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Dual Lyapunov Stability Analysis in Behavioral Approach

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Abstract— **This paper considers a Lyapunov stability analysis for continuous-time systems described by high order difference-algebraic equation from the viewpoint of the semidefinite programming (SDP) duality. In the behavioral system theory, a Lyapunov function is described by a quadratic differential form (QDF) and equivalently characterized by a two-variable polynomial matrix. We first develop the SDP duality to the non-negativity and positivity of two-variable polynomial matrices. Using the duality, we derive an alternative stability condition in terms of the two-variable polynomial matrix equation and QDFs as a main result.**

I. Introduction

The Lyapunov stability theory plays an important role in the stability analysis of a dynamical system. In this paper, we derive an alternative stability condition based on the semidefinite programing duality in the behavioral framework.

In the behavioral approach, a quadratic differential form (QDF) is used for describing a quadratic functional such as a storage function, supply rate and Lyapunov function which play important roles in the dissipation theory and Lyapunov theory [13]. Since there is an one-to-one correspondence between a QDF and a two-variable polynomial matrix, a QDF is useful as an algebraic tool.

Previous works on the behavioral approach to stability analysis of dynamical systems are as follows. Willems and Trentelman [13] proved a generalized Lyapunov stability theorem for a one-dimensional (1-D[1]) continuous-time system based on QDFs. Indeed, in [13], a Lyapunov function is characterized in terms of a QDF which is obtained by solving a two-variable polynomial Lyapunov equation (TVPLE). This condition was developed to 2-D systems as a sufficient stability condition in terms of QDFs and four-variable polynomial Lyapunov equation.

We point out the problems of the above works as follows.

- Based on the QDF conditions in [13], there have been derived necessary and sufficient stability conditions in terms of linear matrix inequalities (LMIs) constructed from the TVPLE [3][4][6]. These LMI conditions can check the stability of a given behavior exactly. But, since we restrict our attention to the systems described by high-order differential algebraic equations, these condition need complex transformations of polynomial matrices in order to describe the condition in terms of LMIs. Hence, it is hard to check the stability by using these LMI conditions in the case where the physical parameters have uncertainties [10].
- In the sufficient stability condition in [5], there exists a stable behavior which does not satisfy the condition, since the condition is a sufficient condition, The reason is that the polynomial matrix which induces a kernel representation has infinite number of zeros. Hence, this indicates the it may be difficult to derive a necessary and sufficient condition for n-D systems.

The above observations suggest the necessity of a condition which assures that the system is not asymptotically stable if it satisfies the condition for the systems which are difficult to deal with.

Under the above observation, we focus our attention on the idea of the semidefinite programming (SDP) [2] which are known as a numerical framework in the optimization theory. As one of the results related to the system and control theory, Balakrishnan, Vandenberghe [1] derived an alternative stability condition which gives a exact decision for the stability based on the SDP duality. They also developed the stability result to a new interpretation to the optimal regulator problem and a new proof of the Kalman-Yakubovich-Popov lemma. Recently, there have been derived the dual stability results for nonlinear systems and algorithmic approach based on the theorem of alternative for the computation of the certificates of infeasibility [7][9][11].

In order to overcome the abate difficulties, we focus on the SDP from a theoretical viewpoint. Moreover, the SDP duality has not been studied in the system theory so far. Hence, we formulate the SDP duality in the behavioral framework. As a main result, we will derive an alternative stability condition in terms of the two-variable polynomial matrix equation and QDFs for some behavior.

The organization of the paper is as follows. In Section II, we review some basic definitions and results of linear continuous-time systems, QDFs and Lyapunov theory in the behavioral approach. We develop the SDP duality to the nonnegativity and positivity of two-variable polynomial matrix in Section III. Based on the duality, we give a main result which gives an alternative stability condition using the two-variable polynomial matrix equation and QDFs in Section IV.

II. Preliminaries

In this section, we will review the basic definitions and results from the behavioral system theory.

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[1]We call the system which has one independent variable as one-dimensional (1-D) system. On the other hand, the system that depend on n-variables ($n \geq 2$) is called n-dimensional (n-D) system.

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A. Linear Continuous-Time System

In the behavioral system theory, a dynamical system is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T}$ is the time axis, and $\mathbb{W}$ is the signal space in which the trajectories take their values on. The behavior $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the set of all possible trajectories. In this paper, we will consider a *linear* time-invariant *continuous-time system* whose time axis is $\mathbb{T} = \mathbb{R}$ and signal space is $\mathbb{W} = \mathbb{C}^q$. Such a Σ is represented by a system of linear differential-algebraic equation as

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_L \frac{d^L}{dt^L} w = 0 \tag{1}$$

where $R_i \in \mathbb{C}^{p \times q}$ $(i = 0, 1, \cdots, L)$ and $L \geq 0$. The variable $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$ is called the *manifest variable*. We call the representation of (1) a *kernel representation* of $\mathfrak{B}$. A short hand notation for (1) is

$$R\left(\frac{d}{dt}\right) w = 0, \tag{2}$$

where $R(\xi) := R_0 + R_1 \xi + \cdots + R_L \xi^L \in \mathbb{C}^{p \times q}[\xi]$. Then, the behavior is given by

$$\mathfrak{B} = \left\{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q) \;\middle|\; R\left(\frac{d}{dt}\right) w = 0 \right\}. \tag{3}$$

