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A coverage algorithm for a class of non-convex regions

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Abstract— **The paper describes a framework for solving the coverage problem for a class of non-convex domains. In [1] we have shown how a diffeomorphism can be used to transform a non-convex coverage problem to a convex one to which the Lloyd's algorithm [2] can be applied. In this paper we show how a diffeomorphism can be constructed for convex regions with obstacles in its interior, so that the solution of the transformed problem yields the solution of the original non-convex problem. As part of this investigation we also identify stationary points of the Lloyd's algorithm in non-convex domains. We provide the formal analysis of the approach and demonstrate its effectiveness through simulations.**

I. Introduction

The need to uniformly cover a set with a finite number of points is common to several branches of engineering. Back in 1957, Lloyd [3] studied how to choose what he called *quanta* so that the *noise power* would be minimized. He proposed to iterate between computing the regions associated with each quantum (Voronoi regions) and the repositioning of each quantum to the centroid of its region.

Several extensions of what is now known as the *Lloyd's Algorithm* have been presented, most notably the generalization [4] in vector quantization. That approach takes into consideration the probabilistic model of the data sequence for which the *quanta* of Lloyd are to be obtained. In general, all the variations of the algorithm implicitly require the convexity of the set in question, so that the repositioning towards the centroid of each Voronoi region is always feasible.

In 2004, Cortés *et al.* [2] proposed a distributed control algorithm motivated by Lloyd's work to achieve (locally) optimal coverage in convex regions using a family of mobile robots. Extensions to this method that include alternative parameters to be optimized have recently been proposed, among which we highlight the work on energy consumption by Kwok and Martínez [5]. Approaches considering different types of sensors and robotic platforms have also been explored [6]. An important feature of the algorithm in [2] is that it is completely distributed and converges to a stationary point of an associated cost function as long as the hypothesis on the convexity of the region is satisfied. These properties have been instrumental for developing distributed control algorithms for networks that need to perform other tasks such as sensing in addition to coverage [7]–[10]. At this point, we mention that the notion of *coverage* is not related with the visibility problems in non-convex regions, known as the *Art Gallery Problem* [11]. For recent results on this problem see [12] and the references therein.

An aspect that has not been addressed in [2] and is the focus of this work is how to perform coverage in domains that are not convex. In [1] we proposed an extension of the original algorithm to non-convex domains by using a diffeomorphic transformations. Although the approach produces a solution of the coverage problem, the diffeomorphism changes the mass distribution for the Lloyd's algorithm so the generated solution does not necessarily correspond to a uniform coverage in the original space. Furthermore, it is difficult to quantify how far the two solutions are. In this paper we address these issues. We first characterize the set of stationary points for the Lloyd's algorithm in general regions. Subsequently, we present a family of diffeomorphisms that guarantee the convergence of the agents to the (locally) optimal points in the interior of the original space under the assumption that the region is convex but has obstacles inside it. We conclude with a discussion of our approach, and the extensions that are currently underway.

II. Preliminaries

In this section we briefly present some background material and the notation that will be used in the rest of the paper.

For a given set $\mathcal{X} \subseteq \mathbb{R}^n$, we denote by $Co(\mathcal{X})$ its convex hull (the intersection of all convex sets containing $\mathcal{X}$). If $\mathcal{X}$ is a compact surface, then $(\mathcal{X})^{int}$ denotes the *interior* of $\mathcal{X}$ (the union of all open sets contained in $\mathcal{X}$), $\overline{\mathcal{X}}$ its closure, and $\partial\mathcal{X} = \overline{\mathcal{X}} \setminus (\mathcal{X})^{int}$ its *boundary*.

We now state the Riemann Mapping Theorem as given in [13], where its proof can also be found.

Theorem 1 (Riemann Mapping Theorem): Given any simply connected open proper subset Ω in the plane and a point $z_0 \in \Omega$, there exists a unique analytic function $f(z)$ in Ω normalized by the conditions $f(z_0) = 0,\ f'(z_0) > 0$, such that $f(z)$ defines a one-to-one mapping of Ω onto the disk $|\omega| < 1$.

A direct consequence of this theorem is that for any two simply connected regions Ω_1 and Ω_2, neither of which is the whole plane, there exists an analytic one-to-one mapping $g : \Omega_1 \to \Omega_2$.

Unfortunately, Theorem 1 only holds in $\mathbb{R}^2$ (which is isomorphic to $\mathbb{C}$) and cannot be extended directly to higher dimensions. Some of our results are thus limited to planar surfaces, although we will outline how the approach can be extended to more general spaces.

