ + m_-}$, such that

$$\tilde{A}\tilde{X}\tilde{A} - \tilde{X} < 0 \qquad (11)$$

The proof is similar to that of Lemma 1. Notice that the $(1,1)$ block of (11) is exactly the well-posedness test (10).

C. Contractiveness

Assume the system (4) is stable, we say the system is *contractive* if the induced gain from input to output is strictly less than one, namely, $||\tilde{d} \mapsto \tilde{z}||_{\tilde{l}_2 \to \tilde{l}_2} < 1$. This is an $\mathcal{H}_\infty$ type criterion, which can be checked in the following way, similar to the KYP lemma.

Lemma 3: The system (4) is stable and contractive if there exists an invertible self-adjoint operator $\tilde{X}$ defined in the same way as in Lemma 2, such that

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}^* \begin{bmatrix} \tilde{X} & 0 \\ 0 & \tilde{I} \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} - \begin{bmatrix} \tilde{X} & 0 \\ 0 & \tilde{I} \end{bmatrix} < 0 \qquad (12)$$

where the operator $\tilde{I}$ is the unit operator on $\mathcal{K}_2^z$.

Notice that the $(1,1)$ block of (12) is exactly the stability test (11).

V. FINITE COMPUTATION

In the previous section, by using similar techniques as in [2], sufficient conditions on well-posedness, stability and contractiveness are given in the form of LOIs: (10), (11), and (12). Unfortunately, they are not immediate for computation. In this section, we reduce these LOIs to computable LMIs based on the technique presented in [10], where the induced norm of a *compression operator* is computed. The cornerstone of this finite reduction is the isometric transformation introduced below:

Lemma 4: ([10]) On the space $\mathcal{K}_2 = L_2[0,h)$, given a constant $\theta \in (-\pi, \pi]$, define

$$\psi_k(t) = h^{-1/2} e^{j\omega_k t}, \text{ for } 0 \le t < h$$

where $\omega_k := \frac{2\pi v_k + \theta}{h}$, and $\{v_k\} = \{0, \pm 1, \pm 2, \cdots\}$. Then $\{\psi_k\}_0^\infty$ form a complete orthonormal basis for $\mathcal{K}_2$.

For an operator $\tilde{K} : \mathcal{K}_2 \mapsto \mathcal{K}_2$, there exists an operator $\bar{K} : \ell_2 \mapsto \ell_2$, such that

$$\langle e_l, \bar{K} e_k \rangle := \langle \psi_l, \tilde{K} \psi_k \rangle \qquad (13)$$

where $\{e_k\}_0^\infty$ are standard bases of ℓ_2.

Definition 6: A *compression operator* $\tilde{K}$ on the Hilbert space $\mathcal{K}_2$ is defined to be of the following form

$$(\tilde{K}u)(t) := \int_0^t Ce^{A(t-\tau)} Bu(\tau) d\tau + Du(t), \quad 0 \le t \le h$$

where A, B, C, D are matrices of appropriately dimensions, and $h > 0$ is a real number.

From Lemma 4 and (13), $\tilde{K}$ has the following equivalent representation on the ℓ_2 space given $e^{j\theta} \notin \mathbf{eig}(e^{Ah})$,

$$\bar{K} = \begin{bmatrix} \bar{C}_0 \\ \vdots \end{bmatrix} (Ie^{j\theta} - \bar{A})^{-1} \begin{bmatrix} \bar{B}_0 & \cdots \end{bmatrix} + \mathbf{diag}(\bar{G}_0, \cdots) \qquad (14)$$

The exact definition of matrices here will be clear as we proceed. It is worth mentioning that $\bar{K}$ is the sum of a finite rank operator and a block diagonal operator.

A. Well-posedness

From Lemma 1, we know that the existence of a self-adjoint operator $\tilde{X}_S$ on $\mathcal{K}_2^{m_+ + m_-}$ which satisfies (10) is sufficient for the system (4) to be well-posed. In this section, we provide a finite-dimensional reduction of the LOI (10). Let us start with the definition of $\tilde{A}_{SS}$ as in (6). It is clearly a compression operator by Definition 6. From Lemma 4 and (14), it has the following equivalent representation on the standard ℓ_2 space:

$$\bar{A}_{SS} = \begin{bmatrix} \bar{C}_0 \\ \vdots \end{bmatrix} M \begin{bmatrix} \bar{B}_0 & \cdots \end{bmatrix} + \begin{bmatrix} \bar{G}_0 & \\ & \ddots \end{bmatrix} \qquad (15)$$

where,

$$\begin{aligned} \check{A} &= e^{A_{TT}h} && (16) \\ M &= (Ie^{j\theta} - \check{A})^{-1} && (17) \\ \bar{B}_k &= (Ij\omega_k - A_{TT})^{-1}(Ie^{j\theta} - \check{A})A_{TS}h^{-1/2} && (18) \\ \bar{C}_k &= A_{ST}(\check{A}e^{-j\theta} - I)h^{-1/2}(Ij\omega_k - A_{TT})^{-1} && (19) \\ \bar{G}_k &= A_{ST}(Ij\omega_k - A_{TT})^{-1}A_{TS} + A_{SS} && (20) \end{aligned}$$

