Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

A Model Based Fault Detection and Prognostic Scheme for Uncertain Nonlinear Discrete-Time Systems

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Abstract— A new fault detection and prognostics (FDP) framework is introduced for uncertain nonlinear discrete time system by using a discrete-time nonlinear estimator which consists of an online approximator. A fault is detected by monitoring the deviation of the system output with that of the estimator output. Prior to the occurrence of the fault, this online approximator learns the system uncertainty. In the event of a fault, the online approximator learns both the system uncertainty and the fault dynamics. A stable parameter update law in discrete-time is developed to tune the parameters of the online approximator. This update law is also used to determine time to failure (TTF) for prognostics. Finally a fourth order translational oscillator with rotating actuator (TORA) system is used to demonstrate the fault detection while a mass damper system is used for demonstrating the prognostics scheme.

I. Introduction

Quantitative fault detection and prognostics (FDP) methodology has gained popularity in the past two decades due to analytical performance guarantees. In this quantitative based approach, a mathematical model of the nonlinear system is developed and its output along with output of the nonlinear system is utilized to generate residuals [1]. A fault is detected if the residual exceeds a predetermined threshold. Since all physical systems are nonlinear, recent work on fault detection [2] is focused around nonlinear systems [2] expressed in continuous-time due to ease of analysis. Various online approximator (OLA) schemes predominantly in continuous-time have been introduced in the literature [2] for characterizing the fault dynamics. It is understood that a system after a fault could still be functional whereas after a failure, it is not operational [3]. In other words, fault detection is a first step in the failure detection.

However, to the best of our knowledge, there are no reported FDP schemes for discrete-time systems even though the FDP schemes have to be implemented on embedded systems that require discrete-time development. Lack of such FDP schemes in discrete-time could be attributed to difficulty in verifying analytically their performance since the first difference of the Lyapunov candidate is not linear with respect to the states in the case of discrete-time in contrast with the case of continuous-time. Recently, a fault detection scheme for nonlinear discrete time systems with actuator faults was developed [4] with the requirement of persistency of excitation (PE) condition.

On the other hand, in our previous work [5], a new fault detection scheme for nonlinear discrete time systems with nonlinear state or process faults was developed by using online approximators. Also, the stability was mathematically analyzed. However, in [5], and [2], the system uncertainty was assumed to be upper bounded and the detection threshold was derived based on this upper bound, which is a stringent assumption. In contrary, in this paper this assumption has been relaxed wherein an online approximator in discrete-time (OLAD) is utilized to approximate the system uncertainty and a suitable threshold is derived to detect the presence of a fault. Subsequently upon detecting the fault, the OLAD learns both the fault dynamics and the system uncertainty.

Stable adaptive parameter update law is developed to tune the parameters of the OLAD schemes online. Relaxing the prior bound on the system uncertainty complicates the performance guarantee whereas it is addressed. After a fault is detected, the time to failure (TTF) needs to be assessed for prognostics. In determining the TTF, previous work [6] assumed a specific degradation model of the system, which is found to be quite limited to the system or material type under consideration. In another technique, a deterministic polynomial and probabilistic methods are developed for prognosis [7] by assuming that only certain parameters affect the fault whereas the fault dynamics are not approximated online making the prediction inaccurate. Under a mild assumption, the TTF is determined by projecting the parameters of the OLAD to their respective maximum limit. The limiting value of a system parameter can be obtained from the system designer [7].

The contributions of this paper involve detecting and learning unknown state or process faults occurring in an uncertain nonlinear discrete time systems while approximating the system uncertainty and the fault dynamics. Additionally, we determine the TTF after the occurrence of the fault using analytical tools. Thus, the development of the fault detection and prognostic framework is unified in this paper. Next in Section II, the system under consideration is discussed.

II. Problem Statement

Consider an uncertain nonlinear discrete time system described by

$$x(k+1) = A_0 x(k) + \varphi(x(k), u(k)) + \eta(k, x(k), u(k)) + g(x(k), u(k)) \quad (1)$$

