Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Optimal Filtering for Crypto-Deterministic Systems with Application to Delay Systems with Unknown Initial Data

Erik I. Verriest
erik.verriest@ece.gatech.edu
School of Electrical and Computer Engineering
Georgia Institute of Technology
Atlanta, GA 30332, USA

***Abstract*— A result of Friedland for efficient filtering in the presence of a static bias is extended to the case where the bias signals are given by the response of persistent autonomous systems with random initial conditions. It is shown that the optimal filter decouples into a bias free and a bias error correction filter. We apply the results to filtering of a system with delay when the initial data is missing. Theses results have potential applications in secure communication, synchronization and networked control.**

I. INTRODUCTION

This short paper considers the problem of estimating in a linear least squares sense, the state of a system, which is driven by an unknown but deterministic bias function superposed on the usual stochastic perturbation. In addition, the output may be corrupted by such a deterministic bias term. We assume that it is known that the bias is generated by an autonomous system for which a state space model is available. Consequently, all uncertainty resides in its initial condition. Such a system is also known as a *crypto-deterministic* system, the roots of this nomenclature taking place in quantum mechanics (more precisely, the hidden variable theories) [1].

Thus let the state space model be

$$\dot{x} = Ax + Bb + w \qquad (1)$$

$$y = Hx + Cb + v \qquad (2)$$

where x is the state to be estimated, w and v are zero mean white noises, with covariance respectively Q and $R > 0$. For simplicity it is assumed that these noises are uncorrelated. The crypto-deterministic bias satisfies

$$\dot{b} = Fb.$$

A special case of this crypto-deterministic model is the case of a constant bias, which was treated in [6]. Another case of interest occurs when the signals of interest are corrupted by sinusoidal interferences. In this case, the matrix F has all its eigenvalues on the imaginary axis. In signal processing (e.g., of EKG signals) there is great interest in removing such interferences, coming for instance from the power net (base frequency and higher harmonics). Besides filtering of unwanted signals the results we derive may also be relevant in synchronization and secure communication. Indeed deterministic observers have already found applicability in this field [3], [5], [11]. Using state augmentation, one obtains a combined model for the extended state $z' = [x', b']$:

$$\dot{z} = \begin{bmatrix} A & B \\ 0 & F \end{bmatrix} z + \begin{bmatrix} I \\ 0 \end{bmatrix} w = \mathscr{F} z + \mathscr{G} w. \qquad (3)$$

The output equation in augmented form is

$$y = \begin{bmatrix} H & C \end{bmatrix} z + v = \mathscr{H} z + v. \qquad (4)$$

The rest of the paper is organized as follows: In Section 2, we show how the Kalman filter equations for the augmented system can be decoupled. We then proceed in Section 3 to obtain the crypto-deterministic filter, and derive an information form in Section 4. Section 5 shows an application for a notch filter which can remove an undesirable harmonic signal. Some comments about observability are made in Section 6. We extend the results in Section 7 for a crypto-deterministic signal generated by a functional differential system (a delay system).

II. DECOUPLING OF AUGMENTED KALMAN FILTER

The Kalman filter equations for this augmented form are

$$\dot{\hat{z}} = \mathscr{F}\hat{z} + \mathscr{P}\mathscr{H}' R^{-1}(y - \mathscr{H}\hat{z}), \qquad (5)$$

where $\mathscr{P}$ satisfies the Riccati equation

$$\dot{\mathscr{P}} = \mathscr{F}\mathscr{P} + \mathscr{P}\mathscr{F}' - \mathscr{P}\mathscr{H}' R^{-1}\mathscr{H}\mathscr{P} + \mathscr{G} Q \mathscr{G}', \qquad (6)$$

with the initialization

$$\mathscr{P}(t_0) = \begin{bmatrix} P_0 & 0 \\ 0 & P_{b0} \end{bmatrix}. \qquad (7)$$

Partition $\mathscr{P}$ consistent with the above as

$$\mathscr{P} = \begin{bmatrix} P_x & P_{xb} \\ P'_{xb} & P_b \end{bmatrix}. \qquad (8)$$