We have the following technical lemma:

Lemma 5: Assume the system before lifting is well-posed, then there exists a positive integer N and an invertible matrix $Y \in \mathbb{S}^{m_+ + m_-}$, such that for all $k > N$,

$$Y - \bar{G}_k^* Y \bar{G}_k > 0 \qquad (21)$$

Proof: Since the system before lifting is well-posed, there exists an invertible matrix $Y \in \mathbb{S}^{m_+ + m_-}$ such that $Y - A_{SS}^* Y A_{SS} > 0$. As $k \to \infty$, $\bar{G}_k \to A_{SS}$. Thus, the positive number N exists. ■

The following lemma follows from the isometry between $\mathcal{K}_2$ and ℓ_2.

Lemma 6: The lifted system is well-posed if there exists a self-adjoint operator X_S on $\ell_2^{m_+ + m_-}$ such that

$$\bar{A}_{SS}^* X_S \bar{A}_{SS} - X_S < 0$$

Given a matrix $Y \in \mathbb{S}^{m_+ + m_-}$ and a positive integer N which satisfies Lemma 5, for an integer $n > N$, we define

$$X_S = \mathbf{diag}(X_n, Y_{\inf}) \qquad (22)$$

where $X_n \in \mathbb{S}^{n(m_+ + m_-)}$, and $Y_{\inf} = \mathbf{diag}(Y, Y, \cdots)$.

Partition the operator $\bar{A}_{SS}$ in (15) as follows,

$$\bar{A}_{SS} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} M \begin{bmatrix} B_1 & B_2 \end{bmatrix} + \begin{bmatrix} G_1 & \\ & G_2 \end{bmatrix} \qquad (23)$$

where matrix $G_1 = \mathbf{diag}(\bar{G}_0, \cdots, \bar{G}_n)$, the operator $G_2 = \mathbf{diag}(\bar{G}_{n+1}, \cdots)$, and operators C_i, B_i, $i = 1, 2$ are partitioned accordingly.

By arithmetic manipulations, we have

$$\begin{aligned}\bar{A}_{SS}^* X_S \bar{A}_{SS} \quad -X_S = &\begin{bmatrix} F_0^*(X_n) \\ F_1^* \end{bmatrix} Q^{-1}(X_n) \begin{bmatrix} F_0(X_n) & F_1 \end{bmatrix} \\ &+ \begin{bmatrix} G_1^* X_n G_1 & \\ & G_2^* Y_{\inf} G_2 \end{bmatrix} - \begin{bmatrix} X_n & 0 \\ 0 & Y_{\inf} \end{bmatrix}\end{aligned} \tag{24}$$

where

$$\begin{aligned}\begin{bmatrix} F_0(X_n) & F_1 \end{bmatrix} &= \begin{bmatrix} C_1^* X_n G_1 & C_2^* Y_{\inf} G_2 \\ B_1 & B_2 \end{bmatrix} \\ Q^{-1}(X_n) &= \begin{bmatrix} 0 & M \\ M^* & M^*(C_1^* X_n C_1 + C_2^* Y_{\inf} C_2) M \end{bmatrix}\end{aligned}$$

It is clear that $(24) < 0$ is equivalent to the feasibility of the following operator inequality

$$\begin{aligned}&\begin{bmatrix} I & 0 \\ 0 & Y_{\inf} - J_1 \end{bmatrix} - \begin{bmatrix} J_0(X_n) & 0 \\ 0 & 0 \end{bmatrix} \\ &- \begin{bmatrix} F_0^*(X_n) \\ F_1^* \end{bmatrix} Q^{-1}(X_n) \begin{bmatrix} F_0(X_n) & F_1 \end{bmatrix} > 0\end{aligned} \tag{25}$$

where $J_0(X_n) = G_1^* X_n G_1 - X_n + I$ and $J_1 = G_2^* Y_{\inf} G_2$.

Now we state the first theorem of the paper on checking the well-posedness property of the lifted system.