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where $x \in \Re^n$ is the state vector, $u \in \Re^m$ is the input vector, $\varphi : \Re^n \times \Re^m \to \Re^n$, $\eta : \Re^+ \times \Re^n \times \Re^m \to \Re^n$, $g : \Re^n \times \Re^m \to \Re^n$ are smooth vector fields whereas $A_0 \in \Re^{n \times n}$ is a known matrix. The known nominal dynamics of the system is given by

$$x(k+1) = A_0 x(k) + \varphi(x(k), u(k))$$

The nonlinear function $\eta(k, x(k), u(k))$ represents the system uncertainties whereas $g(x(k), u(k)) = \Pi(k-T) f(x(k), u(k))$ is the unknown fault function with $f(x(k), u(k))$ representing the nonlinear state or process fault dynamics. The nonlinear fault is modeled in terms of the measurable states and inputs. The diagonal matrices $\Pi \in \Re^{n \times n}$ denote the time profile of the state or process faults, which are given by [2]

$$\Pi(k-T) = diag(\Omega_1(k-T), \Omega_2(k-T), ..., \Omega_n(k-T))$$

where

$$\Omega_i(k-T) = \begin{cases} 0 \text{ if } k < T \\ 1 - e^{-\kappa_i(k-T)}, \text{ if } k \ge T \end{cases} \qquad i=1, 2\ldots n$$

where $\kappa_i > 0$ is a unknown constant, which represent the rate at which the fault in the state x_i evolves. For small values of κ_i, this term describes incipient fault whereas for large values they represent abrupt faults. Also, T denotes the unknown time of occurrence of state or process faults. It is assumed that the initial system states are available i.e. $x(0) = x_0$. Next the following assumption, which is standard in the literature, is introduced.

Assumption 1: The states, which are measurable, and the input vectors are bounded prior to and after the fault occurrence [2].

In the next section, we present the fault detection scheme and in the subsequent sections, the TTF prediction (prognostics) will be introduced.

III. Fault Detection Framework

To monitor and detect faults in the system discussed in the previous section, consider the following nonlinear estimator

$$\hat{x}(k+1) = A_0 \hat{x}(k) + K(x(k) - \hat{x}(k)) + \varphi(x(k), u(k)) + \hat{\eta}(k, x(k), u(k); \hat{\theta}(k)) \qquad (2)$$

where $\hat{x} \in \Re^n$ is the estimated state vector, $\hat{\eta} : \Re^+ \times \Re^n \times \Re^m \times \Re^{p_s} \to \Re^n$ is the OLAD used to learn the system uncertainty and the fault dynamics, K is a design matrix and $\hat{\theta} \in \Re^{p_s}$ is the vector of adjustable parameters for approximating the system uncertainty. Note after detecting a fault, the OLAD is augmented with additional parameters i.e. $\hat{\theta} \in \Re^l$, where $l = p_s + p_f$ in order to approximate both the system uncertainty and the fault dynamics. This intuitively means that the same OLAD with the newly defined size could be used in learning both the system uncertainty and the fault dynamics instead of using two OLADs. However, to understand the fault characteristics, one could always use the knowledge of the learned system uncertainty prior to the fault in order to obtain the fault dynamics. For clarity, we reiterate that prior to the occurrence of the fault, the size of the OLAD parameter vector is $\hat{\theta} \in \Re^{p_s}$. Hence (1) and (2) reduces to

$$x(k+1) = A_0 x(k) + \varphi(x(k), u(k)) + \eta(k, x(k), u(k)) \qquad (3)$$

and

$$\hat{x}(k+1) = A_0 \hat{x}(k) + K(x(k) - \hat{x}(k)) + \varphi(x(k), u(k)) + \hat{\eta}(k, x(k), u(k); \hat{\theta}(k)) \qquad (4)$$

Next define the state estimation error as $e = x - \hat{x}$, hence from (3) and (4), we get

$$e(k+1) = Ae(k) + \eta(k, x(k), u(k)) - \hat{\eta}(k, x(k), u(k); \hat{\theta}(k))$$

where K is selected such that the matrix $A = A_0 - K$ has all its eigen values within the unit disc. Hence by rewriting the above equation, we get

$$e(k+1) = Ae(k) + \hat{\eta}(k, x(k), u(k); \theta^*(k)) - \hat{\eta}(k, x(k), u(k); \hat{\theta}(k)) + v(k) \qquad (5)$$

where $v(k) = \eta(k, x(k), u(k)) - \hat{\eta}(k, x(k), u(k); \theta^*(k))$ is the approximation error, and θ^* is the optimal chosen value of $\hat{\theta}$ such that the L_2 norm between $\eta(k, x(k), u(k))$ and $\hat{\eta}(k, x(k), u(k); \hat{\theta}(k))$ for all (x, u) in some domain $\chi \times U$ is minimized. The optimal value is used for mathematical analysis only. Using the smoothness assumption on $\hat{\eta}$, (5) could be written as