For the above Σ with $\mathfrak{B}$ in (3), we define the *dual system of* Σ by $\Sigma' = (\mathbb{R}, \mathbb{C}^p, \mathfrak{B}')$. This is formulated as the system whose behavior $\mathfrak{B}'$ has the kernel representation

$$R\left(\frac{d}{dt}\right)^* v = 0, \; v \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^p), \tag{4}$$

where $R(\xi)^*$ denotes the Hermite conjugate of $R(\xi)$, i.e. $R(\xi)^* := R_0^* + R_1^* \xi + \cdots + R_L^* \xi^L$. Then, $\mathfrak{B}'$ is given by

$$\mathfrak{B}' = \left\{ v \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^p) \;\middle|\; R\left(\frac{d}{dt}\right)^* v = 0 \right\}. \tag{5}$$

We call $\mathfrak{B}'$ as the *dual behavior of* $\mathfrak{B}$.

We define the autonomy and asymptotic stability of a behavior $\mathfrak{B}$.

Definition 1: [8]

(i) A behavior $\mathfrak{B}$ is said to be *autonomous* if

$$w_1(t) = w_2(t) \; (t < 0) \implies w_1 = w_2$$

holds for all $w_1, w_2 \in \mathfrak{B}$.

(ii) A behavior $\mathfrak{B}$ is said to be *asymptotically stable* if

$$w \in \mathfrak{B} \implies \lim_{t \to \infty} w(t) = 0.$$

The autonomy of $\mathfrak{B}$ is a necessary condition for the asymptotic stability of $\mathfrak{B}$.

We give the definition of the zero of a polynomial matrix and its Hurwitzness.

Definition 2: [13] Let $R \in \mathbb{C}^{p \times q}[\xi]$ be given.

(i) A complex number $\lambda \in \mathbb{C}$ is called the *zero of* $R(\xi)$ if $\mathrm{rank} R(\lambda) < \mathrm{rank} R$.

(ii) A polynomial matrix $R(\xi)$ is called *Hurwitz* if $\mathrm{rank} R = q$ and $\mathrm{rank} R(\lambda) = q$ for all $\lambda \in \mathbb{C}$ such that Re $\lambda \geq 0$.

The behavior (3) is autonomous if and only if $\mathrm{rank} R = q$ holds [12]. Moreover, $\mathfrak{B}$ in (3) is asymptotically stable if and only if $R(\xi)$ is Hurwitz.

B. Quadratic Differential Forms

Consider the two-variable polynomial matrix in $\mathbb{C}^{q_1 \times q_2}[\zeta, \eta]$ described by

$$\Phi(\zeta, \eta) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \Phi_{i,j} \zeta^i \eta^j, \tag{6}$$

where $\Phi_{i,j} \in \mathbb{C}^{q_1 \times q_2}$ $(i = 0, 1, \cdots, N_1; \; j = 0, 1, \cdots, N_2)$ and $N_1, N_2 \geq 0$. For this $\Phi(\zeta, \eta)$, we define $\Phi^\star \in \mathbb{C}^{q \times q}[\zeta, \eta]$ by $\Phi^\star(\zeta, \eta) := \Phi(\bar{\eta}, \bar{\zeta})^*$.

We call $\Phi(\zeta, \eta)$ Hermite if $\Phi^\star(\zeta, \eta) = \Phi(\eta, \zeta)$ (implying $q_1 = q_2 =: q$ and $N_1 = N_2 =: N$). In this case, $\Phi(\zeta, \eta)$ induces a *quadratic differential form (QDF)*

$$\mathsf{Q}_\Phi(\ell) := \sum_{i=0}^{N} \sum_{j=0}^{N} \left(\frac{d^i \ell}{dt^i} \right)^* \Phi_{i,j} \frac{d^j \ell}{dt^j}.$$

Define the *rate of change* of the QDF $\mathsf{Q}_\Phi(\ell)(t)$ by

$$\nabla \mathsf{Q}_\Phi(\ell) := \frac{d}{dt} \mathsf{Q}_\Phi(\ell).$$

It follows immediately that it is also a QDF. Let $\nabla \Phi \in \mathbb{H}^{q \times q}[\zeta, \eta]$ denote the two-variable polynomial matrix which induces $\nabla \mathsf{Q}_\Phi(\ell)$, namely $\nabla \mathsf{Q}_\Phi(\ell) = \mathsf{Q}_{\nabla \Phi}(\ell)$. Then, $\nabla \Phi(\zeta, \eta)$ is expressed as

$$\nabla \Phi(\zeta, \eta) = (\zeta + \eta) \Phi(\zeta, \eta).$$

We define the nonnegative and positive definiteness of two-variable polynomial matrices.

Definition 3: [13] Let $\Phi \in \mathbb{H}^{q \times q}[\zeta, \eta]$ be given.

(i) A two-variable polynomial matrix $\Phi(\zeta, \eta)$ is called *nonnegative definite*, denoted by $\Phi \geq 0$, if $\mathsf{Q}_\Phi(\ell) \geq 0$ holds for all $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^q)$.

(ii) A two-variable polynomial matrix $\Phi(\zeta, \eta)$ is called *nonzero nonnegative definite*, denoted by $\Phi \gneqq 0$, if $\Phi \geq 0$ and $\Phi(\zeta, \eta) \neq 0_{q \times q}$.