Let $\mathcal{R}$ be a convex region in $\mathbb{R}^2$. As mentioned before, even though the results in [2] are trivially extensible to $\mathbb{R}^m$,

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in order to guarantee the existence of a diffeomorphism using Theorem 1, we restrict ourselves to planar surfaces. Let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ be a collection of n robots in the interior of $\mathcal{R}$. Denote their configuration at time $t \geq 0$ by $\mathcal{P}_t = \{p_1(t), p_2(t), \ldots, p_n(t)\}$, respectively. For simplicity, we will drop the explicit dependency on time, and will refer to a particular robot interchangeably by a_i or p_i.

For each time t, the agents in $\mathcal{A}$ induce a *Voronoi partition* of $\mathcal{R}$, where the Voronoi region V_i generated by a_i is given by:

$$V_i = \{q \in \mathcal{R} |\ \|q - p_i\| \leq \|q - p_j\|, 1 \leq j \leq n, j \neq i\}. \tag{1}$$

Observe that in our definition, there are agents a_i, a_j for which $V_i \cap V_j \neq \emptyset$. This intersection is a segment of the perpendicular bisector of the line segment with the endpoints p_i and p_j. It thus has measure zero and is irrelevant for integration purposes. We refer the reader to [14]–[16] for more information on Voronoi partitions and related subjects.

We say that the agents a_i and a_j are (Voronoi) *neighbors* if $V_i \cap V_j \neq \emptyset$. We denote by $\mathcal{N}_i$ the set of all the neighbors of an agent a_i:

$$\mathcal{N}_i = \{a_j \in \mathcal{A} |\ V_i \cap V_j \neq \emptyset\}.$$

Observe that with our definition, a_i is a neighbor of itself, so the relation of "being neighbors" is reflexive.

The following theorem tells us that for any simply connected manifold[1] in $\mathbb{R}^m$, there is an ϵ-strip around its boundary such that for each point in such strip the concept of "closest point" to the set is well-defined. Although we don't present the complete statement of the ϵ-Neighborhood Theorem (as presented, for instance, in [18]), we will refer to this result by that name.

Theorem 2 (ϵ-Neighborhood Theorem [18]): For a compact manifold $\mathcal{Y} \subset \mathbb{R}^m$ and a positive number ϵ, let Y_ϵ be the open set of points in $\mathbb{R}^m$ with distance less than ϵ from Y. If ϵ is sufficiently small, then each point $w \in Y_\epsilon$ possesses a unique closest point in Y, denoted by $\pi(w)$.

We next present the concept of a tangent vector and a tangent plane. Informally, we will think of a tangent vector $\mathbf{v}_x$ to $x \in \mathbb{R}^m$ as a vector that *starts* at x. Let $\alpha : [0, 1] \to \mathbb{R}^m$ be a curve in $\mathbb{R}^m$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. The *velocity vector* of α at $x = \alpha(t_0)$ is the tangent vector

$$\alpha'(t_0) = \left(\frac{d\alpha_1}{dt}(t), \frac{d\alpha_2}{dt}(t), \ldots, \frac{d\alpha_n}{dt}(t)\right)_{t=t_0}.$$

For a point $p \in M \subset \mathbb{R}^m$, a tangent vector $\mathbf{v}_p$ is tangent to M at p provided that v_p is a velocity vector at p for some curve in M. We denote the set of all tangent vectors to M at p by $T_p(M)$ (or simply T_p when M is clear from the context), and we call this set the *tangent plane* to M at p.

III. Description of the Problem

We start by considering a robotic network as defined in [19] with n different agents. As discussed earlier, we denote the position of the ith robot with respect to a common reference frame by p_i, and we denote by $\mathcal{P}$ the set of such configurations. Let $\mathcal{R} \subset \mathbb{R}^m$ be a compact subset. As in [2], we want to find a configuration set $\mathcal{P}$ such that the following function of $\mathcal{P}$ attains a local minimum:

$$J(\mathcal{P}) = \int_{\mathcal{R}} \min_{1 \leq i \leq n} \left[\|q - p_i\|^2 \varphi(q)\right] dq, \tag{2}$$

where $\varphi : \mathcal{R} \to \mathbb{R}$ is an a.e. smooth function, which describes the *density* of $\mathcal{R}$.

An essential assumption in [2] and for the application of the Lloyd's algorithm in general is that $\mathcal{R}$ is convex. Because of the nature of the control law the centroids need to be *reachable* for each of the agents, and this is only guaranteed if the domain is convex. In fact, it is not difficult to construct a non-convex region whose centroid belongs to its exterior. We would thus like to investigate how to deal with domains that are not convex. As part of this investigation, we characterize the set of points where the minima for (2) is attained. We then offer a solution to the problem by extending our previous work in [1].

IV. Points where the minima is attained in general regions

We will describe the set of points in $\mathcal{R} \subset \mathbb{R}^m$ where the function given in (2) attains its minima.