$$e(k+1) = Ae(k) + \frac{\partial \hat{\eta}(k, x, u; \hat{\theta})}{\partial \hat{\theta}} (\theta^* - \hat{\theta}) + w(x, u, \hat{\theta}, \theta^*) + v \qquad (6)$$

where

$$w(x, u, \hat{\theta}, \theta^*) = \hat{\eta}(k, x, u; \theta^*) - \hat{\eta}(k, x, u; \hat{\theta}) - \frac{\partial \hat{\eta}(k, x, u; \hat{\theta})}{\partial \hat{\theta}} (\theta^* - \hat{\theta})$$

intuitively represents the higher order terms of the Taylor series expansion. For approximator with linear in parameter (LIP) assumption w is identically zero. For nonlinearly parameterized approximator, it can be shown that $w \to 0$ as $\hat{\theta} \to \theta^*$. Equation (6) is further manipulated to get

$$e(k+1) = Ae(k) + Z^T \tilde{\theta} + \delta \qquad (7)$$

where $\delta = v + w(x, u, \hat{\theta}, \theta^*)$, $Z = \left[\frac{\partial \hat{\eta}(k, x, u, \hat{\theta})}{\partial \hat{\theta}} \right]^T$, and $\tilde{\theta} = \theta^* - \hat{\theta}$ is the parameter estimation error. Next the following parameter update law is proposed for tuning the OLAD

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \alpha Z \, D[e(k)] - \gamma \left\| I - \alpha Z Z^T \right\| \hat{\theta}(k) \qquad (8)$$

where $\alpha > 0$ is the learning rate or adaptation gain, $0 < \gamma < 1$ is a design parameter and Z is a $p_s \times n$ matrix and $D[.]$ is the deadzone operator defined next.

Remark 1: A fault is detected only if the norm of the residual $\|e(k)\|$ is greater than a predetermined threshold in order to improve the sensitivity of the detection scheme to the fault. The threshold or dead zone for the residual is selected based on equation (7) and it is given as

$$D[e(k)] = \begin{cases} 0, \text{ if } \|e(k)\| \le \varepsilon \\ e(k), \text{ if } \|e(k)\| > \varepsilon \end{cases}.$$

where ε is the fault detection threshold and is derived in the next section.

After the fault occurs, the system equation in (1) is modified as

$$x(k+1) = A_0 x(k) + \varphi(x(k), u(k)) + h(k, x(k), u(k))$$

where $h(k, x(k), u(k)) = \eta(k, x(k), u(k)) + g(x(k), u(k))$ with the size of the OLAD parameter vector in (2) as $\hat{\theta} \in \Re^l$ and may not necessary be prior to the fault. Now the state estimation error would be given as

$$e(k+1) = Ae(k) + h(k, x(k), u(k)) - \hat{\eta}(k, x(k), u(k); \hat{\theta}(k))$$

by performing mathematical manipulations similar to (5) and (6). The residual is obtained as

$$e(k+1) = Ae(k) + Z_f^T \tilde{\theta} + \omega \tag{9}$$

where $Z_f = \left[\dfrac{\partial \hat{\eta}(k, x(k), u(k); \hat{\theta}(k))}{\partial \hat{\theta}} \right]^T$ is a $l \times n$ matrix,

$, \omega = \delta + f(k, x, u, \hat{\theta}, \theta^*) + \upsilon_1,$

$$f(k, x, u, \hat{\theta}, \theta^*) = \hat{\eta}(k, x, u; \theta^*) - \hat{\eta}(k, x, u; \hat{\theta}) - \frac{\partial \hat{\eta}(k, x(k), u(k); \hat{\theta}(k))}{\partial \hat{\theta}} (\theta^* - \hat{\theta})$$

and $\upsilon_1 = h(k, x(k), u(k)) - \hat{\eta}(k, x(k), u(k); \theta^*(k))$ is the approximation error. Additionally, note that the update law in (8) will again be used for tuning the parameters of the modified OLAD. Since we modify the OLAD after the detection of a fault, thus we don't need to apply the deadzone operator in the update law in (8). But it is obvious that the size would be larger i.e. $\hat{\theta} \in \Re^l$. Hence as discussed earlier, the modified OLAD would learn both the system uncertainty and the fault dynamics after the detection of the fault.