Its block components satisfy the coupled ODE's

$$\begin{aligned} \dot{P}_x &= AP_x + P_x A' + BP'_{xb} + P_{xb}B' + \\ &\quad -(P_x H' + P_{xb}C')R^{-1}(HP_x + CP'_{xb}) + Q \qquad (9) \\ \dot{P}_{xb} &= [A - (P_x H' + P_{xb}C')R^{-1}H]P_{xb} + \\ &\quad +[B - (P_x H' + P_{xb}C')R^{-1}C]P_b + P_{xb}F' \qquad (10) \\ \dot{P}_b &= FP_b + P_b F' - (P'_{xb}H' + P_b C')R^{-1}(HP_{xb} + CP_b). \qquad (11) \end{aligned}$$

It is readily seen that these equations are homogeneous in P_{xb} and P_b. Therefore, if $P_{xb}(0)=0$ and $P_b(0)=0$, then for all t, it follows that $P_{xb}(t)=0$ and $P_b(t)=0$, and

$$\dot{P}_x = AP_x + P_xA' - P_xH'R^{-1}HP_x + Q. \tag{12}$$

This may be exploited in a similar fashion as in Friedland's paper [6]. Let $\widetilde{\mathscr{P}} = \begin{bmatrix} \widetilde{P}_x & 0 \\ 0 & 0 \end{bmatrix}$ be the solution to the *unbiased* variance equation. and set

$$\mathscr{P} = \widetilde{\mathscr{P}} + VMV', \tag{13}$$

where V and M satisfy

$$\begin{aligned} \dot{V} &= (\mathscr{F} - \widetilde{\mathscr{P}}\mathscr{H}R^{-1}\mathscr{H})V & (14)\\ \dot{M} &= -MV'\mathscr{H}'R^{-1}\mathscr{H}VM. & (15) \end{aligned}$$

Letting this V be partitioned in a consistent way as

$$V = \begin{bmatrix} V_x \\ V_b \end{bmatrix} \tag{16}$$

one obtains

$$\begin{aligned} \dot{V}_x &= (A - \widetilde{P}_xH'R^{-1}H)V_x + (B - \widetilde{P}_xH'R^{-1}C)V_b & (17)\\ \dot{V}_b &= FV_b, & (18) \end{aligned}$$

and

$$\dot{M} = -M(V_x'H' + V_b'C')R^{-1}(HV_x + CV_b)M. \tag{19}$$

As long as the following consistency conditions hold, freedom remains in the choice for their initial conditions.

$$\begin{aligned} V_x(0)M(0)V_x'(0) &= P_x(0) - \widetilde{P}_x(0) = 0 & (20)\\ V_x(0)M(0)V_b'(0) &= P_{xb}(0) & (21)\\ V_b(0)M(0)V_b'(0) &= P_b(0). & (22) \end{aligned}$$

It seems reasonable to assume that $P_{xb}(0)=0$. Therefore, a simple compatible choice is provided by

$$\begin{aligned} M(0) &= P_b(0) & (23)\\ V_x(0) &= 0 & (24)\\ V_b(0) &= I. & (25) \end{aligned}$$

It follows from the latter and the equation $\dot{V}_b = FV_b$ that V_b is the transition matrix of F. For instance, if F is the system matrix of an oscillatory or quasi-oscillatory system, i.e., if the bias vector contains the pure oscillations at circular frequencies $\omega_1, \omega_2, \ldots, \omega_k$, then

$$V_{b,k} = \text{Block diag}\left\{ \begin{bmatrix} \cos\omega_i t & \sin\omega_i t \\ -\sin\omega_i t & \cos\omega_i t \end{bmatrix} \right\}_{i=1,\ldots,k} \tag{26}$$

Using the identity $\frac{\mathrm{d}}{\mathrm{d}t}M^{-1} = -M^{-1}\dot{M}M^{-1}$, it is readily seen that the M-equation has the solution

$$M(t) = \left[\int_0^t (V_x'H' + V_b'C')R^{-1}(HV_x + CV_b)\,\mathrm{d}t + P_b^{-1}(0)\right]^{-1}. \tag{27}$$

Consequently, solving (12) for $\widetilde{P}_x$ and (18) for V_b, one readily obtains V_x, and then M. Finally, this results in

$$\begin{aligned} P_{xb} &= V_xMV_b' & (28)\\ P_b &= V_bMV_b', & (29) \end{aligned}$$

which is oscillatory in the limit if F is quasi oscillatory.