Assume that $\partial\mathcal{R}$ is smooth. For a point $p \in \mathcal{R}$ we denote by $T_p(\partial\mathcal{R})$ the tangent space to $\partial\mathcal{R}$ at the point p. Let the density $\varphi : \mathcal{R} \to \mathbb{R}$ in (2) be an a.e. smooth function. Consider its extension $\psi : Co(\mathcal{R}) \to \mathbb{R}$ defined as:

$$\psi(q) = \begin{cases} \varphi(q) & q \in \mathcal{R} \\ 0 & q \notin \mathcal{R}. \end{cases} \tag{3}$$

Observe that such function ψ is differentiable a.e. inside the convex set $Co(\mathcal{R})$, which implies that we can define the coverage problem given in (2) for this domain. For simplicity, let $J(\mathcal{U})$ be the cost function induced in the region $\mathcal{U}$.

$$J(Co(\mathcal{R})) = \int_{Co(\mathcal{R})} \min_{1 \leq i \leq n} \left[\|q - p_i\|^2 \psi(q)\right] dq. \tag{4}$$

Let $\mathcal{M}_J(\mathcal{U})$ denote the set of stationary points for the function $J(\mathcal{U})$. Since $Co(\mathcal{R})$ is convex, the elements of $\mathcal{M}_J(Co(\mathcal{R}))$ are centroidal Voronoi configurations. Observe that since $\psi|_{Co(\mathcal{R})\setminus\mathcal{R}} = 0$,

$$J(Co(\mathcal{R})) = J(\mathcal{R}). \tag{5}$$

Despite the equality in (5), note that every configuration in $\mathcal{M}_J\left((\mathcal{R})^{int}\right)$ is also a local minimum in $\mathcal{M}_J(Co(\mathcal{R}))$ but the opposite is not necessarily true. However we can say, following the arguments in [2], that the elements in $\mathcal{M}_J(\mathcal{R})$ are defined by either configurations in which the agents are located at the centroids of their respective Voronoi regions or on the boundary $\partial\mathcal{R}$, or for cases in which all the centroids belong to $(\mathcal{R})^{int}$, they are centroidal configuration on $Co(\mathcal{R})$.

[1] Loosely speaking, a smooth set with no holes in its interior. For a formal definition see [17].

We now introduce the concept of *orthogonality point*. These are points on the boundary that attain the minima for (2).

Definition 1 (Orthogonality point): The point $p \in \partial\mathcal{R}$ is an *orthogonality point* to $q \in \mathbb{R}^m$ if for any tangent vector $v_p \in T_p(\partial\mathcal{R})$ it is true that $(q - p) \cdot v_p = 0$.

Remark 1: It can be proven following the arguments we present next, that if a point in the boundary $\partial\mathcal{R}$ belongs to a stationary configuration for the cost function given in (2), then that point is an orthogonality point for the centroid of its Voronoi region.

Remark 2: Observe that, for a given centroid C, there might be several orthogonality points to C, which will belong to different stationary points for the cost function (2).

In the rest of this section, we will show that the points on the boundary that attain a local minima for (2) are orthogonality points to the centroid of the respective Voronoi region when such a centroid is not reachable under the control law proposed in [2] from the current configuration.

Let f be a smooth real-valued function $f : \mathcal{X} \to \mathbb{R}$, where $\mathcal{X}$ is a metric space its associated metric d. Recall that the point $x \in \mathcal{X}$ is a local minimum for f if there exists some $\epsilon > 0$ such that $f(x) \leq f(x')$ for all $x' \in B_\epsilon(x) = \{y \in \mathcal{X} : d(x, y) < \epsilon\}$. Note that if x is a stationary point for f, then for every smooth trajectory $\gamma : [0, 1] \to B_\epsilon(x)$ such that $\gamma(1) = x$, it is true that $f(\gamma)$ attains its minimum at $\gamma(1)$. This implies that there is some $\epsilon > 0$ such that for every $t \in (1 - \epsilon, 1)$

$$[f(\gamma(t))]' = f'(\gamma(t)) \cdot \gamma'(t) \leq 0, \tag{6}$$

and, when extended to $[1 - \epsilon, 1]$, the respective limit does exist at the end points and also satisfies the inequality (6).

Suppose that the configuration $\mathcal{P}$ is a local minimum for the cost function $J(\mathcal{R})$. Let $p \in \mathcal{P}$ and assume $p \in \partial\mathcal{R}$. Let $\gamma : [0, 1] \to \mathcal{R}$ be a smooth curve with $\gamma(1) = p$. If we fix the positions $\mathcal{P} \setminus \{p\}$ and we travel along γ, variations in J will be induced as a consequence of the variations of p while traversing γ.