(iii) A two-variable polynomial matrix $\Phi(\zeta, \eta)$ is called *positive definite*, denoted by $\Phi > 0$, if $\Phi \geq 0$, and if $\mathsf{Q}_\Phi(\ell) = 0$ implies $\ell = 0$.

The nonnegative and positive definiteness of a two-variable polynomial matrix are extended to the case where the trajectory is constrained in a behavior.

Definition 4: [13] Let $\Phi \in \mathbb{H}^{q \times q}[\zeta, \eta]$ be given.

(i) A two-variable polynomial matrix $\Phi(\zeta, \eta)$ is called *nonnegative definite along* $\mathfrak{B}$, denoted by $\Phi \overset{\mathfrak{B}}{\geq} 0$, if $\mathsf{Q}_\Phi(w) \geq 0$ holds for all $w \in \mathfrak{B}$.

(ii) A two-variable polynomial matrix $\Phi(\zeta, \eta)$ is called *nonzero nonnegative definite along* $\mathfrak{B}$, denoted by $\Phi \overset{\mathfrak{B}}{\gneqq} 0$, if $\Phi \overset{\mathfrak{B}}{\geq} 0$ and $\Phi(\zeta, \eta) \neq 0_{q \times q}$.

(iii) A two-variable polynomial matrix $\Phi(\zeta, \eta)$ is called *positive definite along* $\mathfrak{B}$, denoted by $\Phi \overset{\mathfrak{B}}{>} 0$, if $\Phi \overset{\mathfrak{B}}{\geq} 0$, and if $\mathsf{Q}_\Phi(w) = 0$ implies $w = 0$.

We introduce the equivalence of two-variable polynomial matrices along a behavior.

Definition 5: [13] Two-variable polynomial matrices $\Phi_1 \in \mathbb{H}^{q\times q}[\zeta,\eta]$ and $\Phi_2 \in \mathbb{H}^{q\times q}[\zeta,\eta]$ are *equivalent along* $\mathfrak{B}$ if

$$\mathsf{Q}_{\Phi_1}(w) = \mathsf{Q}_{\Phi_2}(w) \quad \forall\, w \in \mathfrak{B}.$$

We denote this by $\Phi_1 \overset{\mathfrak{B}}{=} \Phi_2$.

The equivalence along $\mathfrak{B}$ is characterized by the polynomial matrix which induces the kernel representation of $\mathfrak{B}$.

Lemma 1: [13] Let $R \in \mathbb{C}^{p\times q}[\xi]$ induce the kernel representation of $\mathfrak{B}$. Let $\Phi_1, \Phi_2 \in \mathbb{H}^{q\times q}[\zeta,\eta]$ be given. Then, $\Phi_1 \overset{\mathfrak{B}}{=} \Phi_2$ holds if and only if there exists a $Y \in \mathbb{C}^{p\times q}[\zeta,\eta]$ satisfying

$$\Phi_2(\zeta,\eta) = \Phi_1(\zeta,\eta) + R(\zeta)^* Y(\zeta,\eta) + Y^\star(\eta,\zeta) R(\eta).$$

C. Stability condition using QDFs

In this section, we review necessary and sufficient conditions for continuous-time behaviors using QDFs from the reference [13].

Lemma 2: [13] Let $R \in \mathbb{C}^{p\times q}[\xi]$ induce the kernel representation of $\mathfrak{B}$. Let $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ be an arbitrary two-variable polynomial matrix satisfying $\Phi \overset{\mathfrak{B}}{<} 0$. Then, the following statements (i), (ii), (iii) and (iv) are equivalent.

(i) The behavior $\mathfrak{B}$ is asymptotically stable.

(ii) There exists a two-variable polynomial matrix $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ satisfying

$$\Psi \overset{\mathfrak{B}}{\geq} 0 \quad \text{and} \quad \nabla\Psi \overset{\mathfrak{B}}{<} 0. \tag{7}$$

(iii) There exists a two-variable polynomial matrix $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ satisfying $\Psi \overset{\mathfrak{B}}{\geq} 0$ and

$$\nabla\Psi \overset{\mathfrak{B}}{=} \Phi.$$

(iv) There exist two-variable polynomial matrices $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ and $Y \in \mathbb{C}^{p\times q}[\zeta,\eta]$ satisfying $\Psi \overset{\mathfrak{B}}{\geq} 0$ and the *two-variable polynomial Lyapunov equation (TVPLE)*

$$\nabla\Psi(\zeta,\eta) = \Phi(\zeta,\eta) - Y^\star(\eta,\zeta)R(\eta) - R(\zeta)^* Y(\zeta,\eta). \tag{8}$$

■

The QDF $\mathsf{Q}_\Psi(w)$ satisfying the equation (7) is called a *Lyapunov function for* $\mathfrak{B}$. In fact, if we regard $Q_\Psi(w)$ as the energy function of $\mathfrak{B}$, the condition in (7) states that the energy settles to its steady-state value as time goes to infinity ($t \to \infty$).

Remark 1: From Lemma 2, we can check the stability of $\mathfrak{B}$ by finding the solution $\Psi(\zeta,\eta)$ and $Y(\zeta,\eta)$ to the TVPLE (8). But the similar method becomes difficult in 2-D systems, which is one of the motivation of this paper. It is explained as follows.

