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Robust Analysis with respect to Real Parameter Uncertainty

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Abstract— **This paper revisits the problem of assessing robust stability of linear uncertain systems. It focus on systems with a single uncertain real parameter, in which case finite dimensional necessary and sufficient conditions for robust stability can be constructed in the form of LMIs (Linear Matrix Inequalities). Among other things, we revisit and propose alternative LMI tests for a problem of computing the structured singular value with (D, G)-scalings and show that an existing condition for the existence of an affine parameter-dependent Lyapunov function which is known to be sufficient is also necessary in the case of a single uncertainty.**

I. INTRODUCTION

This paper is mostly concerned with the problem of assessing asymptotic stability of uncertain linear time-invariant systems whose uncertainty is a single real scalar parameter

$$\dot{x}(t) = A(r)\, x(t), \qquad r \in \mathbb{R}, \qquad |r| \leq 1. \tag{1}$$

The dependence of A in r is assumed to be polynomial. It is known that uncertain linear systems of the form (1) are asymptotically stable if and only if there exists a polynomial parameter-dependent Lyapunov function $V(x, r) = x^* P(r) x$ where $P(r)$ is a polynomial matrix of high enough degree in r satisfying the Lyapunov inequalities

$$A^*(r)\, P(r) + P(r)\, A(r) \prec 0, \qquad P(r) \succ 0, \tag{2}$$

for all $r \in \mathbb{R}$, $|r| \leq 1$ (see, for instance [1], [2]). This is indeed the case even if more than one uncertainty is considered [3].

In the case of a single real uncertainty, explicit bounds on the maximum required degree of $P(r)$ are available [1], [2]. These bounds depend on the size and some other structural properties of the matrix A (see Section III for more details). Furthermore, a variation of a result from μ-analysis [4] makes it possible to construct finite dimensional LMI (Linear Matrix Inequality) conditions that are both necessary and sufficient for verifying the existence of such polynomial parameter-dependent Lyapunov functions. A first contribution of this paper is to revisit these results, providing a set of alternative LMI conditions which are also necessary and sufficient.

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Earlier results on robust stability focused mostly on sufficient robust stability conditions. For instance, the well-known concept of quadratic stability [5], in which $P(r) = P$ is assumed to be parameter independent. The works [6], [7] introduced tests for the case when $P(r)$ is affine (see also [8]). All such results hold for the more general case of many uncertainties, and have the nice feature that they can be often modified to provide LMI conditions for control and filtering design problems [9], [10], [11], [12], a fact that has not yet proved possible in the case of higher order parameter-dependent polynomial Lyapunov function conditions, not even in the case of a single real uncertainty. A second contribution of this paper is to revisit such conditions in the light of our alternative set of LMI conditions. In particular, we prove that the sufficient conditions for the existence of an affine parameter-dependent Lyapunov function of [6] is both necessary and sufficient in the case of a single real uncertainty.

A. Notation

The following notation will be used throughout the paper. The scalar $j = \sqrt{-1}$. For a matrix $X \in \mathbb{C}^{n \times n}$: $\overline{X}$, X^* are the complex-conjugate and complex-conjugate transpose of the matrix X respectively and X^{-1}, $X_\perp$ are full rank matrices such that $XX^{-1} = I$ and $XX_\perp = 0$. $\mathrm{He}\{X\}$ is short-hand notation for $X + X^*$. Finally $\mathbb{HC}^n$ ($\mathbb{AC}^n$) denotes the set of Hermitian (anti-symmetric) matrices of dimension n.

II. STRUCTURED SINGULAR VALUE AND (D, G) SCALING

For some matrix structure $\mathbf{\Delta}$ the *structured singular value* of a square matrix $M \in \mathbb{C}^{p \times p}$ denoted $\mu_{\mathbf{\Delta}}(M)$ is defined as

$$\mu_{\mathbf{\Delta}}(M) := \left(\inf_{\Delta \in \mathbf{\Delta}} \{ \|\Delta\| : \det(I - \Delta M) = 0 \} \right)^{-1}.$$

In case there exists no $\Delta \in \mathbf{\Delta}$ which makes $(I - \Delta M)$ singular then $\mu_{\mathbf{\Delta}}(M) = 0$. For more details see, for instance [13], [14].