Theorem 1: Assume the system before lifting is well-posed, and $e^{j\theta} \notin \textbf{eig}(e^{A_{TT}h})$. Let $N > 0$ be an integer that satisfies Lemma 5. If for some $n > N$, there exists a finite-dimensional matrix $X_n \in \mathbb{S}^{n(m_+ + m_-)}$ such that the following LMI (26) is feasible,

$$\begin{aligned}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} &- \begin{bmatrix} J_0(X_n) & 0 \\ 0 & 0 \end{bmatrix} \\ &- \begin{bmatrix} F_0^*(X_n) \\ E_1^* \end{bmatrix} Q^{-1}(X_n) \begin{bmatrix} F_0(X_n) & E_1 \end{bmatrix} > 0\end{aligned} \tag{26}$$

where $E_1 E_1^* = F_1 (Y_{\inf} - J_1)^{-1} F_1^*$ is a matrix, then the lifted system (4) is well-posed.

Proof: Since we choose $n > N$, then $Y_{\inf} - J_1 > 0$ by Lemma 5. Therefore $S := (Y_{\inf} - J_1)^{-1/2}$ is well-defined. From [10], we know that if the LMI (26) is feasible then the LOI (25) is feasible, therefore $(24) < 0$ is feasible with the operator $X_S = \textbf{diag}(X_n, Y_{\inf})$. By Lemma 6, the lifted system is well-posed. ∎

B. Stability

To guarantee stability, we need the LOI (11) to be feasible for some operator $\tilde{X} = \textbf{diag}(X_T, \tilde{X}_S)$, where $X_T \in \mathbb{S}^{m_0}$ is a positive definite matrix, and $\tilde{X}_S$ is a self-adjoint operator on $\mathcal{K}_2^{m_+ + m_-}$ as in the well-posedness part.

Let us start with the structure of $\tilde{A}$ defined in (8), which is an operator on $\mathbb{R}^{m_0} \oplus \mathcal{K}_2^{m_+ + m_-}$. From Lemma 4, it can also be represented on ℓ_2 as follows,

$$\begin{aligned}\bar{A} \;=\; &\begin{bmatrix} -M^{-1} \\ -\bar{C}_0 \\ \vdots \end{bmatrix} \times M \times \begin{bmatrix} M^{-1} & -\bar{B}_0 & \cdots \end{bmatrix} \\ &+\textbf{diag}(L, \bar{G}_0, \cdots)\end{aligned}$$

where $\check{A}$, M, $\bar{C}_k$, $\bar{B}_k$, and $\bar{G}_k$ are defined in (16) through (20), and $L = Ie^{j\theta}$ is a constant matrix for a fixed θ.

Similar to Lemma 6, we have the following lemma:

Lemma 7: The system (4) is stable if there exists a self-adjoint operator X on $\mathbb{R}^{m_0} \oplus \ell_2^{m_+ + m_-}$ with the structure $\textbf{diag}(X_T, X_S)$ such that

$$\bar{A}^* X \bar{A} - X < 0$$

Define $X_S = \textbf{diag}(X_n, Y_{\inf})$ as in (22), follow the same derivation as in the well-posedness case, partition $\bar{A}$ per the structure of X_T and X_S, we have

$$\begin{aligned}\bar{A} \;=\; &\begin{bmatrix} -M^{-1} \\ -C_1 \\ -C_2 \end{bmatrix} \times M \times \begin{bmatrix} M^{-1} & -B_1 & -B_2 \end{bmatrix} \\ &+\textbf{diag}(L, G_1, G_2)\end{aligned}$$

Then we have the following equality:

$$\begin{aligned}\bar{A}^* X \bar{A} - X = &\begin{bmatrix} L^* X_T L & & \\ & G_1^* X_n G_1 & \\ & & G_2^* Y_{\inf} G_2 \end{bmatrix} \\ &+ \begin{bmatrix} F_0^*(X_T) \\ F_1^*(X_n) \\ F_2^* \end{bmatrix} Q^{-1}(X_T, X_n) \begin{bmatrix} F_0(X_T) & F_1(X_n) & F_2 \end{bmatrix} \\ &- \begin{bmatrix} X_T & & \\ & X_n & \\ & & Y_{\inf} \end{bmatrix}\end{aligned} \tag{27}$$

where

$$\begin{aligned}&\begin{bmatrix} F_0(X_T) & F_1(X_n) & F_2 \end{bmatrix} \\ &:= \begin{bmatrix} -(M^{-1})^* X_T L & -C_1^* X_n G_1 & -C_2^* Y_{\inf} G_2 \\ M^{-1} & -B_1 & -B_2 \end{bmatrix}\end{aligned}$$

$$Q^{-1}(X_T, X_n) = \begin{bmatrix} 0 & M \\ M^* & X_T + M^*(C_1^* X_n C_1 + C_2^* Y_{inf} C_2) M \end{bmatrix}$$