In the reference [5], Kojima and Takaba proposed a sufficient stability condition in terms of the four-variable polynomial Lyapunov equation which is a generalization of the TVPLE (8) to 2-D systems. But it often happens the case that there does not exist the upper bound of the degree of the solutions to the equation. This fact makes the stability checking of the system quite difficult. ■

III. SDP Duality of Two-variable Polynomial Matrices

In Section II-B, we introduced the nonnegativity and positivity of two-variable polynomial matrices. We establish the SDP duality [1] to this property in this section. The duality will be used when we derive an alternative stability condition in the next section. Please see the reference [1] for the detail of the SDP duality.

Suppose that $\mathbb{H}^{q\times q}[\zeta,\eta]$ is a space of block diagonal Hermite two-variable polynomial matrices with some given dimensions, namely

$$\begin{aligned}&\mathbb{H}^{q\times q}[\zeta,\eta]\\ &\quad = \mathbb{H}^{q_1\times q_1}[\zeta,\eta] \times \mathbb{H}^{q_2\times q_2}[\zeta,\eta] \times \cdots \times \mathbb{H}^{q_K\times q_K}[\zeta,\eta].\end{aligned}$$

We define the inner product over $\mathbb{H}^{q\times q}[\zeta,\eta]$ by

$$\begin{aligned}&\langle \mathrm{diag}\,(A_1, A_2, \cdots, A_K)\,, \mathrm{diag}\,(B_1, B_2, \cdots, B_K)\rangle\\ &\quad = \sum_{i=1}^{K} \mathrm{trace}\,(A_i(\zeta,\eta)B_i(\zeta,\eta))\,,\end{aligned}$$

where $A_i, B_i \in \mathbb{H}^{q_i\times q_i}[\zeta,\eta]$ $(i = 1, 2, \cdots, K)$. We can discuss the nonnegativity and positivity of the inner product of two-variable polynomial matrices by regarding it as a two-variable polynomial.

Consider a linear mapping

$$\mathcal{A} : \mathbb{C}^K[\zeta,\eta] \to \mathbb{H}^{q\times q}[\zeta,\eta]$$

defined by[2]

$$\mathcal{A}(x) = x_1 A_1 + x_2 A_2 + \cdots + x_K A_K, \tag{9}$$

where $A_i \in \mathbb{H}^{q\times q}$ $(i = 1, 2, \cdots, K)$ and $x \in \mathbb{C}^K[\zeta,\eta]$ is a two-variable polynomial vector defined by

$$\begin{aligned}x(\zeta,\eta) &:= \mathrm{col}\,(x_1(\zeta,\eta), x_2(\zeta,\eta), \cdots, x_K\zeta,\eta))\,,\\ x_i &\in \mathbb{C}[\zeta,\eta] \ (i = 1, 2, \cdots, K).\end{aligned}$$

Since $\mathcal{A}(x)$ in (9) can be regarded as a Hermite two-variable matrix in $\mathbb{H}^{q\times q}[\zeta,\eta]$, we can formulate the nonnegativity and positivity $\mathcal{A}(x)$ in the context of Section II-B.

We introduce the adjoint mapping of $\mathcal{A}$ which plays a crucial role in this paper. The *adjoint mapping of* $\mathcal{A}$ is defined by

$$\begin{aligned}&\mathcal{A}^{\mathrm{adj}} : \mathbb{H}^{q\times q}[\zeta,\eta] \to \mathbb{C}^K[\zeta,\eta],\\ &\langle \mathcal{A}(x), \Psi' \rangle = \left\langle x, \mathcal{A}^{\mathrm{adj}}(\Psi') \right\rangle.\end{aligned}$$

As we will see in the above definition, the superscript " ′ " means that the two-variable polynomial matrix is defined in the "dual" domain.

We give a following proposition about the SDP duality to the positivity and nonnegativity of two-variable polynomial matrices.

Proposition 1: Let $\mathcal{A}(x) \in \mathbb{H}^{q\times q}[\zeta,\eta]$ be defined by (9). Let $\Phi_0 \in \mathbb{H}^{q\times q}[\zeta,\eta]$ be given.

[2] We omit the indeterminates ζ and η in (9) for the simplicity of the notation.

(i) Exactly one of the following statements (a) and (b) holds.
  (a) There exists an $x \in \mathbb{C}^K[\zeta,\eta]$ satisfying

$$\mathcal{A}(x) + \Phi_0 > 0. \tag{10}$$

  (b) There exists a $\Psi' \in \mathbb{H}^{q\times q}[\zeta,\eta]$ satisfying

$$\Psi' \gneqq 0,\ \mathcal{A}^{\mathrm{adj}}(\Psi') = 0 \text{ and } \langle \Phi_0, \Psi'\rangle \leq 0 \tag{11}$$

(ii) At most one of the following statements (c) and (d) holds.
  (c) There exists an $x \in \mathbb{C}^K[\zeta,\eta]$ satisfying

$$\mathcal{A}(x) + \Phi_0 \geq 0$$

  (d) There exists a $\Psi' \in \mathbb{H}^{q\times q}[\zeta,\eta]$ satisfying

$$\Psi' \geq 0,\ \mathcal{A}^{\mathrm{adj}}(\Psi') = 0 \text{ and } \langle \Phi_0, \Psi'\rangle < 0$$

Moreover, if $\Phi_0 = \mathcal{A}(x_0)$ for some $x_0 \in \mathbb{C}^K$, then exactly one of the statements (c) and (d) holds.