In general the structured singular value cannot be computed in polynomial time [15]. A common practice is the introduction of *scalings* or *multipliers* through duality theory that can provide computable upper bounds for $\mu_{\mathbf{\Delta}}$ in

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polynomial time. For instance, define the sets

$$\mathbf{D} := \{D : \quad D \in \mathbb{HC}^p, \quad D \succ 0\}, \tag{3}$$
$$\mathbf{G} := \{G : \quad G \in \mathbb{AC}^p\}. \tag{4}$$

A pair of multiplier matrices $(D, G) \in \mathbf{D} \times \mathbf{G}$ is called a (D, G) *scaling*. Now let the matrix $M \in \mathbb{C}^{p+q \times p+q}$ be partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{5}$$

and consider the matrix inequality

$$\begin{bmatrix} M_{11} & M_{12} \\ I & 0 \end{bmatrix}^* \begin{bmatrix} D & G^* \\ G & -D \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ I & 0 \end{bmatrix} + \begin{bmatrix} M_{21} & M_{22} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} M_{21} & M_{22} \\ 0 & I \end{bmatrix} \prec 0. \tag{6}$$

In the particular class of a single real scalar and a single complex block uncertainty, i.e.

$$\mathbf{\Delta} := \{\operatorname{diag}(rI_p, \Delta) : \quad r \in \mathbb{R}, \quad \Delta \in \mathbb{C}^{q \times q}\}, \tag{7}$$

the work [4] established the following result.

Lemma 1: Let the matrix $M \in \mathbb{C}^{p+q \times p+q}$ be partitioned as in (5) and the uncertainty structure $\mathbf{\Delta}$ as in (7) be given. $\mu_{\mathbf{\Delta}}(M) \leq 1$ if and only if there exists $(D, G) \in \mathbf{D} \times \mathbf{G}$ satisfying the LMI (6).

This is a remarkable result, since the use of (D, G) scalings is only sufficient for other uncertainty structures [14] not much more complicated then (7) such as, for instance, any combination of two real or complex scalar uncertainties [14], [16], [4].

A. (D, G) Scalings Revisited

The condition to be verified in Lemma 1 is a pair of LMIs. In this section we provide an alternative pair of LMIs that can be used as an alternative necessary and sufficient condition for $\mu_{\mathbf{\Delta}}(M)$ to be less or equal than one. The key is the following two technical lemmas.

Lemma 2: Let $(D, G) \in \mathbf{D} \times \mathbf{G}$. Then

$$\begin{bmatrix} D & G^* \\ G & -D \end{bmatrix} \succeq \begin{bmatrix} -G \\ -D \end{bmatrix} \begin{bmatrix} rI & I \end{bmatrix} + \begin{bmatrix} rI \\ I \end{bmatrix} \begin{bmatrix} G & -D \end{bmatrix},$$

for all $r \in \mathbb{R}$, $|r| \leq 1$.

Proof: Note that

$$|r| \leq 1 \qquad \iff \qquad \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix} \succeq 0.$$

Hence, because $D \succeq 0$,

$$\begin{bmatrix} D & rD \\ rD & D \end{bmatrix} = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix} \otimes D \succeq 0.$$

Therefore

$$\begin{aligned} \begin{bmatrix} D & G^* \\ G & -D \end{bmatrix} &\succeq \begin{bmatrix} D & G^* \\ G & -D \end{bmatrix} - \begin{bmatrix} D & rD \\ rD & D \end{bmatrix}, \\ &= \begin{bmatrix} -G \\ -D \end{bmatrix} \begin{bmatrix} rI & I \end{bmatrix} + \begin{bmatrix} rI \\ I \end{bmatrix} \begin{bmatrix} G & -D \end{bmatrix}, \end{aligned}$$

which concludes the proof. ■

The following is related to a slight generalization of inequalities of the form (6) which we will use extensively.

Lemma 3: Let the matrices $\Sigma \in \mathbb{HC}^k$ and $M, N \in \mathbb{C}^{p \times k}$ be given with $k > p$. The following statements are equivalent.

(i) $\xi^* \Sigma \xi < 0$, for all nonzero vectors $\xi \in \mathbb{R}^k$ which satisfy $(N - r\,M)\,\xi = 0$ for some $|r| \leq 1$.