Let C_V be the centroid of the Voronoi region generated by p, and let M_V be its hypervolume. To simplify the notation we will drop the dependency on time for both of these quantities. By following the reasoning in [2], we thus obtain that

$$\frac{dJ}{dt} = 2M_V(\gamma(t) - C_V) \cdot \gamma'(t). \tag{7}$$

By virtue of (6), and since $\mathcal{P}$ is a local minimum for J, then for every smooth curve γ as above, it holds that at $t = 1$

$$\frac{dJ}{dt} = 2M_V(p - C_V) \cdot \gamma'(1) \leq 0. \tag{8}$$

Observe that the right hand side in (8) can be written as

$$2M_V\|p - C_V\|\,\|\gamma'(1)\|\cos\alpha, \tag{9}$$

where α is the angle between $\gamma'(1)$ and $p - C_V$, as shown in Figure 1. For (8) to hold, it is thus necessary that $\pi/2 \leq \alpha \leq \pi$.

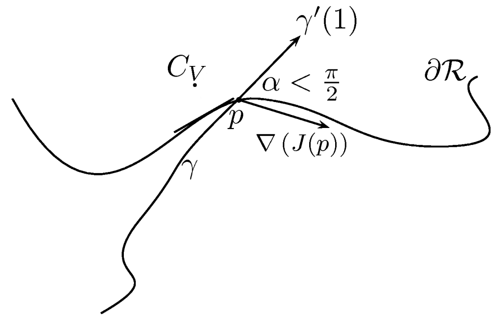


Fig. 1. If p is not an orthogonality point to the centroid (i.e. the orthogonal projection of C_V on $\partial\mathcal{R}$), there is a trajectory γ for which $\gamma(1) = p$ does not give a minimum.

If p is *not* an orthogonality point to C_V then, since ∂R is smooth, there exists a curve γ such that the induced angle α is acute, violating the inequality in (8). We can thus state the following theorem:

Theorem 3: The elements of the set $\mathcal{M}_J(\mathcal{R})$ consist of those formations $\mathcal{P} \subset \mathcal{R}^n$ for which the elements in $\mathcal{P}$ either coincide with the centroids of their respective Voronoi regions, or are at an orthogonality point to such a centroid.

Remark 3: If we allow $\mathcal{R}$ to be a polygonal region, then $\partial\mathcal{R}$ is only a.e. smooth. In that case, if $p \in \mathcal{R}$ is such that $\partial\mathcal{R}$ is not smooth at p, p would be an extreme point for J if there exists a small $\epsilon > 0$ such that if we consider a curve γ as before in $B_\epsilon(p)$, then for $q \in B_\epsilon(p) \setminus \{p\}$ it holds that $2M_V(q - C_V) \cdot \gamma'(t_q) \leq 0$, where $\gamma(t_q) = q$.

Remark 4: Note that an agent might be at an orthogonality point even if its centroid is inside $\mathcal{R}$ (when a boundary prevents it to move towards the centroid). This phenomenon does not occur when $\mathcal{R}$ is convex.

So far, we have characterized the stationary points for (2), but we have not presented any control law that would allow the agents to approach such configurations. In the next section, we will present how the configurations that lie in the interior of $\mathcal{R}$ can be reached by using a suitable diffeomorphism when $\mathcal{R}$ is convex but not simply-connected.

V. Transformation of the Problem

In this section, we propose a diffeomorphism that drives the agents to a stationary configuration, as long as the location of each agent is in the interior of the region. We first present the scenario when a single obstacle is inside a convex region $\mathcal{R} \subset \mathbb{R}^2$, and then we generalize the approach to multiple obstacles.

A. Dealing with a single hole

Let $\mathcal{H} \subset \mathcal{R} \subset \mathbb{R}^2$ be a bounded, simply connected manifold with boundary, such that $\partial\mathcal{H} \cap \partial\mathcal{R} = \emptyset$.

Let $\mathcal{H}_\epsilon$ be an ϵ-neighborhood of $\mathcal{H}$. Each point $h \in \mathcal{H}_\epsilon$ has a unique closest point in $\partial\mathcal{H}$ according to Theorem 2. We assume that ϵ is small enough, so the result still holds for $\partial\mathcal{H}_\epsilon$. Let $h^\star$ be the closest point in $\partial\mathcal{H}$ to h; i.e. $h^\star = \pi(h)$.

Claim 4: Let h' be a point on the line segment $\overline{hh^\star}$. Then $h^\star$ is also the closest point to h' in $\partial\mathcal{H}$.

Proof: This follows from the triangle inequality. ■

Let $\psi_1 : (\mathcal{H})^{int} \to \mathbb{R}^2$ be a diffeomorphism between the interior of $\mathcal{H}$ and an open disk $\mathcal{S} \subset \mathcal{H}$. Such a diffeomorphism exists according to Theorem 1. Because of our hypothesis on $\mathcal{R}$, the boundary of $\mathcal{H}$ is locally connected, which implies that we can extend ψ_1 to $\partial\mathcal{H}$ in such a way that it induces a 1-1 mapping to ∂S.