The update law in (8) guarantees bounded stability of the OLAD scheme as would be shown in the next section. It is also very important to note that the update law given in (8) is different from controller design [8], whereas to the best of the knowledge this law is not used for estimation. Additionally, the update law complicates the stability proof for fault detection. Hence a guaranteed performance for controller design might not guarantee a stable performance for fault detection. Next, the performance of the fault detection scheme is examined mathematically.

IV. Analytical Results

Consider the system in (1) with uncertainties and prior to fault occurrence, the system is given by

$$x(k+1) = Ax(k) + \varphi(x(k), u(k)) + \eta(k, x(k), u(k))$$

The following theorem is proposed to show the stability of the OLAD scheme and the parameter update law in (8) prior to the occurrence of the fault. It is emphasized that the PE condition is relaxed in deriving the stability.

Theorem 1: (PE condition not required) let the initial conditions for the OLAD be bounded in a region $D \subset \Re^{p_s}$. Consider the parameter update law (8), the state estimation error $e(k)$ and the parameter estimation error $\tilde{\theta}(k)$ are uniformly ultimately bounded (UUB).

Proof: let the Lyapunov function be

$$V = e^T(k)e(k) + \tilde{\theta}^T(k)\tilde{\theta}(k)$$

The first difference of the Lyapunov function is given by

$$\Delta V = \underbrace{e^T(k+1)e(k+1) - e^T(k)e(k)}_{\Delta V_1} + \underbrace{\tilde{\theta}^T(k+1)\tilde{\theta}(k+1) - \tilde{\theta}^T(k)\tilde{\theta}(k)}_{\Delta V_2}$$

Next we substitute the error equation (7) in ΔV_1, and by taking $\Psi = Z^T \tilde{\theta}$, we get

$$\Delta V_1 = \left(Ae(k) + \Psi + \delta\right)^T . \left(Ae(k) + \Psi + \delta\right) - e^T(k)e(k)$$

Next we expand the above equation to get

$$\Delta V_1 = e^T(k)A^T Ae(k) + \Psi^T\Psi + 2e^T(k)A^T\delta + 2e^T(k)A^T\Psi + \delta^T\delta + 2\Psi^T\delta - e^T(k)e(k) \tag{10}$$

Next we substitute the parameter update law (8) in ΔV_2, we get

$$\Delta V_2 = \Big\{ \left((I - \gamma\|I - \alpha ZZ^T\| I)\tilde{\theta}(k) - (\alpha Z\, e(k) - \gamma\|I - \alpha ZZ^T\|\theta^*)\right)^T \times \left((I - \gamma\|I - \alpha ZZ^T\| I)\tilde{\theta}(k) - (\alpha Z\, e(k) - \gamma\|I - \alpha ZZ^T\|\theta^*)\right)^T - \tilde{\theta}^T(k)\tilde{\theta}(k) \Big\}$$

Next, we apply the results of Cauchy-Schwartz inequality ($(a_1 + a_2)^T(a_1 + a_2) \le 2(a_1^T a_1 + a_2^T a_2)$), we get

$$\Delta V_2 \le 2(1 - \gamma\|I - \alpha ZZ^T\|)^2 \tilde{\theta}^T(k)\tilde{\theta}(k) + 2(\alpha Z\, e(k) - \gamma\|I - \alpha ZZ^T\|\theta^*)^T . (\alpha Z\, e(k) - \gamma\|I - \alpha ZZ^T\|\theta^*) - \tilde{\theta}^T(k)\tilde{\theta}(k) \tag{11}$$

Next $\Delta V = \Delta V_1 + \Delta V_2$, combine (10) and (11), apply the Cauchy-Schwartz equality on the 3rd, 4th and 6th terms in equation (10) to get

$$\Delta V \le 3e^T(k)A^T Ae(k) + 3\Psi^T\Psi + 3\delta^T\delta - e^T(k)e(k) + 2(1 - \gamma\|I - \alpha ZZ^T\|)^2 \tilde{\theta}^T(k)\tilde{\theta}(k) + 2(\alpha Z\, e(k) - \gamma\|I - \alpha ZZ^T\|\theta^*)^T . (\alpha Z\, e(k) - \gamma\|I - \alpha ZZ^T\|\theta^*) - \tilde{\theta}^T(k)\tilde{\theta}(k)$$