Proof: For the proof, see Appendix A. ■

IV. Stability Condition Based on the SDP Duality

In this section, we will derive an alternative condition for the asymptotic stability based on the the SDP duality.

We suppose that a kernel representation of $\mathfrak{B}$ is induced by the square polynomial matrix $R \in \mathbb{R}^{q\times q}[\xi]$ throughout this section. Moreover, we assume that the two-variable polynomial matrix $\Phi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ satisfies $\Phi < 0$ which is stronger assumption than $\Phi \overset{\mathfrak{B}}{<} 0$.

From Lemma 2 (iv), $\mathfrak{B}$ is asymptotically stable if and only if there exist a $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ and $Y \in \mathbb{C}^{q\times q}[\xi]$ satisfying

$$\Psi \overset{\mathfrak{B}}{\geq} 0 \tag{12}$$

and the TVPLE

$$\nabla\Psi(\zeta,\eta) = \Phi(\zeta,\eta) - Y(\zeta,\eta)^\star R(\eta) - R(\zeta)^\star Y(\zeta,\eta). \tag{13}$$

It follows from Lemma 1 that (12) is equivalent to the existence of $X \in \mathbb{C}^{q\times q}[\zeta,\eta]$ satisfying

$$\Psi(\zeta,\eta) + X^\star(\zeta,\eta)R(\eta) + R(\zeta)^* X(\zeta,\eta) \geq 0. \tag{14}$$

Moreover, (13) is equivalently rewritten by

$$\begin{aligned} -(\zeta+\eta)\Psi(\zeta,\eta) - Y^\star(\zeta,\eta)R(\eta) - R(\zeta)^* Y(\zeta,\eta) \\ = -\Phi(\zeta,\eta) \\ > 0 \end{aligned} \tag{15}$$

In order to rewrite (14) and (15) in the form of (9), we express $\Psi(\zeta,\eta)$, $X(\zeta,\eta)$ and $Y(\zeta,\eta)$ in the form

$$\begin{aligned} &\Psi(\zeta,\eta) := \sum_{i=1}^{K} \psi_i(\zeta,\eta) S_i, \\ &X(\zeta,\eta) := \sum_{i=1}^{q^2} x_i(\zeta,\eta) E_i,\ Y(\zeta,\eta) := \sum_{i=1}^{q^2} y_i(\zeta,\eta) E_i, \end{aligned} \tag{16}$$

$$\psi_i \in \mathbb{H}[\zeta,\eta]\ (i = 1,2,\cdots,K),$$
$$x_i, y_i \in \mathbb{C}[\zeta,\eta]\ (i = 1,2,\cdots,q^2)$$

where $S_i \in \mathbb{S}^{q\times q}$ $(i = 1,2,\cdots,K; K := \frac{q(q+1)}{2})$ and $E_i \in \mathbb{R}^{q\times q}$ $(i = 1,2,\cdots,q^2)$ span the bases for $\mathbb{S}^{q\times q}$ and $\mathbb{R}^{q\times q}$, respectively. Substituting (16) into (14) and (15), we obtain the positivity of two-variable polynomial matrix $\mathcal{A}(\psi,x,y)$ given by

$$\mathcal{A}(\psi,x,y) > 0, \tag{17}$$

$$\mathcal{A}(\psi,x,y) := \begin{bmatrix} \mathcal{A}_1(\zeta,\eta) & 0 \\ 0 & \mathcal{A}_2(\zeta,\eta) \end{bmatrix}, \tag{18}$$

$$\mathcal{A}_1(\zeta,\eta) := \Psi(\zeta,\eta) + X^\star(\zeta,\eta)R(\eta) + R(\zeta)^* X(\zeta,\eta)$$
$$\mathcal{A}_2(\zeta,\eta) := -(\zeta+\eta)\Psi(\zeta,\eta) - Y^\star(\zeta,\eta)R(\eta) - R(\zeta)^* Y(\zeta,\eta),$$

where $\psi \in \mathbb{H}^K[\zeta,\eta]$ and $x, y \in \mathbb{C}^{q^2}[\zeta,\eta]$ are defined by

$$\begin{aligned} \psi(\zeta,\eta) &:= \mathrm{col}\left(\psi_1(\zeta,\eta), \psi_2(\zeta,\eta), \cdots, \psi_K(\zeta,\eta)\right), \\ x(\zeta,\eta) &:= \mathrm{col}\left(x_1(\zeta,\eta), x_2(\zeta,\eta), \cdots, x_{q^2}(\zeta,\eta)\right), \\ y(\zeta,\eta) &:= \mathrm{col}\left(y_1(\zeta,\eta), y_2(\zeta,\eta), \cdots, y_{q^2}(\zeta,\eta)\right), \end{aligned}$$

respectively. Thus, we get the following stability condition in terms of the positive definiteness of $\mathcal{A}(\psi,x,y)$.

Proposition 2: Let $R \in \mathbb{C}^{q\times q}[\xi]$ be square and given. Then, the behavior $\mathfrak{B}$ in (3) is asymptotically stable if and only if there exist two-variable polynomials $\psi_i \in \mathbb{H}[\zeta,\eta]$ $(i = 1,2,\cdots,K)$ and $x_i, y_i \in \mathbb{C}[\zeta,\eta]$ $(i = 1,2,\cdots,q^2)$ satisfying (17). ■

Proof: The solvability of (17) is equivalent to the solvability of (12) and (13). Hence, the proof follows immediately by applying Lemma 2, ■

From Propositions 1, 2, we obtain an alternative stability condition based on the SDP duality as a main result.