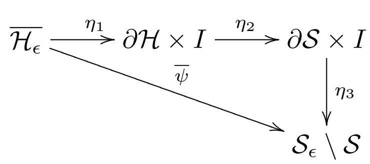
For every point $s \in \mathcal{S}$ we induce a homotopy path from s to S_0, the center of S. This homotopy path is given by the contraction F induced on the different radii of $\mathcal{S}$.

We now describe the construction of a diffeomorphism ϕ_ϵ that will take the ϵ-strip around $\mathcal{H}$ to cover the hole.

We want to define a diffeomorphism $\phi_\epsilon : \mathcal{H}_\epsilon \to \mathcal{H}_\epsilon \cup \partial\mathcal{H}$. Observe that because of our conditions on $\mathcal{R}$ and $\mathcal{H}$, $\partial\mathcal{H}_\epsilon$ is locally connected, and hence ϕ_ϵ can be extended to this boundary. Furthermore, it will act as the identity in $\partial\mathcal{H}_\epsilon$. This allows us to extend the transformation to $\mathcal{R}$, where it will act as the identity map in $\mathcal{R} \setminus \mathcal{H}_\epsilon$. Such ϕ_ϵ is differentiable a.e.

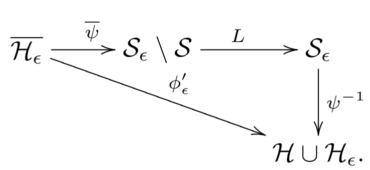
Now we describe how ϕ_ϵ acts on each point $x \in \overline{\mathcal{H}_\epsilon}$. Let $d_1 : \mathcal{H}_\epsilon \to I = [0, 1]$ be such that $d_1(x) = \|x - \pi(x)\|/\epsilon$. Let $\eta_1 : \overline{\mathcal{H}_\epsilon} \to \partial\mathcal{H} \times I$ be defined as $\eta_1(x) = (\pi(x), d_1(x))$. Recall that under the action of ψ_1, we have a conformal transformation between ∂H and $\partial\mathcal{S}$. Let $\eta_2 : \overline{\mathcal{H}_\epsilon} \times I \to \partial\mathcal{S} \times I$ be such that $\eta_2(x, r) = (\psi_1(\pi(x)), r)$. In particular, $\eta_2(x, d_1(x)) = (\psi_1(\pi(x)), d_1(x))$. Let $\mathcal{S}_\epsilon$ be a closed ϵ-strip in $\mathcal{R}$ around $\mathcal{S}$; i.e $\mathcal{S}_\epsilon = \{x \in \mathcal{R} : \inf_{s \in \mathcal{S}} \|x - s\| \le \epsilon\}$. Consider now a map $\eta_3 : \partial S \times I \to \mathcal{S}_\epsilon \setminus \mathcal{S}$ such that if $x \in \partial S$, we can write x in polar form as $x = S_0 + re^{i\theta}$, then $\eta_3(x, \rho) = S_0 + (1 + \rho\epsilon/r)re^{i\theta}$.

All the maps we have introduced up to this point are 1 to 1 homeomorphisms. We can thus induce a continuous map $\overline{\psi} : \overline{\mathcal{H}_\epsilon} \to \mathcal{S}_\epsilon \setminus \mathcal{S}$ such that the following diagram commutes:



Let $s \in \mathcal{S}_\epsilon \setminus \mathcal{S}$. Let $s = S_0 + (1 + \rho\epsilon/r)(r)e^{i\theta}$. Let $L : \mathcal{S}_\epsilon \setminus \mathcal{S} \to \mathcal{S}_\epsilon$ be given by $L(s) = S_0 + \rho(r + \epsilon)e^{i\theta}$, a map that takes the annulus of radius ϵ and *fills* the closed disk $\mathcal{S}_\epsilon$.

Observe that $\overline{\psi}$ can be extended to $\psi : \mathcal{H}_\epsilon \cup \mathcal{H} \to \mathcal{S}_\epsilon$ such that $\psi|_{\mathcal{H}} = \psi_1$ and $\psi|_{\mathcal{H}_\epsilon} = \overline{\psi}$. This is a 1-1 continuous transformation with continuous inverse ψ^{-1}. Hence, we can induce the map ϕ'_ϵ:



ϕ'_ϵ is continuous, 1-1 in $\mathcal{H}_\epsilon \setminus \partial\mathcal{H}$ and onto. Also, from its construction, it follows it is differentiable everywhere except on $\partial\mathcal{H}$. Since ϕ'_ϵ acts as the identity map in $\partial(\mathcal{H}_\epsilon \cup \mathcal{H})$, we can extend it to $\phi_\epsilon\ \mathcal{R} \setminus \mathcal{H} \to \mathcal{R}$ such that, restricted to $\mathcal{R} \setminus \mathcal{H}_\epsilon$ acts as the identity map, and restricted to $\mathcal{H}_\epsilon$ acts as ϕ'_ϵ. Observe that this transformation is invertible everywhere in $(\mathcal{R} \setminus \mathcal{H})^{int}$, and differentiable everywhere in $\mathcal{R}$, except (possibly) in $\partial\mathcal{H}_\epsilon$, which is a set of measure zero.