(ii) There exist matrices $(D, G) \in \mathbf{D} \times \mathbf{G}$ such that

$$\begin{bmatrix} M \\ N \end{bmatrix}^* \begin{bmatrix} D & G^* \\ G & -D \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} + \Sigma \prec 0. \tag{8}$$

(iii) There exist matrix $X \in \mathbb{C}^{k \times p}$ such that

$$X\,(N \pm M) + (N \pm M)^*\,X^* + \Sigma \prec 0. \tag{9}$$

Proof: That (ii) $\Leftrightarrow$ (i) has been shown in [1, Lemma 4.2].

(iii) $\Rightarrow$ (i): Construct the convex combination of (9)

$$X\,(N + r\,M) + (N + r\,M)^*\,X^* + \Sigma \prec 0,$$

which holds for all $|r| \leq 1$. Define a nonzero vector $\xi \in \mathbb{C}^k$ such that $(N - r\,M)\xi = 0$. Multiply the above inequality by ξ on the right and by ξ^* on the left to obtain (i).

(ii) $\Rightarrow$ (iii): Suppose (ii) is feasible, then use Lemma 2 to conclude that

$$\operatorname{He}\left\{ \begin{bmatrix} M \\ N \end{bmatrix}^* \begin{bmatrix} -G \\ -D \end{bmatrix} \begin{bmatrix} rI & I \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} \right\} + \Sigma \prec 0, \tag{10}$$

is feasible for all $|r| \leq 1$. Now define

$$X = -\,M^* G - N^* D.$$

Since (10) is feasible for all $|r| \leq 1$, then it is feasible for the endpoints $r_0 = -1$ and $r_1 = 1$, thus producing (9). ■

The above lemma is a version of the result of [17]. In [17], which establishes alternative LMI conditions for the KYP (Kalman-Yakubovich-Popov) Lemma, the "uncertainty" is a frequency variable on the imaginary axis. The relation between Lemma 3 and [17] is the same that exists between [4] and the finite frequency KYP Lemma in [18].

The following corollary relates Lemma 3 to stability analysis via μ-analysis, that is Lemma 1.

Corollary 1: Let the matrix $M \in \mathbb{C}^{p+q \times p+q}$ be partitioned as in (5) and the uncertainty structure $\mathbf{\Delta}$ as in (7) be given. $\mu_{\mathbf{\Delta}}(M) \leq 1$ if and only if there exists $X \in \mathbb{C}^{2p\times p}$ such that the LMI

$$\mathrm{He}\left\{X\begin{bmatrix}I \pm M_{11} & \pm M_{12}\end{bmatrix}\right\} + \begin{bmatrix}M_{21} & M_{22}\\ 0 & I\end{bmatrix}^* \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}M_{21} & M_{22}\\ 0 & I\end{bmatrix} \prec 0, \quad (11)$$

is satisfied.

Proof: Follows from Lemma 3 applied with $M = \begin{bmatrix}M_{11} & M_{12}\end{bmatrix}$, $N = \begin{bmatrix}I & 0\end{bmatrix}$ and

$$\Sigma = \begin{bmatrix}M_{21} & M_{22}\\ 0 & I\end{bmatrix}^* \begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}\begin{bmatrix}M_{21} & M_{22}\\ 0 & I\end{bmatrix}.$$

■

III. Robust Stability

As mentioned previously in the introduction, testing robust stability of the uncertain system (1) amounts to finding computable conditions for the existence of a polynomial parameter-dependent Lyapunov matrix $X(r)$ of high enough degree that satisfy the Lyapunov inequalities (2). The problem now will be to convert such inequalities to a form that is amenable to the results of the previous section. We will start by discussing the affine case. For that assume that $A(r)$ depends affinely on r, that is

$$A(r) = A_0 + rA_1. \quad (12)$$

Unfortunately it is known, by virtue of a counter-example given in [19], that one may have to consider $P(r)$ of degree higher then one in order to prove stability of the affine uncertain system. The works [1], [2] provide upper bounds on the degree of $P(r)$ of the form

$$\mathrm{degree}(P(r)) \leq \min\left\{2\,n\,\rho - \rho^2 + \rho,\ n\,(n+1) - 2\right\}$$

where $\rho = \mathrm{rank}(A_1)$ and n is the dimension of A. We investigate in the next sections two cases: quadratic stability, i.e. $P(r) = P$ parameter independent and $P(r)$ affine before we consider the most general case of $A(r)$ and $P(r)$ of arbitrary degree.