Since $L^* X_T L = Ie^{-j\theta} X_T I e^{j\theta} = X_T$, then $(27) < 0$ is equivalent to the feasibility of following LOI

$$\begin{aligned}&\begin{bmatrix} I & & \\ & I & \\ & & Y_{\inf} - J_1 \end{bmatrix} - \begin{bmatrix} I & & \\ & J_0(X_n) & \\ & & 0 \end{bmatrix} - \\ &\begin{bmatrix} F_0^*(X_T) \\ F_1^*(X_n) \\ F_2^* \end{bmatrix} Q^{-1}(X_T, X_n) \begin{bmatrix} F_0(X_T) & F_1(X_n) & F_2 \end{bmatrix} > 0\end{aligned} \tag{28}$$

where $J_0(X_n) = G_1^* X_n G_1 - X_n + I$ and $J_1 = G_2^* Y_{\inf} G_2$.

We have the following theorem on the stability.

Theorem 2: Assume the system before lifting is stable, and $e^{j\theta} \notin \textbf{eig}(e^{A_{TT}h})$. Let $N > 0$ be an integer that satisfies Lemma 5. If for some $n > N$, there exist finite-dimensional matrices $X_T \in \mathbb{S}^{m_0}$, and $X_n \in \mathbb{S}^{n(m_+ + m_-)}$ such that the following LMI (29) is feasible,

$$\begin{aligned}&\begin{bmatrix} I & & \\ & I & \\ & & I \end{bmatrix} - \begin{bmatrix} I & & \\ & J_0(X_n) & \\ & & 0 \end{bmatrix} - \\ &\begin{bmatrix} F_0^*(X_T) \\ F_1^*(X_n) \\ E_2^* \end{bmatrix} Q^{-1}(X_T, X_n) \begin{bmatrix} F_0(X_T) & F_1(X_n) & E_2 \end{bmatrix} > 0\end{aligned} \tag{29}$$

where $E_2^* E_2 = F_2 (Y_{\inf} - J_1)^{-1} F_2^*$ is a matrix, then the lifted system (4) is stable.

The proof is similar to that of Theorem 1.

C. Contractiveness

Now let us consider the reduction of the LOI (12).

Let us start with the simplest version of (12) by assuming the system has no temporal dynamics; Then it can be written as follows,

$$M_{SS}^* \tilde{X} M_{SS} - \tilde{X} < 0$$

where $M_{SS} := \begin{bmatrix} \tilde{A}_{SS} & \tilde{B}_S \\ \tilde{C}_S & \tilde{D} \end{bmatrix}$, and $\tilde{X} = \begin{bmatrix} \tilde{X}_S & \\ & \tilde{I} \end{bmatrix}$. $\tilde{X}_S$ is a self-adjoint operator on $\mathcal{K}_2^{m_+ + m_-}$, and $\tilde{I}$ is the unit operator on $\mathcal{K}_2^z$.

Our goal is to find such a feasible operator $\tilde{X}$ by using finite computation, the difficulty here is that both $\tilde{X}_S$ and $\tilde{I}$ are infinite-dimensional operators.

From (6) and (7), it is clear that M_{SS} is just a regular compression operator. By reloading the notation $\bar{B}_k$, $\bar{C}_k$, $\bar{G}_k$, we can represent M_{SS} on the standard ℓ_2 space as we did for the operator $\tilde{A}_{SS}$.

$$\bar{M}_{SS} = \begin{bmatrix} \bar{C}_0 \\ \bar{C}_1 \\ \vdots \end{bmatrix} M \begin{bmatrix} \bar{B}_0 & \bar{B}_1 & \cdots \end{bmatrix} + \begin{bmatrix} \bar{G}_0 & & \\ & \bar{G}_1 & \\ & & \ddots \end{bmatrix} \quad (30)$$

where $\check{A}$ and M are defined in (16) and (17), and

$$\bar{B}_k = (Ij\omega_k - A_{TT})^{-1}(Ie^{j\theta} - \check{A}) \begin{bmatrix} A_{TS} & B_S \end{bmatrix} h^{-1/2} \quad (31)$$

$$\bar{C}_k = \begin{bmatrix} A_{ST} \\ C_T \end{bmatrix} (\check{A}e^{-j\theta} - I) h^{-1/2} (Ij\omega_k - A_{TT})^{-1} \quad (32)$$

$$\bar{G}_k = \begin{bmatrix} A_{ST} \\ C_T \end{bmatrix} (Ij\omega_k - A_{TT})^{-1} \begin{bmatrix} A_{TS} & B_S \end{bmatrix} + \begin{bmatrix} A_{SS} & B_S \\ C_S & D \end{bmatrix} \quad (33)$$