Finally, we take the norm to get

$$\Delta V \le -(1 - 3A_{\max}^2 - 4\alpha^2 Z_{\max}^2)\|e(k)\|^2 - \left(1 - 2\|(1 - \gamma\|I - \alpha ZZ^T\|)\|^2 - 3Z_{\max}^2\right)\|\tilde{\theta}(k)\|^2 + \beta^2$$

where $\beta^2 = 3\|\omega\|^2 + 4\gamma^2 \|I - \alpha ZZ^T\|^2 \theta_{\max}^2$. Therefore, $\Delta V \le 0$ provided the following conditions hold

$$\|e(k)\| \ge \frac{\beta}{\sqrt{(1 - 3A_{\max}^2 - 4\alpha^2 Z_{\max}^2)}}$$

(or)

$$\|\tilde{\theta}(k)\| \ge \frac{\beta}{\sqrt{\left(1 - 2(1 - \gamma(1 - \alpha Z_{\max}^2))^2 - 3Z_{\max}^2\right)}}$$

and $\alpha\|Z\|^2 < 1/4$, $0 < \gamma < 1$, $\|Z\| \le Z_{\max}$, $A_{\max} \le 1/\sqrt{3}$. This completes the proof.

Remark 2: The principal advantage of such Lyapunov theory based approach is that the threshold (ε) is derived in the process of proving stability and thus it guarantees robust ness of the fault detection when compared to other schemes [4], where the threshold is chosen analytically or by assuming fixed bound on system uncertainties [2]. Hence we have chosen the threshold as

$$\varepsilon = \lambda \frac{\beta}{\sqrt{(1 - 3A_{\max}^2 - 4\alpha^2 Z_{\max}^2)}} \qquad (12)$$

where $\lambda > 0$ is the fault sensitivity design parameter. From Remark 1, we see that a fault is detected only if $\|e(k)\| > \varepsilon$.

The following theorem shows the boundness of the state estimation error (residual) and the parameters of the OLAD, after the occurrence of the fault, while relaxing the PE condition.

Theorem 2 (PE condition not required) let the initial values of the augmented parameters of the OLAD be bounded in a region $B \subset \Re^l$. In the presence of bounded uncertainties as shown in Theorem 1, consider the following parameter update law (8) for the augmented parameter, then the residual and the parameter estimation errors, $e(k)$ and $\tilde{\theta}(k)$ respectively are uniformly ultimately bounded (UUB).

Proof: One could always see that the proof of this theorem is very similar to Theorem 1. However the parameter vector size would now be $\hat{\theta} \in \Re^l$.

Remark 3: For stability, the adaptation gain has to satisfy $\alpha\|Z\|^2 < 1/4$ for Theorems 1 and 2 which is different for control. From an initial glance, these conditions appear to be strong whereas most commonly used parameterized online approximators satisfy the above conditions [2, 8].

The next section elaborates on the development of the prognostics scheme.

V. Prognostic Scheme

The prognostics scheme is developed by using the behavior of the parameter trajectories before and after the occurrence of the fault. The following assumption holds in deriving the time to failure.

Assumption 2: The parameter $\hat{\theta}(k)$ is an estimate of the actual system parameter.

Remark 4: This assumption is satisfied when a system can be expressed as linear in the unknown parameters (LIP). For example in a mass damper system or civil infrastructure such as a bridge, the mass, damping and spring constants can be expressed as unknown parameters. Hence in the event of a fault, we assume that system parameters change and tend to reach their limits defined by the designer. When any one of the parameters exceeds its limit, it is considered unsafe to operate. TTF will be defined as the time that the first parameter reaches its limit. Here the TTF analysis can be done with lower limits as well.

In order to estimate the system parameter in real-time, we use the parameter update law given in (8) and solve it iteratively by fixing all other quantities and projecting the TTF under the current state estimation error. Next we present the following theorem.

Theorem 3 (Time to failure): Assume that the parameter update law can be treated time invariant during the time interval k and $k+1$ and consider system (1) can be expressed as LIP, the TTF for the i^{th} system parameter could be iteratively determined by solving

$$k_{f_i} = \frac{\left|\log\left(\left|\frac{\left(\gamma\|I - \alpha zz^T\|\theta_{i_{\max}} - \alpha\sum_{j=1}^{n} z_{ij}e_j\right)}{\left(\gamma\|I - \alpha zz^T\|\theta_{i_0} - \alpha\sum_{j=1}^{n} z_{ij}e_j\right)}\right|\right)\right|}{\left|\log(1 - \gamma\|I - \alpha zz^T\|)\right|} + k_{0_i} \qquad (13)$$

where k_{f_i} is the TTF, k_{0_i} is the time instant when the prediction starts (starts at k_d and incremented with time), $\theta_{i_{\max}}$ is the maximum value of the system parameter, and θ_{i_0} is the value of the system parameter at the time instant k_{0_i}.