A. Quadratic stability

It is convenient to define the following matrices

$$B_1 = A_0 + A_1, \qquad B_2 = A_0 - A_1. \quad (13)$$

The uncertain system (1) with the affine matrix (12) is quadratically stable, i.e. there exists a parameter-independent Lyapunov matrix $P(r) = P$ satisfying the parameter-dependent Lyapunov inequalities (2) if and only if (see [5], [9])

$$B_i^*\,P + P\,B_i \prec 0, \qquad P \succ 0, \qquad i = \{1,2\}.$$

We will now obtain this result as a particular case of Lemma 3. First note that in the case $P(r) = P$ that the inequality (2) can be rearranged as

$$\begin{bmatrix}A(r)\\ I\end{bmatrix}^* \begin{bmatrix}0 & P\\ P & 0\end{bmatrix}\begin{bmatrix}A(r)\\ I\end{bmatrix} \prec 0, \qquad X \succ 0. \quad (14)$$

Using the methods of [20], the first inequality above is equivalent to the inequality

$$\xi^*\begin{bmatrix}0 & P\\ P & 0\end{bmatrix}\xi < 0, \quad \begin{bmatrix}I & -A(r)\end{bmatrix}\xi = 0, \quad \xi \neq 0, \quad (15)$$

which should hold for all vectors $\xi \in \mathbb{C}^{2n}$ and $|r| \leq 1$. This inequality is in the form of statement (i) in Lemma 3 with

$$M = \begin{bmatrix}0 & A_1\end{bmatrix}, \quad N = \begin{bmatrix}I & -A_0\end{bmatrix}, \quad \Sigma = \begin{bmatrix}0 & P\\ P & 0\end{bmatrix}. \quad (16)$$

Application of Lemma 3 then provides the equivalence with the existence a matrix X such that

$$X\begin{bmatrix}I & -B_i\end{bmatrix} + \begin{bmatrix}I & -B_i\end{bmatrix}^* X^* + \Sigma \prec 0, \quad i = \{1,2\},$$

where Σ is as in (16). Furthermore, defining the partition $X = \begin{bmatrix}-Y^* & -Z^*\end{bmatrix}^*$ gives the following LMI for quadratic stability,

$$\begin{bmatrix}-Y - Y^* & YB_i - Z^* + X\\ B_i^*Y^* - Z + X & Z^*B_i^* + B_iZ\end{bmatrix} \prec 0, \quad i = \{1,2\}, \quad (17)$$

which should hold for some Y, Z and $P \succ 0$. It has been shown in [6] that (17) is completely equivalent to (14). Indeed, multiplication of (17) by $\begin{bmatrix}B_i^* & I\end{bmatrix}$ on the left and by its conjugate transpose on the right for $i = \{1,2\}$ produces (14). Conversely, the choice $Z = Z^* = P$ and $Y = \epsilon I$ for a sufficiently small $\epsilon > 0$ produce feasible inequalities (17).

B. Affine parameter-dependent Lyapunov functions

Following our previous discussions, the parameter-dependent Lyapunov inequality (2) can be rewritten as the condition

$$\xi^*\begin{bmatrix}0 & P(r)\\ P(r) & 0\end{bmatrix}\xi < 0, \quad \begin{bmatrix}I & -A(r)\end{bmatrix}\xi = 0, \quad (18)$$

with $P(r) \succ 0$ for all vectors $\xi \in \mathbb{C}^{2n}$ and $|r| \leq 1$. The parameter independent case when $P(r) = P$ has been discussed in the previous section. In this section we focus on the affine parameter-dependent Lyapunov function, that is

$$P(r) = P_0 + rP_1. \quad (19)$$

We seek to establish the following result.

Theorem 1: Let $A(r)$ and $P(r)$ be affine in $r \in \mathbb{R}$ as in (12) and (19), respectively. There exists $P(r) \succ 0$ such that $A^*(r)P(r) + P(r)A(r) \prec 0$ for all $|r| \leq 1$ if and only if there exist matrices $Y, Z \in \mathbb{C}^{n\times n}$ and $P_1, P_2 \in \mathbb{HC}^n$ such that

$$\begin{bmatrix} -Y - Y^* & P_i - Z^* + YB_i \\ P_i - Z - B_i^* G^* & B_i^* Z^* + ZB_i \end{bmatrix} \prec 0, \quad P_i \succ 0 \quad (20)$$

for all $i = \{0, 1\}$ where B_1, B_2 are as in (13).