Theorem 1: Let $R \in \mathbb{C}^{q\times q}[\xi]$ be square and given. Then, exactly one of the following statements (a) and (b) holds.

(a) The behavior $\mathfrak{B}$ in (3) is asymptotically stable.
(b) There exist $\Phi', \Psi' \in \mathbb{H}^{q\times q}[\zeta,\eta]$ and $Z \in \mathbb{C}^{q\times q}[\zeta,\eta]$ satisfying

$$\begin{aligned} &\mathrm{diag}\left(\Phi', \Psi'\right) \gneqq 0, \\ &\nabla\Psi'(\zeta,\eta) = \Phi'(\zeta,\eta) - Z(\zeta,\eta)R(\eta)^* - R(\eta)Z^\star(\zeta,\eta) \end{aligned} \tag{19}$$

Proof: The statement (a) holds if and only if there exist two-variable polynomial matrices $\Psi \in \mathbb{H}^{q\times q}[\zeta,\eta]$ and $X, Y \in \mathbb{C}^{q\times q}[\zeta,\eta]$ satisfying (17). Hence, it is sufficient to compute the adjoint mapping from Proposition 1 with $\Phi_0(\zeta,\eta) = 0_{q\times q}$.

We express $\Psi', \Phi' \in \mathbb{H}^{q\times q}[\zeta,\eta]$ and $Z_i \in \mathbb{H}^{q\times q}[\zeta,\eta]$ $(i = 1,2)$ similar to Proposition 2 as

$$\Psi'(\zeta,\eta) = \sum_{i=1}^{K} \psi_i'(\zeta,\eta) S_i,\ \Phi'(\zeta,\eta) = \sum_{i=1}^{K} \phi_i'(\zeta,\eta) S_i,$$

where $\psi_i', \phi_i' \in \mathbb{H}[\zeta,\eta]$ $(i = 1,2,\cdots,K)$. Moreover, observe that there hold the properties

$$\begin{aligned} \langle \nabla\Psi(\zeta,\eta), \Psi'(\zeta,\eta)\rangle &= \mathrm{trace}\left\{(\zeta+\eta)\Psi(\zeta,\eta)\cdot\Psi'(\zeta,\eta)\right\} \\ &= \mathrm{trace}\left\{\Psi(\zeta,\eta)\cdot(\zeta+\eta)\Psi'(\zeta,\eta)\right\} \\ &= \left\langle \Psi(\zeta,\eta), \nabla\Psi'(\zeta,\eta)\right\rangle \end{aligned}$$

where $\Psi, \Psi' \in \mathbb{H}^{q\times q}[\zeta,\eta]$. We also have

$$\mathrm{trace}\,\{Y(\zeta,\eta)R(\eta)\cdot Z(\zeta,\eta)\} = \mathrm{trace}\,\{Y(\zeta,\eta)\cdot R(\eta)Z(\zeta,\eta)\}$$

for $Y, Z \in \mathbb{C}^{q\times q}[\zeta,\eta]$. Hence, we obtain the adjoint mapping of $\mathcal{A}(\psi, x, y)$ expressed as

$$\begin{aligned}
&\mathcal{A}^{\mathrm{adj}} : \mathbb{H}^K[\zeta,\eta]\times\mathbb{C}^{q^2}[\zeta,\eta]\times\mathbb{C}^{q^2}[\zeta,\eta],\\
&\quad\to \mathbb{H}^K[\zeta,\eta]\times\mathbb{H}^K[\zeta,\eta]\times\mathbb{C}^{q^2}[\zeta,\eta]\times\mathbb{C}^{q^2}[\zeta,\eta]\\
&\mathcal{A}^{\mathrm{adj}}(\phi',\psi',z_1,z_2)\\
&:= \Phi'(\zeta,\eta) - (\zeta+\eta)\Psi'(\zeta,\eta) + Z_1(\zeta,\eta)R(\eta)^* + R(\eta)Z_1^\star(\zeta,\eta)\\
&\quad - Z_2(\zeta,\eta)R(\eta)^* - R(\eta)Z_2^\star(\zeta,\eta),
\end{aligned}$$

where $Z_i \in \mathbb{C}^{q\times q}[\zeta,\eta]$ and $z_i \in \mathbb{C}[\zeta,\eta]$ $(i = 1,2)$ are defined by

$$\begin{aligned}
Z_i(\zeta,\eta) &:= \sum_{j=1}^{q^2} z_{i,j}(\zeta,\eta)E_j,\\
z_i(\zeta,\eta) &:= \mathrm{col}\left(z_{i,1}(\zeta,\eta), z_{i,2}(\zeta,\eta),\cdots,z_{i,q^2}(\zeta,\eta)\right)\\
z_{i,j} &\in \mathbb{C}[\zeta,\eta]\ (i=1,2; j=1,2\cdots,q^2).
\end{aligned}$$

