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Equivalence to Dissipative Hamiltonian Realization

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Abstract— **This paper considers the problem of deriving a generalized Hamiltonian potential for autonomous dynamical systems. For a given vector field, the objective is to construct a locally defined dissipative Hamiltonian generating function for the system. The proposed approach consists of studying the deviation of the given vector field from a canonically defined Hamiltonian vector field. First, we obtain a one-form by taking the interior product of a nonvanishing two-form with respect to the vector field. We then construct a homotopy operator on a star-shaped region that decomposes the system into an exact part and an anti-exact one. Equivalence between the exact part and an exact one-form generated from a known potential is then used to compute the locally defined dissipative potential of the original system. An example is presented to illustrate the method.**

I. INTRODUCTION

Generalized Hamiltonian systems are an important tool for stability studies and controller design for nonlinear control systems [16]. In recent years, several physical problems were studied using generalized Hamiltonian (or pseudo-Hamiltonian) systems [3]. For example, an admissible nonequilibrium thermodynamical representations was obtained by lifting a Hamiltonian dissipative structure to the so-called thermodynamic phase space [5]. Another example was given recently in [15], where the stability of a closed reaction network was studied using its pseudo-Hamiltonian realization. One of the main drawbacks associated to the study of nonlinear systems using Hamiltonian representation is to derive a suitable Hamiltonian generating function for the problem, in particular when studying non-mechanical problems such as RLC circuits [10].

The general problem of deriving a generalized Hamiltonian realization for a known system was considered from a feedback equivalence point of view in [17] for control-affine systems (see also [4] for feedback equivalence to port-controlled Hamiltonian systems). In [2], conditions for approximate Hamiltonian realizations were given in terms of a normal form. Sufficient conditions and a constructive algorithm for a generalized Hamiltonian realization for time-invariant nonlinear systems were presented in [19]. In particular, the method proposed in [19], reviewed briefly in Section II-A, seeks to decompose the vector field along the gradient direction $\nabla H(x)$ and the tangential direction of the equivalue surfaces of $H(x)$, for a regular positive-definite function $H(x)$. Extensions to port-controlled time-varying systems were carried out in [8] using an error dynamic system and in [20] using Poisson structures. The relationship between the concepts of Lyapunov stability and Hamiltonian with dissipation was discussed in [13] using Morse theory and in [14] using Poisson structures. Recently, following the work in [11] and [12] which related port-controlled Hamiltonian systems to the construction of Lyapunov functions, it was shown in [21] how k-th degree approximate dissipative Hamiltonian systems can be used to solve the realization problem and how associated k-th degree approximate Lyapunov functions can be used to study the stability of such systems.

In this paper, we propose to study the local equivalence problem between a known autonomous vector field and a pre-defined Hamiltonian dissipative realization, viewed as a reference system. The method seeks to develop a local change of coordinates resulting in a local dissipative potential for the system. First, we obtain a one-form (which is possibly non-closed) by taking the interior product of a nondegenerate two-form with respect to the given vector field. Then, a homotopy operator is constructed on a star-shaped domain to decompose the (possibly) non-closed one-form into its exact and anti-exact parts. Following [6], the exact part is used to derive a dissipative potential, while the anti-exact part is associated with a nondissipative potential that does not contribute to the dissipative potential on the star-shaped region. To compute the dissipative potential, we study the equivalence problem [9] between the closed one-form and a reference closed one-form derived from a known dissipative Hamiltonian realization. The locally defined dissipative potential for the original system is then expressed in coordinates.

The paper is organized as follows. Section II provides mathematical background for the problem considered, including definitions of dissipative systems and background on homotopy operators. The main developments of the paper are presented in Section III. An application to a simple dissipative dynamical system is considered in Section IV to illustrate the method. Conclusions and future areas of investigation are outlined in Section V.

II. MATHEMATICAL BACKGROUND

A. Generalized Hamiltonian Systems

In this section, the problem of dissipative Hamiltonian realization for the time-invariant case is summarized following [2] and [19]. First, we give the definition of a generalized Hamiltonian realization.