One can verify that P and P_r are related to the variables P_1 and P_2 through

$$P = \frac{1}{2}(P_1 + P_2), \qquad P_r = \frac{1}{2}(P_1 - P_2).$$

Sufficiency of the above LMI condition has long been established, for instance in [6], [20]. It amounts to producing convex combinations of (20) so as to conclude that $P(r) \succ 0$ and

$$\begin{bmatrix} -Y - Y^* & P(r) - Z^* + GA(r) \\ P(r) - Z - A(r)^* Y^* & A(r)^* Z^* + ZA(r) \end{bmatrix} \prec 0. \quad (21)$$

from where (2) is produced after multiplication by $\begin{bmatrix} A(r)^* & I \end{bmatrix}$ on the left and by its conjugate transpose on the right.

A proof that the above condition is also necessary will be developed in Section IV.

C. The General Case

Back to the case when $A(r)$ and $P(r)$ are polynomials of arbitrary degree, one will find in the literature several ways of recasting the parameter-dependent LMI (2) as a finite dimensional LMI. For instance, see [1], [2]. Most of these results transform the original polynomial problem into some form of affine dependence of higher dimensional system. This practice is customary also when many uncertainties are involved, as for instance in [21].

For example, an interesting result is that of [2, Theorem 1]. Let N be an even integer and partition the polynomial matrix

$$A(r) = A_0 + rA_1 + \cdots + r^N A_N$$

of degree N into its odd and even components

$$A(r) = A_e(r^2) + rA_o(r^2)$$

with

$$A_e(r^2) = A_0 + r^2 A_2 + \cdots + r^N A_N,$$
$$A_o(r^2) = A_1 + r^2 A_3 + \cdots + r^{N-2} A_{N-1}.$$

If N is odd a similar partitioning is possible (see [2] for details). Then robust stability of $A(r)$ for all $|r| \leq 1$ is equivalent to robust stability of the polynomial matrix

$$\tilde{A}(\rho) = \begin{bmatrix} A_e(\rho) & \rho A_o(\rho) \\ A_o(\rho) & A_e(\rho) \end{bmatrix}, \qquad \rho = \frac{r+1}{2}$$

of degree $N/2$. By applying this idea recursively to $\tilde{A}(\rho)$ one will eventually find an equivalent system which is now affine in r. For example, if $N = 2^k$ then after k iterations. By the same token, the resulting equivalent affine system matrix will have dimension $2^k n$. The present Theorem 1 provides the necessary and sufficient conditions for the existence of an affine parameter-dependent Lyapunov function of degree N for the original system. If a higher order polynomial dependence is needed in the Lyapunov function then the system matrices can be appropriately augmented so as to accomplish that. Again, see [2] for details.

The conclusion is that Theorem 1, now shown to be a necessary and sufficient condition could be used to provide alternative LMI tests for robust stability of uncertain linear systems polynomial in a single real uncertain parameter. Indeed, the LMIs in [1], [2] are all based on (D, G) scalings, an alternative version of which has been discussed in Section II-A is at the heart of Theorem 1. We are currently investigating whether these alternative conditions can bring computational advantage to this kind of robust stability analysis and whether they would represent advantage in the problem of controller design.

IV. Necessity of the LMI Conditions in Theorem 1

In order to prove necessity of Theorem 1 we follow methods similar to the ones in [22]. Though the basic ideas are similar, the technical results are quite different. Indeed, the manipulations in [22] have as an essential assumption the fact that a certain matrix $(j\omega I - A)$ needs to be invertible, which makes sense in the context of the KYP Lemma. In the context of the present paper one can verify that this matrix is not invertible, so that the results of [22] do not readily apply. The necessary changes are described in the remainder of this section. Some passages are only sketched and the interested reader is referred to [22] for more details. We start with the following lemma.

Lemma 4: Let $A(r) \in \mathbb{R}^n$ be affine in $r \in \mathbb{R}$ as in (12) and $\Omega(r) \in \mathbb{HC}^k$, $\Omega(r) = \Omega_0 + r\Omega_1$. Define

$$H = \begin{bmatrix} A_0 & 0 & -A_1 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_0 & -\Sigma_1/2 \\ -\Sigma_1/2 & 0 \end{bmatrix}. \quad (22)$$

The following statements are equivalent.

(i) $\xi^* \Omega(r) \xi < 0$, for all nonzero vectors $\xi \in \mathbb{R}^k$ which satisfy $\begin{bmatrix} I & -A(r) \end{bmatrix} \xi = 0$.