Lemma 8: Assume the system before lifting (3) is contractive, then there exists an positive integer N and an invertible matrix $\mathbf{diag}(Y, I)$, where $Y \in \mathbb{S}^{m_+ + m_-}$ and I is an identity matrix on $\mathbb{R}^z$, such that for all $k > N$

$$\begin{bmatrix} Y & \\ & I \end{bmatrix} - \bar{G}_k^* \begin{bmatrix} Y & \\ & I \end{bmatrix} \bar{G}_k > 0 \quad (34)$$

where $\bar{G}_k$ is defined in (33).

Again, from the isometry between $\mathcal{K}_2$ and ℓ_2, we have,

Lemma 9: The system without temporal dimension is contractive if there exists a self-adjoint operator $X \in \ell_2^{m_+ + m_- + z}$ such that

$$\bar{M}_{SS}^* X \bar{M}_{SS} - X < 0$$

Again we want to choose $X = \mathbf{diag}(X_n, Y_{\text{inf}})$, but we have to be careful this time since now both X_n and Y_{inf} are structured, as we shall see in the following remark.

Remark 4: In the above procedure, an implicit coordinate transformation has been performed. Originally, we want to find an $\tilde{X} = \mathbf{diag}(\tilde{X}_S, \tilde{I})$ on $\mathcal{K}_2^{m_+ + m_-} \oplus \mathcal{K}_2^z$. However, $\bar{M}_{SS}$ in (30) comes from an operator on $\mathcal{K}_2^{m_+ + m_- + z}$. This introduces an equivalent coordinate transformation from $\ell_2^{m_+ + m_-} \oplus \ell_2^z \mapsto \ell_2^{m_+ + m_- + z}$. We have to take that into account by choosing $\tilde{X}$ in the form of $\mathbf{diag}(X_n, Y_{\text{inf}})$, with X_n of the following structure:

$$X_n = \begin{bmatrix} X_{11} & 0 & \cdots & X_{1n} & 0 \\ 0 & I & \cdots & 0 & 0 \\ \cdots & & & & \\ X_{n1} & 0 & \cdots & X_{nn} & 0 \\ 0 & 0 & \cdots & 0 & I \end{bmatrix} \quad (35)$$

Each $X_{ij} \in \mathbb{S}^{m_+ + m_-}$, where $0 \leq i, j \leq n$; And each I is an identity matrix on $\mathbb{R}^z$. Operator $Y_{\text{inf}} = \mathbf{diag}\left(\begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}, \cdots \right)$ for some Y which satisfies the previous Lemma 8.

We have the following theorem on the contractiveness.

Theorem 3: Assume the system before lifting is contractive, and $e^{j\theta} \notin \mathbf{eig}(e^{A_{TT}h})$. Let $N > 0$ be an integer that satisfies Lemma 8. If for some $n > N$, there exists a finite-dimensional matrix $X_n \in \mathbb{S}^{n(m_+ + m_- + z)}$ with structure (35) such that the following LMI (36) is feasible,

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} J_0(X_n) & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} F_0^*(X_n) \\ E_1^* \end{bmatrix} Q^{-1}(X_n) \begin{bmatrix} F_0(X_n) & E_1 \end{bmatrix} > 0 \quad (36)$$

where J_0, J_1, F_0, F_1 and E_1 are defined in the same way as in well-posedness part but using redefined operators (31) to (33), then the lifted system is contractive.

The proof is similar to the proof of theorem 1.

Remark 5: We can now add the temporal dimension back, by following the exact procedure from well-posedness test to stability test but using the reloaded notation (31) to (33). Interested readers are referred to [7] for details.

VI. CONCLUSION REMARKS

In this paper we have formulated and solved the analysis problem of sampled-data interconnected systems. An infinite-dimensional state-space model is employed to represent the lifted interconnected system exactly. A group of LOIs is provided as sufficient conditions on well-posedness, stability, and contractiveness of the lifted interconnected system, which are further reduced to computable LMIs.

Future works include but not limited to: controller synthesis, and extending current results to the heterogenous case, where the system is composed of non-uniform basic building blocks.

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