Remark 4: The mathematical equation (13) is derived for the i'$^{\text{th}}$ system parameter. In general for a given system, the time to failure would be $k_{ft} = \min(k_{f_i}), i = 1, 2, ...l$, where l the number of system parameters. This also implies that for a fault that is occurring in the system, the TTF is obtained as the time that the first parameter reaches its limit.

Proof: In general for any system satisfying *Assumption 2*, the maximum value of the system parameter in the event of a fault is determined via physical limitation. Hence we take $\hat{\theta}_i(k_{f_i}) = \theta_{i_{\max}}$. Note that the equation (13) holds only in the time interval $k \in [k_d, k_f]$ when the state estimation error and other terms are held constant at each k. Thus the values of Z and e are known and would be held fixed for the k^{th} time instant. Under the assumption, the parameter update law shown in (8) could be written as

$$\hat{\theta}(m+1) = (I - \gamma\|I - \alpha ZZ^T\| I)\hat{\theta}(m) + \alpha Z\ e$$

where we use m as the time index to simplify the understanding of the theorem, and the above defined equation could be written as

$$\overline{x}(m+1) = \overline{A}.\overline{x}(m) + \overline{B}.\overline{u} \tag{14}$$

where $\overline{x}(m+1) = \hat{\theta}(m+1)$, $\overline{A} = (I - \gamma\left\|I - \alpha ZZ^T\right\| I)$ is a diagonal matrix, $\overline{x}(m) = \hat{\theta}(m)$, and $\overline{B} = \alpha$, and $\overline{u} = Z\,e$. Since the above defined $\overline{A}$ matrix is diagonal, (14) could be written as

$$\overline{x}_i(m+1) = \overline{a}_{ii}\overline{x}_i(m) + \overline{b}_i\overline{u}_i \tag{15}$$

where $\overline{a}_{ii} = 1 - \gamma\left\|I - \alpha ZZ^T\right\|$, $\overline{b}_i = \alpha$, and $\overline{u}_i = \sum_{j=1}^{n} z_{ij}e_j$ with the elements of input being constant between the time instant k and $k+1$.

Solving (15) to determine TTF using [10], we get

$$\overline{x}_i(m) = \overline{a}_{ii}^{(m-m_0)}\overline{x}_i(m_0) + \sum_{j=m_0+1}^{m} \overline{b}_i\overline{a}_{ii}^{(m-j)}\overline{u}_i \tag{16}$$

Since at a given instance k, u_i is time-invariant in (16), thus we get

$$\overline{x}_i(m) = \overline{a}_{ii}^{(m-m_0)}\overline{x}_i(m_0) + \overline{b}_i\overline{u}_i \sum_{j=m_0+1}^{m} \overline{a}_{ii}^{(m-m_0)}$$

Now using results of geometric series, the above equation become

$$\overline{x}_i(m) = \overline{a}_{ii}^{(m-m_0)}\overline{x}_i(m_0) + \overline{b}_i\overline{u}_i\left(\frac{1-\overline{a}_{ii}^{\,m-m_0}}{1-\overline{a}_{ii}}\right)$$

After performing some simple mathematical manipulation, one obtains

$$\overline{a}_{ii}^{\,m-m_0} = \frac{[\overline{x}_i(m)(1-\overline{a}_{ii}) - \overline{b}_i\overline{u}_i]}{[\overline{x}_i(m_0)(1-\overline{a}_{ii}) - \overline{b}_i\overline{u}_i]}$$

Since $0 < \overline{a}_{ii} < 1$, take absolute value and logarithm on both sides and apply absolute value to get

$$m = \frac{\left|\log\left(\left|\frac{\overline{x}_i(m)(1-\overline{a}_{ii}) - \overline{b}_i\overline{u}_i}{\overline{x}_i(m_0)(1-\overline{a}_{ii}) - \overline{b}_i\overline{u}_i}\right|\right)\right|}{\left|\log(\overline{a}_{ii})\right|} + m_0$$

Next we take $m = k_{f_i}$, and $m_0 = k_{0_i}$. Additionally, we have $\overline{x}_i(m) = \overline{x}_i(k_{f_i}) = \theta_{i_{\max}}$, $\overline{x}_i(m_0) = \overline{x}_i(k_{0_i}) = \theta_{i_0}$, and we know that $\overline{a}_{ii} = 1 - \gamma\left\|I - \alpha ZZ^T\right\|$, $\overline{b}_i = \alpha$, and $\overline{u}_i = \sum_{j=1}^{n} z_{ij}e_j$. Thus, we get equation (13). This completes the proof.