(ii) $v^* H^* \Omega H v < 0$, for all nonzero vectors $v \in \mathbb{R}^{2k}$ which satisfy $\begin{bmatrix} rI & I \end{bmatrix} H v = 0$.

Proof: Let us first investigate the structure of the null space of the matrix $\begin{bmatrix} rI & I \end{bmatrix} H$. Note that

$$\left(\begin{bmatrix} rI & I \end{bmatrix} H\right)_\perp = \begin{bmatrix} rA_0 & I & -rA_1 \\ rI & 0 & I \end{bmatrix}_\perp = \begin{bmatrix} I \\ -rA_0 - r^2 A_1 \\ -rI \end{bmatrix}.$$

Now multiply the above on the left by H to obtain

$$H\left(\begin{bmatrix} rI & I \end{bmatrix} H\right)_\perp = \begin{bmatrix} A_0 + rA_1 \\ I \\ -rA_0 - r^2 A_1 \\ -rI \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \\ -rI & 0 \\ 0 & -rI \end{bmatrix} \begin{bmatrix} A_0 + rA_1 \\ I \end{bmatrix}.$$

Finally noting that

$$\begin{bmatrix} I \\ -rI \end{bmatrix}^* \Omega \begin{bmatrix} I \\ -rI \end{bmatrix} = \Omega(r) \tag{23}$$

establishes the equivalence between (i) and (ii). ■

Feasibility of the inequality in statement (i) where $\Omega(r)$ depends affinely on r for all $|r| \leq 1$ can then be verified by checking feasibility of the augmented inequality in statement (ii) with H given by (22). In the context of Theorem 1, just define

$$\Omega(r) = \begin{bmatrix} 0 & P(r) \\ P(r) & 0 \end{bmatrix}. \tag{24}$$

with $P(r)$ affine. The consequence is that verifying statement (ii) for all $|r| \leq 1$ can be done with Lemma 3. For that just define

$$\Sigma = H^* \Omega H.$$

and

$$M = \begin{bmatrix} -A_0 & 0 & A_1 \\ -I & 0 & 0 \end{bmatrix}, \qquad N = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Then item (ii) in Lemma 3 provides an LMI test for the statements in Lemma 4 in the form of the existence of scaling matrices $D, G \in \mathbf{D} \times \mathbf{G}$ such that

$$\begin{bmatrix} M \\ N \end{bmatrix}^* \begin{bmatrix} D & G \\ G^* & -D \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} + H^* \Omega H \prec 0. \tag{25}$$

Note that

$$\begin{bmatrix} M \\ N \end{bmatrix}^* \begin{bmatrix} D & G \\ G^* & -D \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} -M \\ N \end{bmatrix}^* \begin{bmatrix} D & G^* \\ G & -D \end{bmatrix} \begin{bmatrix} -M \\ N \end{bmatrix}, = H^* \begin{bmatrix} D & G^* \\ G & -D \end{bmatrix} H,$$

so that (25) can be rewritten as

$$H^* \left(\begin{bmatrix} D & G^* \\ G & -D \end{bmatrix} + \Omega \right) H \prec 0. \tag{26}$$

As in [22], the particular structure of the LMI (26) can be explored by the Elimination Lemma.

Lemma 5 (Elimination Lemma): Let matrices $\mathcal{Q} \in \mathbb{HC}^n$, $\mathcal{B} \in \mathbb{C}^{k \times n}$ such that $\operatorname{rank}(\mathcal{B}) < n$, and $\mathcal{C} \in \mathbb{C}^{m \times n}$ such that $\operatorname{rank}(\mathcal{C}) < n$ be given. Then the following statements are equivalent.

(i) The two conditions hold

$$\mathcal{B}_\perp^* \mathcal{Q} \mathcal{B}_\perp \prec 0, \quad \text{and} \quad \mathcal{C}_\perp^* \mathcal{Q} \mathcal{C}_\perp \prec 0. \tag{27}$$

(ii) There exist a matrix $\mathcal{X} \in \mathbb{C}^{m \times k}$ such that

$$\mathcal{C}^* \mathcal{X} \mathcal{B} + \mathcal{B}^* \mathcal{X}^* \mathcal{C} + \mathcal{Q} \prec 0. \tag{28}$$

Proof: See [9], [23]. ■

Application of the above lemma on the inequality (26) produces the following result.

Lemma 6: Let matrices $H \in \mathbb{C}^{4n \times 3n}$ and $\Omega \in \mathbb{HC}^{2n}$ be given as in (22). The following are equivalent statements.