Definition 2.1: A dynamical system

$$\dot{x} = f(x) \tag{1}$$

with $x \in \mathbb{R}^n$ is said to have a generalized Hamiltonian realization if there exists a suitable coordinate chart on $\mathbb{R}^n$

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and a Hamiltonian function $H : \mathbb{R}^n \to \mathbb{R}$ such that (1) can be expressed as

$$\dot{x} = T(x)\nabla H, \tag{2}$$

where $T(x)$ is an $n \times n$ matrix called the structure matrix and $\nabla H = (\frac{\partial H}{\partial x})^T$. If the structure matrix can be expressed as $T(x) = J(x) - R(x)$ with $J(x)$ skew-symmetric, $i.e.$ $J(x) = -J^T(x)$, and $R(x)$, a symmetric positive semi-definite, then (2) is called a dissipative Hamiltonian realization. Furthermore, if $R(x) > 0, \forall x$, (2) is called a strict dissipative realization.

As mentioned in the introduction, the procedure proposed in this paper is related to the orthogonal decomposition used in [19]. Their key observation is based on the following theorem.

Theorem 2.2: [19] Consider the system (1) with $f(0) = 0$. Then if there exists a function $H(x)$ ($\nabla H(x) \neq 0$) such that $L_f H(x) = 0$, $\forall x \in \mathbb{R}^n$, then the system can be expressed as

$$\dot{x} = J(x)\nabla H \tag{3}$$

where

$$J(x) = \frac{1}{\|\nabla H\|^2}\left(f(\nabla H)^T - \nabla H(f)^T\right) \tag{4}$$

is an $n \times n$ skew-symmetric matrix.

For completeness of this review, we report the following definition from [19].

Definition 2.3: A function $H(x)$ is said to be a regular positive function if (i) $H(x) > 0$ for $x \neq 0$, (ii) $H(0) = 0$, (iii) $\frac{\partial H}{\partial x}|_{x=0} = 0$ and (iv) $\frac{\partial H}{\partial x}|_{x\neq 0} \neq 0$.

For a regular positive definite function $H(x)$, it can be shown that at any point $x \neq 0$, the autonomous vector field $f(x)$ can be decomposed as

$$f(x) = f_{\text{gd}}(x) + f_{\text{td}}(x) \tag{5}$$

where the gradient direction is given by

$$f_{\text{gd}}(x) = \frac{\langle f, \nabla H\rangle}{\|\nabla H\|^2}\nabla H, \tag{6}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product in $\mathbb{R}^n$. The tangential direction is then defined as

$$f_{\text{td}}(x) = f(x) - f_{\text{gd}}(x) \tag{7}$$

which is $J(x)\nabla H$ from the preceding theorem. By setting $S(x) = \frac{\langle f, \nabla H\rangle}{\|\nabla H\|^2} I_{n\times n}$, the orthogonal decomposition Hamiltonian realization was given in [19] as

$$\dot{x} = (J(x) + S(x))\nabla H. \tag{8}$$

In [3], it was shown that such function $H(x)$ exists under mild assumptions. Different approaches were reported in [19] to construct $H(x)$, for example

$$H(x) = \frac{1}{2}\sum_{i=1}^{n} f_i^2(x) \tag{9}$$

in the case where the Jacobian in nonsingular.

In the present paper, we develop an approach that parallels this development using differential forms. Our approach uses a target dissipative realization to built the Hamiltonian function for the system under study.

B. Exterior Calculus

In this section, basic elements of exterior calculus on $\mathbb{R}^n$ are introduced. A complete account of exterior calculus can be found in [6]. Let the tangent space at a point $x \in \mathbb{R}^n$ be denoted by $T_x\mathbb{R}^n$. Since the tangent space $T_x\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$, it has a natural vector space structure. We denote a smooth vector field $X \in \Gamma^\infty(\mathbb{R}^n)$ as a smooth map

$$X : \mathbb{R}^n \to T\mathbb{R}^n, \qquad X|_x = \sum_{i=1}^{n} v^i(x)\partial_{x_i}|_x, \tag{10}$$

i.e. a map taking a point $x \in \mathbb{R}^n$ and assigning a tangent vector $X|_x \in T_x\mathbb{R}^n$. The cotangent (dual) space $T_x^*\mathbb{R}^n$ is the set of all linear functionals on $T_x\mathbb{R}^n$,

$$T_x^*\mathbb{R}^n = \{\omega|_x : T_x\mathbb{R}^n \to \mathbb{R}\} \tag{11}$$

where each $\omega|_x$ is linear, *i.e.*

$$(a\omega_1|_x + b\omega_2|_x)(X_x) = a\omega_1|_x(X|_x) + b\omega_2|_x(X|_x). \tag{12}$$