Remark 5: In other words, TTF is obtained by treating the estimation error and system matrix to be time invariant which may not necessarily true in practice whereas this is a first step. This assumption will be relaxed in the future. Similarly, one can use the state estimator to determine TTF by using the same idea provided states can be related to the physical quantities.

To iteratively determine the time to failure (k_{ft}), we propose the following algorithm.

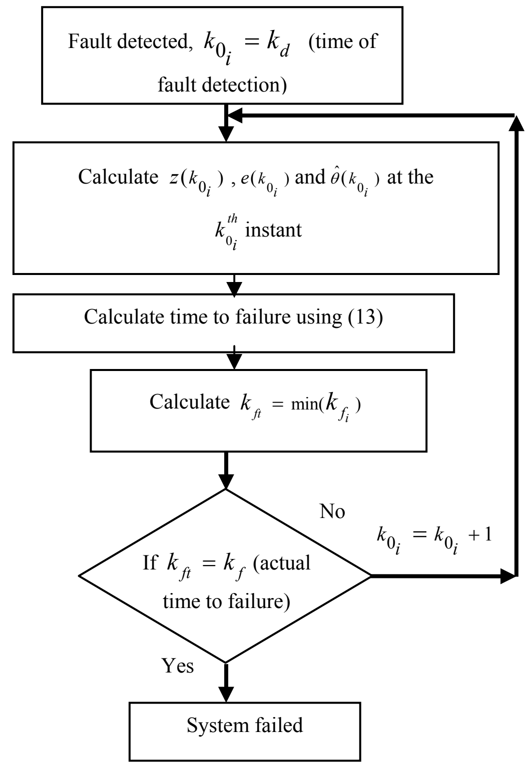

Figure 1: Procedure to iteratively update the time to failure.

The next section details the simulation results.

VI. Simulation Results

The modified discrete time state space model of a fourth order translational oscillator with rotating actuator (TORA) system is given by [9]

$$x(k+1) = A_0x(k) + \varphi(x(k),u(k)) + \eta(k,x(k),u(k)) + g(x(k),u(k)) \tag{17}$$

where the states are given by $x(k) = \left[x_1(k), x_2(k), x_3(k), x_4(k)\right]^T$, $A_0 = I$, the nominal dynamics of the system are given by

$$\varphi = \begin{vmatrix} t_s\left(x_2(k)\right) \\ t_s\left(\frac{1}{d}\right).\left((m+M)u - mL\cos(x_1(k)).(mL.x_2^2(k)\sin(x_1(k)) - k_sx_3(k))\right) \\ t_s\left(x_4(k)\right) \\ t_s\left(\frac{1}{d}\right)\left(-mL\cos(x_1(k)) + (I+mL^2).(mL.x_2^2(k)\sin(x_1(k)) - k_sx_3(k))\right) \end{vmatrix}$$

with $d = (I + mL^2)(m+M) - m^2L^2\cos^2(x_1(k))$, and input $u(k) = 0.5 * \sin(kt_s)$. The system uncertainty is given by $\eta(k,x(k),u(k)) = [0,0,0,0.2\sin(0.01x_3(k))]^T$, and the induced fault is taken as $g(x(k),u(k)) = \left[0,0,0,(0.1\left(1 - e^{-K_4(k-T)}\right)(x_2^3(k)))\right]^T$.