III. Filter for Crypto-Deterministic Systems

The above exact least squares filter (5) in partitioned form yields the equations for the estimates

$$\begin{aligned} \dot{\widehat{x}} &= A\widehat{x} + B\widehat{b} + (P_xH' + P_{xb}C')R^{-1}(y - H\widehat{x} - C\widehat{b}) & (30)\\ \dot{\widehat{b}} &= F\widehat{b} + (P_{xb}'H' + P_bC')R^{-1}(y - H\widehat{x} - C\widehat{b}). & (31) \end{aligned}$$

With (12) this yields

$$\begin{aligned} \dot{\widehat{x}} &= A\widehat{x} + B\widehat{b} + (P_xH' + V_xMV_b'C') \\ &\quad \times R^{-1}(y - H\widehat{x} - C\widehat{b}) & (32)\\ \dot{\widehat{b}} &= F\widehat{b} + (V_bMV_x'H' + V_bMV_b'C') \\ &\quad \times R^{-1}(y - H\widehat{x} - C\widehat{b}). & (33) \end{aligned}$$

or, exploiting the relation (13) between P_x and $\widetilde{P}_x$

$$\begin{aligned} \dot{\widehat{x}} &= A\widehat{x} + B\widehat{b} + [\widetilde{P}_xH' + V_xM(V_x'H' + V_b'C')] \\ &\quad \times R^{-1}(y - H\widehat{x} - C\widehat{b}) & (34)\\ \dot{\widehat{b}} &= F\widehat{b} + V_bM(V_x'H' + V_b'C') \\ &\quad \times R^{-1}(y - H\widehat{x} - C\widehat{b}). & (35) \end{aligned}$$

Clearly, the $\widehat{x}$ and $\widehat{b}$ update equations are coupled. In order to reduce the complexity of the computation, we try to decouple them by seeking a solution in the form

$$\widehat{x} = x^* + S\widehat{b} \tag{36}$$

where x^* is the optimal estimate in the absence of the bias. This estimate is given by

$$\dot{x}^* = Ax^* + \widetilde{P}_xH'R^{-1}(y - Hx^*). \tag{37}$$

Substituting in the equation for $\hat{x}$, one finds after a bit of algebra

$$\begin{aligned} &\big[\dot{S} + SF - AS - SV_bM(V_x'H' + V_b'C')R^{-1}C - B + \\ &\quad + \widetilde{P}_xH'R^{-1}(HS + C) + V_xM(V_x'H' + V_b'C')R^{-1}C\big]\widehat{b} + \\ &+ \big[(SV_b - V_x)M(V_x'H' + V_b'C')R^{-1}\big](y - H\widehat{x}) = 0. \end{aligned}$$

Since initial conditions can be arbitrary, and the above must be identically satisfied for arbitrary $\widehat{b}$ and $(y - H\widehat{x})$, it follows that

$$(SV_b - V_x)M(V_x'H' + V_b'C')R^{-1} \equiv 0$$

and

$$\begin{aligned} \dot{S} &= AS - SF + B + (SV_b - V_x)M(V_x'H' + V_b'C')R^{-1}C + \\ &\quad - \widetilde{P}_xH'R^{-1}(HS + C). \end{aligned}$$