Define $Z \in \mathbb{C}^{q\times q}[\zeta,\eta]$ as

$$Z(\zeta,\eta) := Z_1(\zeta,\eta) - Z_2(\zeta,\eta),$$

then the adjoint mapping is rewritten by

$$\begin{aligned}
&\mathcal{A}^{\mathrm{adj}}(\phi',\psi',z_1,z_2)\\
&= \Phi'(\zeta,\eta) - (\zeta+\eta)\Psi'(\zeta,\eta) + Z(\zeta,\eta)R(\eta)^* + R(\eta)Z^\star(\zeta,\eta).
\end{aligned}$$

Hence, the statement of the theorem follows from Propositions 1 and 2. ■

Remark 2: The above theorem gives a necessary and sufficient condition that $\mathfrak{B}$ is not asymptotically stable. ■

Remark 3: In Proposition 2 of [1], an alternative stability condition was proved for the behavior $\mathfrak{B}_{\mathrm{state}}$ whose kernel representation is expressed as the state-space equation.

$$\mathfrak{B}_{\mathrm{state}} = \left\{ w \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C}^q) \;\middle|\; \frac{d}{dt}w = Aw,\ A\in\mathbb{C}^{q\times q}\right\} \tag{20}$$

Namely, exactly one of the following statements (a) and (b) holds.

(a) The behavior $\mathfrak{B}_{\mathrm{state}}$ in (20) is asymptotically stable, i.e. there exists a $P \in \mathbb{S}^{q\times q}$ satisfying

$$A^*P + PA < 0 \text{ and } P > 0.$$

(b) There exist $P', Q' \in \mathbb{S}^{q\times q}$ satisfying

$$\mathrm{diag}\,(P',Q') \gneqq 0 \text{ and } P'A^* + AP' - Q' = 0.$$

The alternative condition in Theorem 1 recovers this condition due to [1]. It is explained as follows. By setting

$$R(\xi) = A - \xi I_q,\ \Psi'(\zeta,\eta) = Z(\zeta,\eta) = P',\ \Phi'(\zeta,\eta) = Q'.$$

in (19), then we get

$$(\zeta+\eta)P' = Q' - P'\left(A^* - \eta I_q\right) - \left(A - \zeta I_q\right)P'.$$

Substituting $\zeta = -\xi$ and $\eta = \xi$, the above equation is rewritten by

$$P'A^* + AP' - Q' = 0.$$

Since $\mathrm{diag}\,(P',Q') \gneqq 0$ is clear from Theorem 1 (b), we obtain the statement (b). ■

The next theorem gives an another alternative condition in terms of QDFs for the dual behavior.

Theorem 2: Let $R \in \mathbb{C}^{q\times q}[\xi]$ be square and given. Let $\mathfrak{B}'$ be described by the kernel representation (4). Then, exactly one of the following statements (a) and (b) holds.

(a) The behavior $\mathfrak{B}$ in (3) is asymptotically stable.
(b) There exist $\Psi' \in \mathbb{H}^{q\times q}[\zeta,\eta]$ satisfying

$$\Psi' \overset{\mathfrak{B}'}{\gneqq} 0 \text{ and } \nabla\Psi' \overset{\mathfrak{B}'}{\gneqq} 0.$$

Proof: The proof follows immediately from Theorem 1 and Lemma 1. ■

Remark 4: If we regard the QDF $\mathsf{Q}_{\Psi'}(v)$ $(v \in \mathfrak{B}')$ as the energy function for the dual behavior $\mathfrak{B}'$, the condition (b) in Theorem 1 shows that the energy diverges as time goes to infinity $(t \to \infty)$. This means that $\mathfrak{B}'$ is not asymptotically stable. ■

V. CONCLUSIONS

In this paper, we have derived an alternative condition for the asymptotic stability of a behavior described by a high-order differential-algebraic equation (kernel representation) based on the SDP duality. This condition gives a necessary and sufficient condition that the behavior is not asymptotically stable using the two-variable polynomial matrix equation and the QDFs for the dual behavior. Note that this condition recovers an alternative stability condition due to [1] in the sense that the behavior is described by the kernel representation. Since such a study has not been considered so far in the behavioral system theory, our result gives another theoretical knowledge.

As future works, we need to develop our results the dissipation theory and the optimal control in the behavioral framework. Moreover, it should be noted that the set of our dual analyzes must be extended to the Lyapunov stability analysis in n-D systems.

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NOTATIONS

- $\mathbb{S}^{q\times q}$: the set of $q \times q$ real symmetric matrices
- $\mathbb{H}^{q\times q}$: the set of $q \times q$ Hermite matrices
- $\mathbb{C}[\zeta,\eta]$: the set of complex coefficient polynomials in the indeterminates ζ and η
- $\mathbb{H}[\zeta,\eta]$: the set of complex coefficient Hermite polynomials in the indeterminates ζ and η
- $\mathbb{C}^q[\zeta,\eta]$: the set of q-dimensional complex coefficient polynomial vectors in the indeterminates ζ and η
- $\mathbb{H}^q[\zeta,\eta]$: the set of q-dimensional complex coefficient Hermite polynomial vectors in the indeterminates ζ and η
- $\mathbb{C}^{p\times q}[\xi]$: the set of $p\times q$ complex coefficient polynomial matrices in the indeterminate ξ
- $\mathbb{C}^{p\times q}[\zeta,\eta]$: the set of $p \times q$ complex coefficient polynomial matrices in the indeterminates ζ and η
- $\mathbb{H}^{q\times q}[\zeta,\eta]$: the set of $q\times q$ Hermite polynomial matrices in the indeterminates ζ and η
- $\mathbb{W}^{\mathbb{T}}$: the set of maps from $\mathbb{T}$ to $\mathbb{W}$
- $\mathcal{C}^\infty(\mathbb{R},\mathbb{C}^q)$: the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{C}^q$
- $\mathrm{col}(A_1, A_2, \cdots, A_n) := [\, A_1^\top \quad A_2^\top \quad \cdots \quad A_n^\top \,]^\top$
- $\mathrm{diag}(A_1, A_2, \cdots, A_n)$: $q \times q$ (block) diagonal matrix with (block) diagonal elements $\{A_1, A_2, \cdots, A_n\}$
- $\mathrm{rank}R$: the rank of polynomial matrix $R(\xi)$
- $\mathrm{rank}R(\lambda)$: the rank of constant matrix $R(\lambda)$