The standard basis of $T_x^*\mathbb{R}^n$ is given by $\{dx_1, \ldots, dx_n\}$, where $dx_i(\partial_{x_j}) = \delta_j^i$, δ_j^i being the Kronecker delta. An element $\omega|_x$ in the cotangent space $T_x^*\mathbb{R}^n$ can be written as

$$\omega|_x = \sum_{i=1}^{n} \omega_i dx_i, \quad \omega_i \in \mathbb{R}. \tag{13}$$

In the sequel, differential one-forms will be used. They are generated the following way. A differential one-form ω on $\mathbb{R}^n$ is a smooth map taking a point $x \in \mathbb{R}^n$ and assigning an element of its dual space $T_x^*\mathbb{R}^n$. We write

$$\omega = \sum_{i=1}^{n} \omega_i(x)dx_i, \tag{14}$$

where ω_i are smooth functions on $\mathbb{R}^n$. The exterior (wedge) product $\wedge$ is defined on $\Omega^1(\mathbb{R}^n) \times \Omega^1(\mathbb{R}^n)$ by the requirements

$$\begin{aligned} dx_i \wedge dx_j &= -dx_j \wedge dx_i \\ dx_i \wedge f(x)dx_j &= f(x)dx_i \wedge dx_j \end{aligned}$$

for all smooth functions $f(x)$ and

$$\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma, \tag{15}$$

for all $\alpha, \beta, \gamma \in T^*\mathbb{R}^n$. If $\alpha \in \Lambda^k(\mathbb{R}^n)$, then we write $\deg \alpha = k$. Notice that $\Lambda^1(\mathbb{R}^n) = T^*\mathbb{R}^n$ and $\Lambda^0(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n)$.

The differential operator d is the unique operator on $\Lambda(\mathbb{R}^n) = \bigoplus_{k=0}^{n} \Lambda^k(\mathbb{R}^n)$ with the following properties:

$$d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n), \quad 0 \leq k \leq n-1, \tag{16}$$

1. $d(\alpha + \beta) = d\alpha + d\beta$.

2. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$.
3. $df = (\partial f_{x_i}) dx_i, \ \forall f(x) \in \Lambda^0(\mathbb{R}^n)$.
4. $d \circ d\alpha = 0$.

A k-form α is said to be closed if $d\alpha = 0$. It is said to be exact if there exists a $(k-1)$-form β such that $d\beta = \alpha$.
The interior product $\lrcorner$ is a map

$$\lrcorner : \Gamma^\infty \mathbb{R}^n \times \Lambda^k(\mathbb{R}^n) \to \Lambda^{k-1}(\mathbb{R}^n) \tag{17}$$

with the following properties $\forall X \in \Gamma^\infty \mathbb{R}^n$ and $\forall f \in \Lambda^0(\mathbb{R}^n)$:

1. $X \lrcorner f = 0$.
2. $X \lrcorner \omega = \omega(X), \forall \omega \in \Lambda^1(\mathbb{R}^n)$.
3. $X \lrcorner (\alpha + \beta) = X \lrcorner \alpha + X \lrcorner \beta, \forall \alpha, \beta \in \Lambda^k(\mathbb{R}^n), \ k = 1, \ldots, n$.
4. $X \lrcorner (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge (X \lrcorner \beta), \forall \alpha, \beta \in \Lambda(\mathbb{R}^n)$.