The first equation is satisfied for

$$S = V_xV_b^{-1}. \tag{38}$$

When substituted in the second, it gives a Lyapunov like equation

$$\dot{S} = (A - \widetilde{P}_xH'R^{-1}H)S - SF + B - \widetilde{P}_xH'R^{-1}C. \tag{39}$$

for which the initial condition is

$$S(0) = V_x(0)V_b(0)^{-1} = 0. \tag{40}$$

Thus we summarize the filter

$$\widehat{x} = x^* + V_x V_b^{-1} \widehat{b}, \tag{41}$$

where the estimate $\widehat{b}$ of the crypto-deterministic term satisfies

$$\begin{aligned} \dot{\widehat{b}} &= [F - V_b M (V_x' H' + V_b' C') R^{-1} C] \widehat{b} + \\ &\quad + V_b (M (V_x' H' + V_b' C') R^{-1} (y - H\widehat{x}). \end{aligned} \tag{42}$$

or, in terms of the unbiased estimate x^*:

$$\begin{aligned} \dot{\widehat{b}} &= [F - V_b M (V_x' H' + V_b' C') R^{-1} (HS + C)] \widehat{b} + \\ &\quad + V_b M (V_x' H' + V_b' C') R^{-1} (y - Hx^*). \end{aligned}$$

Theorem 1: *The exact least squares filter for the crypto-deterministic system is given by the usual Kalman filter*

$$\begin{aligned} \dot{x}^* &= Ax^* + K(y - Hx^*) \\ K &= \widetilde{P}_x H' R^{-1} \\ \dot{\widetilde{P}}_x &= A\widetilde{P}_x + \widetilde{P}_x A' - KRK' + Q, \quad \widetilde{P}_x(0) = 0. \end{aligned}$$

and the bias-correction filter

$$\dot{S} = AS - SF + B - \widetilde{P}_x H' R^{-1} (HS + C), \quad S(0) = 0$$

$$\dot{V}_b = FV_b, \quad V_b(0) = I$$

$$M = \left[\int_0^t V_b' (S'H' + C') R^{-1} (HS + C) V_b \, \mathrm{d}t + P_b^{-1}(0) \right]^{-1}.$$

$$\begin{aligned} \dot{\widehat{b}} = {} & [F - V_b M V_b' (S'H' + C') R^{-1} (HS + C)] \widehat{b} + \\ & + V_b M V_b' (S'H' + C') R^{-1} (y - Hx^*) \end{aligned}$$

with the combination

$$\hat{x} = x^* + S\widehat{b}.$$

IV. Information form

Further simplification can be obtained by considering the *information state*, $b^* = V_b^{-1} \widehat{b}$. Indeed. one gets then from $\dot{b}^* = -V_b^{-1} \dot{V}_b V_b^{-1} \widehat{b} + V_b^{-1} \dot{\widehat{b}}$, that

$$\frac{\mathrm{d}}{\mathrm{d}t} b^* = M (V_x' H' + V_b' C') R^{-1} (y - Hx^* - (HV_x + CV_b) b^*). \tag{43}$$

The structure of the solution is thus the same as in the special case of a constant bias. First, a bias free estimate x^* is obtained. From the residuals $y - Hx^*$, the bias filter computes the quantity b^*, which in turn is used to update $\widehat{x}$. If the bias states are not necessary, the complexity of the information form is reduced. We summarize:

Unbiased estimator

$$\begin{aligned} \dot{x}^* &= Ax^* + \widetilde{P}_x H' R^{-1} (y - Hx^*) \\ \dot{\widetilde{P}}_x &= A\widetilde{P}_x + \widetilde{P}_x A' + Q - \widetilde{P}_x H' R^{-1} H \widetilde{P}_x. \end{aligned}$$