APPENDIX

A. PROOF OF PROPOSITION 1

(i) From the statements of (i), we obtain the following inequalities.

$$\begin{aligned} 0 &< \langle \mathcal{A}(x), \Psi' \rangle \\ &= \left\langle x, \mathcal{A}^{\mathrm{adj}}(\Psi') \right\rangle + \langle \Phi_0, \Psi' \rangle \\ &= \langle \Phi_0, \Psi' \rangle \\ &\le 0, \end{aligned}$$

Note that the first inequality follows from (10) and the second inequality of (11). Hence, one of the statements (a) and (b) is true at most.

To complete the proof, we show that if the statement (a) does not hold, then the statement (b) must be true. Consider the set $\mathcal{H} \subset \mathbb{H}^{q\times q}[\zeta,\eta]$ defined by

$$\mathcal{H} := \left\{ \Upsilon \in \mathbb{H}^{q\times q}[\zeta,\eta] \;\middle|\; \begin{array}{l} \mathcal{A}(y) + \Upsilon > 0 \\ \text{for some } y \in \mathbb{C}^q[\zeta,\eta] \end{array} \right\}.$$

Suppose that the statement (a) holds, i.e. $\Phi_0(\zeta,\eta) \notin \mathcal{C}$. Since $\mathcal{H}$ is open, nonempty and convex, there exists a nonzero $\Psi' \in \mathbb{H}^{m\times m}[\zeta,\eta]$ satisfying

$$\langle \Phi_0, \Psi' \rangle < \langle \Upsilon, \Psi' \rangle \tag{21}$$

for all $\Upsilon \in \mathcal{C}$. This implies that $\Psi'(\zeta,\eta)$ must satisfy $\Psi'(\zeta,\eta) \neq 0_{q\times q}$ and

$$\begin{aligned} \langle \Phi_0, \Psi' \rangle &< \langle -\mathcal{A}(y) + X, \Psi' \rangle \\ &= -\left\langle y, \mathcal{A}^{\mathrm{adj}}(\Psi') \right\rangle + \langle X, \Psi' \rangle \end{aligned} \tag{22}$$

for all $X > 0$ and $y \in \mathbb{C}^m[\zeta,\eta]$.

Suppose that $\mathcal{A}^{\mathrm{adj}}(\Psi') \neq 0$ holds in (22), then the first term $-\left\langle y, \mathcal{A}^{\mathrm{adj}}(\Psi') \right\rangle$ in the right hand is unbounded below as a function of $y(\zeta,\eta)$. On the other hand, if $\mathcal{A}^{\mathrm{adj}}(\Psi') = 0$ holds, then we have

$$-\left\langle y, \mathcal{A}^{\mathrm{adj}}(\Psi') \right\rangle = 0$$

Therefore, if $\Psi'(\zeta,\eta)$ satisfies (21), it must satisfy $\mathcal{A}^{\mathrm{adj}}(\Psi') = 0$. Also, in (22), the second term is unbounded below as a function of $X > 0$ if $\Psi' \not\geq 0$ holds. This yields a second condition $\Psi' \geq 0$. If $\Psi'(\zeta,\eta)$ satisfies the both conditions, the right-hand side of (22) is positive definite for all $X(\zeta,\eta)$ and $y(\zeta,\eta)$, and can take values arbitrary close to 0. Hence, the inequality holds for all $y(\zeta,\eta)$ and all $X > 0$ if

$$\langle \Phi_0, \Psi' \rangle \le 0.$$

is satisfied. Therefore, $\Psi'(\zeta,\eta)$ satisfies

$$\Psi' \gneqq 0,\ \mathcal{A}^{\mathrm{adj}}(\Psi') = 0,\ \langle \Phi_0, \Psi' \rangle \le 0.$$

This completes the proof of (i).

(ii) It is straightforward to see that the statements (c) and (d) contradict each other from the following inequalities.

$$0 \le \langle \mathcal{A}(x) + \Phi_0, \Psi' \rangle = \langle \Phi_0, \Psi' \rangle < 0$$

Hence, one of the statements (c) and (d) holds at most.

It is sufficient to show that one of the statements (c) and (d) is true at least. This is the case, if $\Phi_0 = \mathcal{A}(x_0)$ for some $x_0(\zeta,\eta)$, then the statement (c) holds with $x(\zeta,\eta) = -x_0(\zeta,\eta)$ and the statement (d) is false. This concludes the proof. ■