Now to monitor the system in (17) and to detect the fault that occurs in the system, we propose the following nonlinear estimator, which is obtained from (2). The nonlinear estimator uses an OLAD scheme given by

$$\hat{x}(k+1) = A_o\hat{x}(k) + K(x(k) - \hat{x}(k)) + \varphi(x(k), u(k)) + \hat{\eta}(k, x(k), u(k); \hat{\theta}(k)) \quad (18)$$

where the estimated states are $\hat{x}(k) = \left[\hat{x}_1(k), \hat{x}_2(k), \hat{x}_3(k), \hat{x}_4(k)\right]^T$, $K = 0.25I$ with I is an identity matrix, $\hat{\eta}(k, x(k), u(k); \hat{\theta}(k)) = \left[0, 0, 0, \hat{\eta}_4(x(k); \hat{\theta}(k))\right]^T$ is OLAD. Initially we assume that only the system uncertainty is present in the system in (17). Hence, the OLAD $\hat{\eta}_4(x(k); \hat{\theta}(k))$ is used for approximating the system uncertainty. In this simulation, we assume that the system uncertainty and the fault affects the system state $x_4(k)$, and the values of the system parameter used for this simulation are $m = 2, M = 10, k_s = 1, L = 0.5, I = 2$, and $t_s = 0.01$.

Under fault free condition, the system in (17) is subjected only to the uncertainty as given above. Initially the OLAD used is a feed forward network with 6 sigmoid activation functions. The OLAD is tuned using the update law in (8). The tuning parameters of the update law in (8) are chosen as $\alpha = 0.01$ and $\gamma = 0.01$. The initial value of the parameters ($\hat{\theta}$) of the OLAD are chosen to be zero. Next we assume that a fault is seeded at $T = 25$ sec and its growth rate is taken as $\kappa_4 = 0.1$. Upon detecting the fault, we modify the size of the OLAD, thus we have a feed forward network with 16 sigmoid activation functions. To tune the OLAD parameters online, we use again the update law in (8), and the tuning parameters are chosen as $\alpha = 0.01$ and $\gamma = 0.1$.

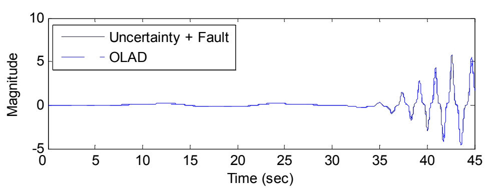

Figure 2: Combined dynamics (uncertainty + fault) response compared with the OLAD response.

Figure 2 shows the evolution of the system uncertainty and the unknown fault, and is compared with the OLAD response. From the figure, the learning of the combined dynamics (system uncertainty and unknown fault) by the OLAD is satisfactory. Thus the proposed fault detection scheme is able to detect an unknown fault in the presence of system uncertainty. Additionally, the scheme is also able to learn the combined dynamics.

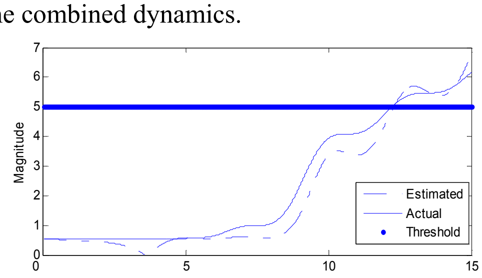

Figure 3: Comparison between the estimated and the actual system parameter, and also shown the safe threshold.

Since the system in (17) cannot satisfy the LIP, the prognostics scheme cannot be directly applied. However, we use another example, a mass damper system (system details omitted due to page constraints, however please refer to [5] for more details). Next we use the algorithm proposed in Theorem 3 to predict the TTF of a mass damper system with a spring stiffness fault. The increasing spring constant due to the fault is shown in Fig. 3, where we use an upper threshold $\theta_{i_{\max}}$ to determine the TTF after the occurrence of the fault. Fig. 4 shows the TTF upon detection of the fault to the actual time of failure. Here TTF reaches zero to indicate that the upper threshold was reached.

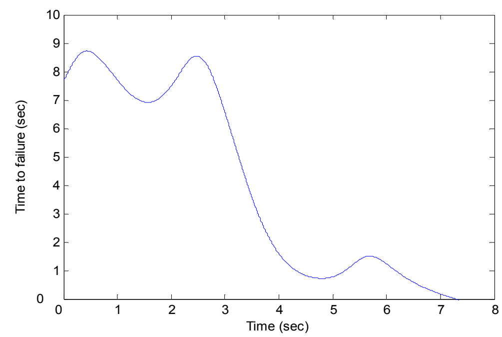

Figure 4: Prediction of the time to failure after the occurrence of the fault.

VI. CONCLUSIONS

In this paper, the development of the fault detection and the prognostics scheme was combined. The scheme is developed by considering a bound on system uncertainty. Additionally the time to failure was also derived. However the scheme was developed with an assumption that all the states are measurable. As part of future work, the requirement of all state measurability will be relaxed.

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