Let $J(\mathcal{P}, \phi)$ be the cost function defined in $(\mathcal{R} \setminus \mathcal{H})^{int}$:

$$J(\mathcal{P}, \phi) = \int_{(\mathcal{R}\setminus\mathcal{H})^{int}} \min_{1 \le i \le n} \|\phi(p_i) - \phi(q)\| \varphi(q)\ dq., \tag{10}$$

equivalently,

$$J(\mathcal{P}, \phi) = \int_{\mathcal{R}} \min_{1 \le i \le n} \|q_i - r\| \varphi\left(\phi^{-1}(r)\right) \left|\det J_{\phi^{-1}}(r)\right| dr, \tag{11}$$

where $q_i = \varphi(q)$ and $\left|\det J_{\phi^{-1}}(r)\right|$ is the determinant of the Jacobian of the inverse transformation. Such Jacobian is defined a.e. in $(\mathcal{R})^{int}$, with the (possible) exception of $\partial\mathcal{H}_\epsilon$. For purposes of integration, this is a set of measure zero, so it would not affect our discussion.

Observe that (10) coincides with (4) when ϕ is the identity map in $\mathbb{R}^2$. In this case, the set of extreme points for the cost function is defined as in Section IV.

We now state two theorems on $J(\mathcal{P}, \phi_\epsilon)$ about its continuity as a function of $\mathcal{P}$ and ϕ_ϵ.

Theorem 5: The map $J(\mathcal{P}, \phi_\epsilon)$ is continuous as a function of $\mathcal{P}$.

In an abuse of notation, given x, we can think of ϕ_ϵ as a continuous function on ϵ. Formally, for ϕ_ϵ to be continuous on ϵ we need it to be continuous under the supremum norm. In our construction of ϕ_ϵ the dependency on ϵ appears in two parts: in the map d_1 and in the set $\mathcal{S}_\epsilon$. When x is fixed, d_1 changes continuously with ϵ. Repeating our argument, we can send $\mathcal{S}_\epsilon \setminus \mathcal{S} \to \mathcal{S}_{\epsilon'} \setminus \mathcal{S}$ continuously. This allows us to state the following theorem:

Theorem 6: The map ϕ_ϵ is a continuous function on ϵ.

We can now merge these two results to obtain a corollary on the continuity of $J(\mathcal{P}, \phi)$.

Corollary 7: The function $J(\mathcal{P}, \phi_\epsilon)$ is a continuous function of both $\mathcal{P}$ and ϕ_ϵ as a function of ϵ.

Solving the problem under the transformation ϕ_ϵ.

Unless we explicitly specify otherwise, throughout this discussion ϵ is assumed to be small enough to satisfy the requirements of Theorem 2. Under the action of ϕ_ϵ we have allowed ourselves to work in a convex space, and hence the method presented in [2] would work. We now proceed to describe the system evolution of the agents under a transformation ϕ. We invoke some results from our previous work [1], which are not included here due to space constraints: Claim 3 in [1] describes the control law to solve the problem under the change of coordinates induced by the transformation. Problems would arise if any of the agents in $\mathcal{P}$, when solving the modified problem in (11), would move towards the singularity point S_0. Claims 6 and 7 in [1] guarantee that the agents can avoid such a singularity, and modify the final cost function by an arbitrarily small number. They state that, given $\gamma > 0$, under the action of $J(\mathcal{P}, \phi_\epsilon)$ the agents will evolve to a configuration $\mathcal{P}' \subset \left((\mathcal{R} \setminus \mathcal{H})^{int}\right)^n$ that is *close* to a local minimum, such that the cost function J induced by this formation differs by at most γ from the local optimal value for the formation.

B. Solution to the original problem

In order to simplify the notation, we will denote by $\epsilon_m = \epsilon/m$ and by ϕ_m the transformation ϕ_{ϵ_m}. Let $\mathcal{P}_m$ be a local minimum solution for the coverage problem (when starting from configuration $\mathcal{P}^\circ$) under the action of the map ϕ_m.