C. Homotopy Operator

In this section, we show how to construct a homotopy operator $\mathbb{H}$, *i.e.*, a linear operator on elements of $\Lambda(\mathbb{R}^n)$ that satisfies the identity

$$\omega = d(\mathbb{H}\omega) + \mathbb{H} d\omega, \tag{18}$$

for a form $\omega \in \Lambda(\mathbb{R}^n)$.
The first step in the construction of a homotopy operator is to define a star-shaped domain on $\mathbb{R}^n$ following [6] (see also [1] for an application to approximate feedback linearization). An open subset S of $\mathbb{R}^n$ is said to be star-shaped with respect to a point $p^0 = (x_1^0, \ldots, x_n^0) \in S$ if the following conditions hold:

- S is contained in a coordinate neighborhood U of p^0.
- The coordinate functions of U assign coordinates $(x_1^0, \ldots, x_n^0)$ to p^0.
- If p is any point in S with coordinates $(x_1, \ldots, x_n)$ assigned by functions of U, then the set of points $(x + \lambda(x - x^0))$ belongs to S, $\forall \lambda \in [0, 1]$.

A star-shaped region S has a natural associated vector field $\mathfrak{X}$, defined by

$$\mathfrak{X}(x) = (x_i - x_i^0) \partial_{x_i}, \quad \forall x \in S. \tag{19}$$

Remark 2.4: In this paper, we will consider the case where the star-shaped domain is centered at the origin for simplicity. For systems where a dissipative realization exists but is not centered at the origin (*e.g.*, see the example considered in [15]), we set the center at $p^0 : (x_1^0, \ldots, x_n^0)$, hence

$$\mathfrak{X}(x) = (x_i - x_i^0) \partial_{x_i}. \tag{20}$$

For a differential form ω of degree k on a star-shaped region S centered at the origin, the homotopy operator will be defined, in coordinates, as

$$(\mathbb{H}\omega)(x) = \int_0^1 \mathfrak{X}(x) \lrcorner \omega(\lambda x) \lambda^{k-1} d\lambda, \tag{21}$$

where $\omega(\lambda x)$ denotes the differential form evaluated on the star-shaped domain in the local coordinates defined above.
The important properties of the homotopy operator that will be used here (see [6] for proofs) are the following:

H1. $\mathbb{H}$ maps $\Lambda^k(S)$ into $\Lambda^{k-1}(S)$ for $k \geq 1$ and maps $\Lambda^0(S)$ identically to zero.
H2. $d\mathbb{H} + \mathbb{H}d =$ identity for $k \geq 1$ and $(\mathbb{H}df)(x) = f(x) - f(x_0)$ for $k = 0$.
H3. $(\mathbb{H}\mathbb{H}\omega)(x_i) = 0, \quad (\mathbb{H}\omega)(x_i^0) = 0$.
H4. $\mathfrak{X} \lrcorner \mathbb{H} = 0, \quad \mathbb{H}\mathfrak{X} \lrcorner = 0$.

The first part of the right hand side of (18), $d(\mathbb{H}\omega)$, is obviously a closed form, since $d \circ d(\mathbb{H}\omega) = 0$. Since by property (H1), for $\omega \in \Lambda^k(S)$, we have $(\mathbb{H}\omega) \in \Lambda^{k-1}(S)$, $d(\mathbb{H}\omega)$ is also exact on S.
We denote the exact part of ω by $\omega_e = d(\mathbb{H}\omega)$ and the anti-exact part by $\omega_a = \mathbb{H} d\omega$. It is possible to show that ω vanishes on $\mathbb{R}^n$ if and only if ω_e and ω_a vanish together [6]. From the decomposition outlined above, we have

$$\omega - \omega_a = \omega_e. \tag{22}$$

Taking the exterior derivative on both sides and using the fact that ω_e is closed, we have

$$d(\omega - \omega_a) = d(\omega_e) = 0. \tag{23}$$

In the sequel, we will apply the homotopy operator on one-forms. Since in our applications, ω_e is an exact one-form, $(\mathbb{H}\omega)$ computed by homotopy is a dissipative potential. A nondissipative potential is associated with the anti-exact part, but on the star-shaped domain S, ω_a does not contribute to the dissipative part of the system. In other words, ω_a belongs to the kernel of $\mathbb{H}$, which can be seen by applying property (H3) from above to the definition of ω_a.
In the following, we will construct a diffeomorphism that preserves the exact one-form to derive the dissipative potential. Stability analysis for the system will be carried on using only the one-form ω_e (see [7] for a complete discussion on Lyapunov one-forms).