(i) $v^* H^* \Omega H v < 0$, for all nonzero vectors $v \in \mathbb{R}^{2k}$ which satisfy $\begin{bmatrix} rI & I \end{bmatrix} H v = 0$ for some $|r| \leq 1$.

(ii) There exist matrices $(D, G) \in \mathbf{D} \times \mathbf{G}$, and $K \in \mathbb{C}^{2n \times n}$ such that

$$\begin{bmatrix} D & G^* \\ G & -D \end{bmatrix} + \operatorname{He}\left\{ \begin{bmatrix} K \\ 0 \end{bmatrix} \begin{bmatrix} I & -A_0 & 0 & A_1 \end{bmatrix} \right\} + \Omega \prec 0. \tag{29}$$

Proof: Equivalence of item (i) with feasibility of the inequality (26) follows from the previous discussion. Note that $D \in \mathbf{D}$ implies that

$$0 \succ -D, = \begin{bmatrix} 0 & I \end{bmatrix} \left(\begin{bmatrix} D & G^* \\ G & -D \end{bmatrix} + \Omega \right) \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{30}$$

since the $(2,2)$-block of Ω is zero. With feasibility of inequalities (26) and (30), apply the Elimination Lemma (Lemma 5) with

$$\mathcal{B}_\perp = \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad \mathcal{C}_\perp = H.$$

This proves the equivalence of item (i) with the existence of a matrix variable $K \in \mathbb{C}^{2n \times n}$ such that

$$\operatorname{He}\left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} K \begin{bmatrix} I & -A_0 & 0 & A_1 \end{bmatrix} \right\} + \begin{bmatrix} D & G^* \\ G & -D \end{bmatrix} + \Omega \prec 0,$$

where we have used the fact that

$$(H^*)_\perp = \begin{bmatrix} I & -A_0 & 0 & A_1 \end{bmatrix},$$

to obtain the final result. ■

In the above lemma, an extra multiplier variable K has been introduced, similarly as in the extended LMI conditions derived for robustness analysis [10], [20]. The inequality (29) remains an LMI in the optimization variables D, G and K, however, on a space of larger dimension and with more optimization variables. Note that Lemma 6 holds for any matrix H such that $(H^*)_\perp$ exists provided that $\Omega(r)$ is affine, see [22].

We can now use Lemma 3 to factor the (D, G)-scalings appearing in (29) as follows.

$$\Omega + \mathrm{He}\left\{ \begin{bmatrix} K \\ 0 \end{bmatrix} \begin{bmatrix} I & -A_0 & 0 & A_1 \end{bmatrix} \right\} + \left(\begin{bmatrix} -G \\ -D \end{bmatrix} \begin{bmatrix} rI & I \end{bmatrix} + \begin{bmatrix} rI \\ I \end{bmatrix} \begin{bmatrix} G & -D \end{bmatrix} \right) \prec 0,$$

Multiply the above inequality on the left hand side by $\begin{bmatrix} I & -rI \end{bmatrix}$ and on the right hand side by the transpose,

$$0 \succ \begin{bmatrix} I \\ -rI \end{bmatrix}^* \left(\mathrm{He}\left\{ \begin{bmatrix} -G \\ -D \end{bmatrix} \begin{bmatrix} rI & I \end{bmatrix} + \begin{bmatrix} K \\ 0 \end{bmatrix} \begin{bmatrix} I & -A_0 & 0 & A_1 \end{bmatrix} \right\} + \Omega \right) \begin{bmatrix} I \\ -rI \end{bmatrix},$$

which reduces to

$$0 \succ \mathrm{He}\left\{ K \begin{bmatrix} I & -A(r) \end{bmatrix} \right\} + \begin{bmatrix} I \\ -rI \end{bmatrix}^* \Omega \begin{bmatrix} I \\ -rI \end{bmatrix}.$$

This can be rewritten as

$$\mathrm{He}\left\{ K \begin{bmatrix} I & -A(r) \end{bmatrix} \right\} + \Omega(r) \prec 0.$$

Now choose

$$K = \begin{bmatrix} -Y \\ -Z \end{bmatrix} \tag{31}$$

to show that (21) is feasible for all $|r| \leq 1$. In particular, (21) holds for $r = -1$ and $r = 1$ to prove that the pair of inequalities (20) are feasible, thus completing this proof.

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