In this section, we will show that if a sub-sequence of the optimal configurations $\mathcal{P}_m$ (when the problem is solved under the action of the map ϕ_m), converges to an interior configuration $\mathcal{P}^\star$ (i.e. a configuration that does not contain any point in the boundary of $\mathcal{R} \setminus \mathcal{H}$), then such an interior configuration is a local minimum for the original problem in (2).

First we prove that there is always a convergent sub-sequence among $\{\mathcal{P}_m\}$: since $\mathcal{P}_m \subset \left((\mathcal{R} \setminus \mathcal{H})^{int}\right)^n \subset \left(\overline{(\mathcal{R} \setminus \mathcal{H})^{int}}\right)^n$, which is a compact set, the existence of a convergent sub-sequence $\{\mathcal{P}_{m_i}\}_{i\in\mathbb{N}}$ such that $\mathcal{P}_{m_i} \to \mathcal{P}^\star$ follows. Assume $\mathcal{P}^\star \subset \left((\mathcal{R} \setminus \mathcal{H})^{int}\right)^n$.

In order to show that $\mathcal{P}^\star$ is a local minimum for $J(\mathcal{P}, Id)$, we will establish the following Theorem:

Theorem 8: The centroids for the Voronoi regions that result under the application of the map $\phi_{\epsilon_{m_i}}$ induced by $\mathcal{P}_{m_i}$ converge to the centroids of the Voronoi region induced by $\mathcal{P}^\star$ under the Identity map.

Proof: For reasons of space we only present the overview of the approach.

Recall that the centroid $C_{m,i}$ for the Voronoi region generated by the agent p_i under the transformation ϕ_m is given by

$$C_{m,i} = \frac{\displaystyle\int_{V_{i,m}} r\,\varphi(\phi_m^{-1}(r)) \left|\det J_{\phi_m^{-1}}(r)\right| dr}{\displaystyle\int_{V_{i,m}} \varphi(\phi_m^{-1}(r)) \left|\det J_{\phi_m^{-1}}(r)\right| dr} \tag{12}$$

where $V_{i,m}$ is the Voronoi region induced by the agent a_i under the transformation ϕ_m. Observe that the centroids are a continuous function of the Voronoi regions and the induced densities. For the points $r \notin \mathcal{H}_{\epsilon_m}$, the transformation acts as the identity map and $\left|\det J_{\phi_m^{-1}}(r)\right| = 1$. Therefore, for the points $r \notin \mathcal{H}_{\epsilon_m}$ the respective integrals would coincide. Now, consider the points $r \in \mathcal{H}_{\epsilon_m}$. From the change of Variables Theorem, given $\mathcal{P}$, the integral in (12) when restricted to those points is uniformly bounded, and hence it goes to 0 as m goes to infinity. Since $\mathcal{V}_i \to \mathcal{V}$ when $\mathcal{P}_{m_i} \to \mathcal{P}^\star \in (\mathcal{R} \setminus \mathcal{H})^{int}$, the centroids also converge, and the result follows. ■

C. Multiple holes

Let $\mathcal{H}^1$, $\mathcal{H}^2$, $\mathcal{H}^N$ be N different holes in $\mathcal{R}$, such that their closures are all disjoint. Let $\mathcal{H}^i_{\epsilon_i}$ be a strip of a suitable ϵ_i width around $\mathcal{H}^i$, as described in Theorem 2. Since, for $1 \le i \le N$ outside $\mathcal{H}_{\epsilon_i}$ the map ϕ_{ϵ_i} acts as the identity map, it is possible to extend each ϕ_{ϵ_i} to a unique map ϕ_ϵ, where $\epsilon = \min_{1\le i\le N} \epsilon_i$, given by

$$\phi_\epsilon(r) = \begin{cases} r & r \notin \displaystyle\bigcup_{i=1}^{N} \mathcal{H}^i_{\epsilon_i} \\ \phi_{\epsilon_i}(r) & r \in \mathcal{H}^i_{\epsilon_i}. \end{cases} \tag{13}$$

The continuity of this function follows from the construction. The same can be said about its differentiability properties: Since being differentiable is a local property, ϕ_ϵ will be differentiable a.e., with the (possible) exception of $\partial\mathcal{H}^i_{\epsilon_i}$, which is a set of measure zero, and hence will not affect the integration results.

The argument for the convergence to an interior locally optimal configuration in this case is a direct extension of our discussion for the scenario with a single hole.

VI. Simulations

In Figure 2 we present the simulation results for a region with a hole. In the example, the region and the hole both have the shape of an equilateral triangle. We show the initial and final configurations of the agents in the transformed space and the trajectories in both cases. It can be observed that for the configurations in the interior, both configurations do coincide in the end.