III. COMPUTATION OF A DISSIPATIVE POTENTIAL

We now present the main construction of this paper, namely using the homotopy operator to discriminate the exact and the anti-exact parts associated to a given autonomous system and then computing a diffeomorphism between the exact part and a normal form of a dissipative Hamiltonian structure to compute a dissipative potential.

A. Homotopy Operator

Let the vector field $X|_x = \sum_{i=1}^n f_i(x) \partial_{x_i}$ be known, $i = 1, \ldots, n$. We assume that X is of class $\mathcal{C}^k$ with $k \geq 2$. It is also assumed that X has an equilibrium point at the origin. First, we define a nonvanishing closed two-form $\Omega = \sum_{1 \leq i < j \leq n} dx_i \wedge dx_j$ on $\mathbb{R}^n$. For example, if $n = 3$, we have,

$$\Omega = dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + dx_2 \wedge dx_3. \tag{24}$$

Remark 3.1: In the present paper, the nonvanishing two-form Ω is not necessarily defined in a canonical way, since the objective is ultimately to compute a dissipative potential (and not a minimal one). A method was presented in [18] to construct sets of $n-1$ nongenerate closed two-forms for n

even. To the best of the authors knowledge, that procedure was not extended for n odd.

Taking the interior product of Ω with respect to the vector field X, we compute a one-form ω as follows

$$\begin{aligned} \omega &= X \lrcorner \Omega && (25)\\ &= \sum_{1 \le i < j \le n} \left(f_i dx_j - f_j dx_i \right). && (26) \end{aligned}$$

Given a star-shaped region centered at the origin, with associated vector field $\mathfrak{X}(x) = x_i \partial_{x_i}$, we have

$$(\mathbb{H}\omega)(x) = \int_0^1 (\mathfrak{X} \lrcorner \omega(\lambda x))\, d\lambda. \qquad (27)$$

Letting $\tilde{f}_i$ denote the values of the components of f after integration with respect to λ, we have

$$(\mathbb{H}\omega)(x) = \sum_{1 \le i < j \le n} \left(\tilde{f}_i \cdot x_j - \tilde{f}_j \cdot x_i \right) := \tilde{F}(x). \qquad (28)$$

Taking the exterior derivative, we have

$$\omega_e = \sum_{i=1}^{n} \frac{\partial \tilde{F}}{\partial x_i} dx_i. \qquad (29)$$

The anti-exact form is then given by

$$\begin{aligned} \omega_a &= \omega - \omega_e \\ &= \sum_{1 \le i < j \le n} \left(f_i - \frac{\partial \tilde{F}}{\partial x_j} \right) dx_j - \left(f_j + \frac{\partial \tilde{F}}{\partial x_i} \right) dx_i. && (30) \end{aligned}$$

Remark 3.2: As a special case, if one defines Ω to be the canonical symplectic two-form and if X_H is the vector field generated by a known Hamiltonian H, ω obtained by the interior product $X_H \lrcorner \Omega$ is closed, and we show that $\omega = \omega_e = -dH$, where the Hamiltonian function H is the potential (see [7]).

Explicit expressions for $(\mathbb{H}\omega)$ and ω_e depend on the particular application. Next, we look at the equivalence problem between ω_e obtained in this section and a closed one-form derived from a simple Hamiltonian dissipative realization.

B. Equivalence of Closed One-Forms

The objective is to compute a change of coordinates to express the exact one-form ω_e obtained above and the dissipative function. To set the problem, we first develop a normal closed one-form for a simple dissipative Hamiltonian realization of the same dimension than the original problem following the development outline above. Consider the system

$$\dot{z} = (J(z) - R(z)) \nabla H(z) \qquad (31)$$

with $z \in \mathbb{R}^n$ and H, the Hamiltonian. Following the argument given in [3], it can be shown that for $f(0) = 0$, the Hamiltonian function

$$H(z) = \frac{1}{2} \sum_{i=1}^{n} z_i^2 \qquad (32)$$

is a suitable locally defined dissipative potential for the system. Assuming for simplicity that n is even, the simplest form for $J = -J^T$ in suitable dimensions is