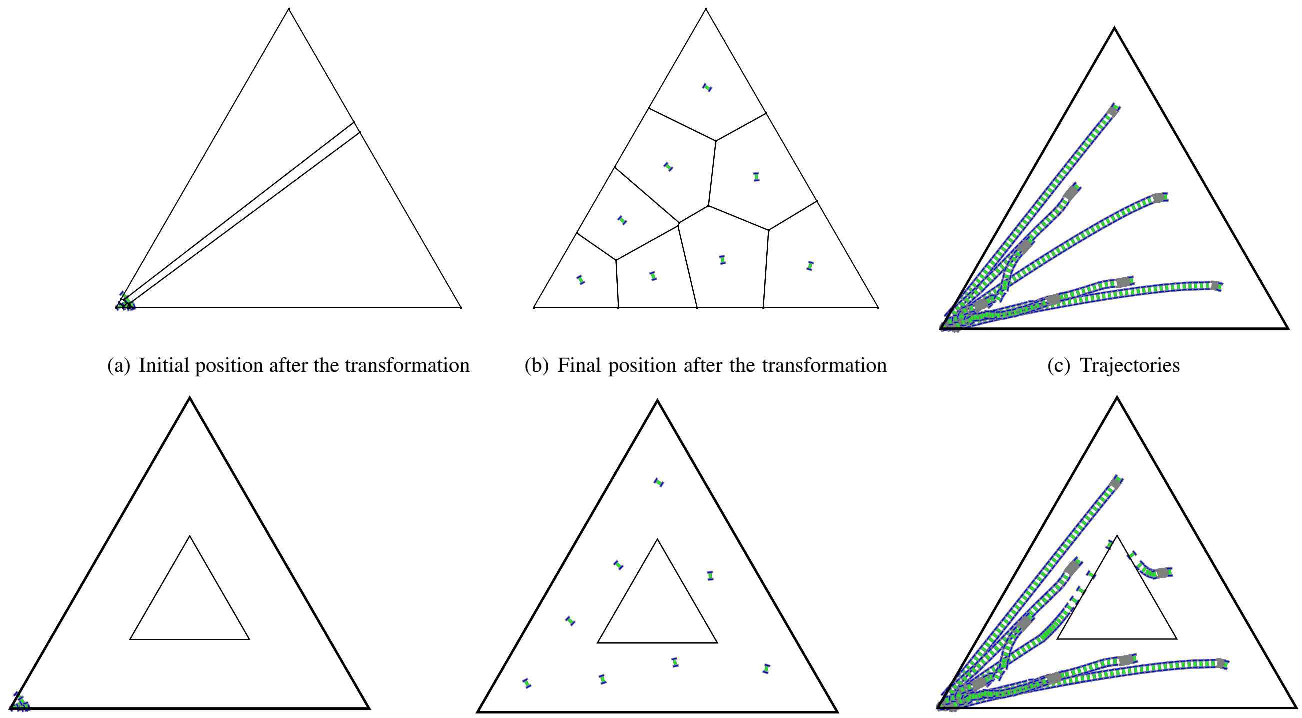
A. Numerical Issues

Although our algorithm is provably correct, a fundamental drawback with our current approach is its computational feasibility. Theorem 1 establishes that the conformal transformation between any two open, bounded, simply connected regions in $\mathbb{R}^2$ exists, but it does not tell us *how to compute it*. There is no general answer to this question. Furthermore, even with the knowledge of such a transformation, when we compute the ϵ-strip around each hole, as $\epsilon \to 0$ the Jacobian matrix becomes an ill-conditioned matrix, making numerical implementation problematic. We are currently investigating motion strategies that avoid this problem.

It should be mentioned that even though we guarantee the convergence to the right configuration for interior points, when the optimal solution lies on the boundary, the evolution of the agents does not necessarily converge to the desired configuration for an arbitrary diffeomorphism. We speculate that since the key point is for the agents on the boundary to be at *orthogonality points* the fact that the map given by Theorem 1 is conformal would be important. Unfortunately, in general such a family of transformations is difficult to find.

VII. Conclusion

We presented a framework for solving the coverage problem for a class of non-convex domains. We characterized the stationary points for the coverage problem for non-convex domains. We then formulated an algorithm that provably approaches the solution of the original problem when such solution belongs to the interior of the set, provided that the original set is convex but with obstacles in its interior. The



(a) Initial position after the transformation (b) Final position after the transformation (c) Trajectories

(d) Real initial position (e) Real final position (f) Trajectories

Fig. 2. Evolution of the formation according to the proposed algorithm when holes are present in the region. As observed, both configurations coincide for optimal configurations which are interior to the region.

algorithm relies on a family of diffeomorphisms that map a class of connected regions in $\mathbb{R}^2$ to an *almost convex* region in $\mathbb{R}^2$ – a convex region from which a finite (possibly empty) set of points has been removed. In particular, we showed that for this family of diffeomorphisms, the solution of the transformed problem converges to the solution in the original non-convex space.

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