$$J = \begin{pmatrix} 0 & -I_{\frac{n}{2} \times \frac{n}{2}} \\ I_{\frac{n}{2} \times \frac{n}{2}} & 0 \end{pmatrix} \qquad (33)$$

where I denotes the identity matrix. in the case where n is odd, J can be complemented with an extra column and an extra row of 0. We let $R = I_{n \times n}$. In that particular case, it can be shown that $(\mathbb{H}\omega)(z) = -H(z)$ and that the closed one-form is given as

$$\bar{\omega}_e = \sum_{i=1}^{n} -z_i dz_i. \qquad (34)$$

It can be observed that the anti-exact part for the problem encodes the same information than the tangential component from [19], as the anti-exact part is

$$\bar{\omega}_a = (J \nabla H) dz. \qquad (35)$$

The problem of equivalence between the two systems, as far as the dissipation component is concerned, consists in finding conditions under which there exists a diffeomorphism $z = \Phi(x)$ preserving the exact form, $i.e.$ a diffeomorphism between the reference exact one-form $\bar{\omega}_e$ (34) and the exact one-form for the system of interest ω_e given in (29). Following [9], the equivalence problem is posed as follows:

$$\begin{aligned} \Phi^*(\bar{\omega}_e) &= \omega_e && (36)\\ \Phi^*(-z^T dz) &= (\frac{\partial \tilde{F}}{\partial x}^T dx). && (37) \end{aligned}$$

Following the procedure exposed in [9], the first step is to complete the coframes (where the coframes are the bases for the cotangent spaces, in this case (dz_i) and (dx_i)). To illustrate, we consider the two-dimensional case and complete the coframes such that the determinant of the matrices are different from 0 (except at the origin). The two dimensional problem of equivalence is

$$\Phi^* \begin{pmatrix} -z_1 & -z_2 \\ -z_2 & z_1 \end{pmatrix} \begin{pmatrix} dz_1 \\ dz_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{F}}{\partial x_1} & \frac{\partial \tilde{F}}{\partial x_2} \\ \frac{-\partial \tilde{F}}{\partial x_2} & \frac{\partial \tilde{F}}{\partial x_1} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix}. \qquad (38)$$

Let

$$\begin{aligned} \Theta &= \begin{pmatrix} -z_1 & -z_2 \\ -z_2 & z_1 \end{pmatrix} \begin{pmatrix} dz_1 \\ dz_2 \end{pmatrix} && (39)\\ \theta &= \begin{pmatrix} \frac{\partial \tilde{F}}{\partial x_1} & \frac{\partial \tilde{F}}{\partial x_2} \\ \frac{-\partial \tilde{F}}{\partial x_2} & \frac{\partial \tilde{F}}{\partial x_1} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} && (40) \end{aligned}$$

and take the exterior derivative on both coframes, we obtain:

$$\begin{aligned} d\Theta &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} dz_1 \wedge dz_2 && (41)\\ d\theta &= \begin{pmatrix} 0 \\ \frac{\partial^2 \tilde{F}}{\partial x_1^2} + \frac{\partial^2 \tilde{F}}{\partial x_2^2}. \end{pmatrix} dx_1 \wedge dx_2. && (42) \end{aligned}$$

The result for the first line of $d\theta$ follows for the equality of mixed partials, $i.e.$ since ω_e is closed, we have

$$\frac{\partial^2 \tilde{F}}{\partial x_1 \partial x_2} = \frac{\partial^2 \tilde{F}}{\partial x_2 \partial x_1}. \quad (43)$$

For the simple case study here (no torsion), it is therefore possible to pick the components of the diffeomorphism to be

$$\phi_i = -\frac{1}{2}\frac{\partial \tilde{F}}{\partial x_i}\left(\sum_{i=1}^{2}\frac{\partial^2 \tilde{F}}{\partial x_i^2}\right)^{-1}. \quad (44)$$

The example in the next section will show the application of this transformation.

IV. APPLICATION EXAMPLE

In this section, we consider a nonlinear dynamical system. Let the system be given by

$$\begin{aligned} \dot{x}_1 &= -x_1^3 - x_2 := f_1(x) && (45)\\ \dot{x}_2 &= x_1 - x_2^3 := f_2(x). && (46) \end{aligned}$$

The vector field associated with this system is expressed as $X|_x = f_1(x)\partial_{x_1} + f_2(x)\partial_{x_2}$. Let the two-form Ω be given by

$$\Omega = dx_1 \wedge dx_2. \quad (47)$$

Computing $\omega = X \lrcorner \Omega$ gives

$$\omega = -f_2 dx_1 + f_1 dx_1. \quad (48)$$

One can check that ω is not closed (*i.e.* $d\omega \neq 0$ since $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \neq 0$). We construct the homotopy operator $\mathbb{H}$ centered at the origin by letting

$$\mathfrak{X}|_x = x_1\partial_{x_1} + x_2\partial_{x_2} \quad (49)$$

and by evaluating the one-form ω on the star-shaped domain. We have

$$\begin{aligned} (\mathbb{H}\omega)(x) &= \int_0^1 \left(\lambda^3 x_2^3 x_1 - \lambda^3 x_1^3 x_2 - \lambda(x_1^2 + x_x^2)\right) d\lambda && (50)\\ \tilde{F} &= \frac{x_1 x_2(x_2^2 - x_1^2)}{4} - \frac{(x_1^2 + x_2^2)}{2}. && (51) \end{aligned}$$

The exact part ω_e of the one-form ω is given by

$$\begin{aligned} (\omega_e)|_x &= d(\mathbb{H}\omega)|_x \\ &= \left(-x_1 + \frac{x_2}{4}(x_2^2 - 3x_1^2)\right) dx_1 \\ &\quad + \left(-x_2 + \frac{x_1}{4}(3x_2^2 - x_1^2)\right) dx_2. \quad (52) \end{aligned}$$

One locally admissible dissipative potential for the system is given by $-\tilde{F}$, as noted at the end of Section II. However, the real interest here is to use a change of coordinates to define a potential that is easier to use, such as the one of Section III-B. Using the construction procedure developed in the last section based on the equivalence of the closed one-form in two dimensions, we apply the transformation (44), with $\tilde{F}$

$$\tilde{F}(x_1, x_2) = \frac{-(x_1^2 + x_2^2)}{2} + \frac{x_1 x_2(x_2^2 - x_1^2)}{4}. \quad (53)$$

We obtain:

$$\begin{aligned} z_1 &= \frac{x_2^3 - 3x_1^2 x_2}{16} - \frac{x_1}{4} && (54)\\ z_2 &= \frac{3x_2^2 x_1 - x_1^3}{16} - \frac{x_2}{4}. && (55) \end{aligned}$$

We remark that the transformation maps the origin of (x_1, x_2) to the origin of (z_1, z_2). As noted above, a suitable dissipative potential for the system is given by

$$H(z) = \frac{1}{2}(z_1^2 + z_2^2). \quad (56)$$

We obtain, in the neighborhood of the origin, a regular positive function that can be used as a dissipative potential for the system.

V. CONCLUSION

In this paper, a procedure to study autonomous systems using local dissipative Hamiltonian realization for nonlinear dynamical systems has been derived. Taking the interior product of a nonvanishing two-form with respect to the vector field defining the system, we obtained a (possibly) non-closed one-form. Constructing a locally defined homotopy operator on a star-shaped domain, we showed how to locally decompose the obtained form into an exact and an anti-exact one-forms. Dissipative realization is studied as an equivalence problem between the exact form and an exact one-form derived from a known potential. The coordinate transformation obtained enables us to explicitly write the dissipative potential for the original system. The obtained anti-exact form is associated to a nondissipative potential with associated tangential dynamics that do not contribute to the value of the dissipative potential on the star-shaped domain. An application to a two-dimensional nonlinear system was provided to illustrate the method. Further studies will focus on the computation of the nondissipative potential associated with the anti-exact form, and implications in terms of stability, especially within the context of nonequilibirum thermodynamics. Extensions to the problem to feedback dissipative equivalence problem for control applications will also be considered.

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