Crypto-deterministic estimator

$$\dot{b}^* = W^{-1} (V_x' H' + V_b' C') R^{-1} (y - Hx^* - (HV_x + CV_b) b^*)$$

$$\dot{V}_b = FV_b, \qquad V_b(0) = I$$

$$\dot{V}_x = (A - \widetilde{P}_x H' R^{-1} H) V_x + (B - \widetilde{P}_x H' R^{-1} C) V_b, \quad V_x(0) = 0$$

$$\dot{W} = (V_x' H' + V_b' C') R^{-1} (HV_x + CV_b), \quad W(0) = P_b^{-1}(0).$$

followed by the correction to obtain the LS estimate

$$\hat{x} = x^* + V_x b^*.$$

It is readily seen that $\beta = Wb^*$ satisfies

$$\dot{\beta} = (V_x' H' + V_b' C') R^{-1} (y - Hx^*).$$

If an estimate is not needed until some time t_1, the the weighted residual $(V_x' H' + V_b' C') R^{-1} (y - Hx^*)$ is integrated to yield $\beta = Wb^*$ at t_1. At this one time (and only then) the matrix $W(t_1)$ is inverted to yield $M(t_1)$ and thus $\hat{x}(t_1)$ and $\hat{b}(t_1)$.

Since the filter has the same structure as for a constant bias, the same computational considerations as described in [6] will hold. A block diagram is shown in Figure 1.

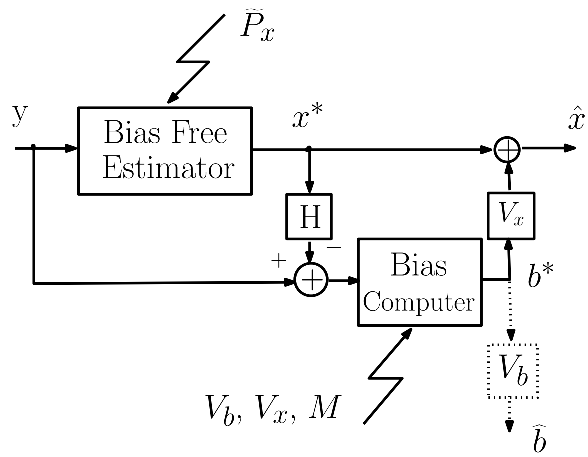

Fig. 1. Structure of the Decoupled Solution

V. Application: Notch Filter

If the bias b is purely harmonic, then F is a 2×2 matrix with eigenvalues at $\pm j\omega$. specified before, and V_b is a rotation matrix. In this case, the steady state solution for the $* - filter$,

$$\widetilde{P}_x = P_\infty.$$

corresponds to the Wiener filter solution, with

$$\dot{x}^* = (A - K_\infty H) x^* + K_\infty y, \tag{44}$$

with constant gain

$$K_\infty = P_\infty H' R^{-1}, \tag{45}$$

where P_∞ is the solution to the algebraic Riccati equation

$$AP_\infty + P_\infty A' + Q - K_\infty R K_\infty' = 0. \tag{46}$$

Let also

$$\begin{aligned} \dot{V}_x &= (A - K_\infty H) V_x + (B - K_\infty C) V_b \quad &(47) \\ \dot{V}_b &= FV_b \quad &(48) \end{aligned}$$

with initial condition $V_b(0) = I$. The solution to the latter is $V_b(t) = \Phi(t) = \exp(Ft)$, which is purely oscillatory. Consider now the periodic regime solution of V_x. Since it is driven by an oscillation, the solution will have the same oscillation frequency. Thus for some L,

$$V_x = LV_b = L\Phi, \tag{49}$$

where in view of the general solution, $L = S_\infty$. Differentiating this relation

$$\dot{V}_x = L\dot{\Phi} = LF\Phi$$

leads to an asymmetric algebraic Lyapunov equation

$$(A - K_\infty H)L + (B - K_\infty C) = LF. \tag{50}$$

Since $(A - K_\infty H)$ is stable, its eigenvalues have negative real parts. The matrix F has its eigenvalues on the imaginary axis. So the equation

$$(A - K_\infty H)L - LF = -(B - K_\infty C)$$

or equivalently, its Kronecker form

$$[(A - K_\infty H) \otimes I - I \otimes F']\mathrm{vec}\,(L) = \mathrm{vec}\,(K_\infty C - B).$$

is solvable for L, since the matrix $[(A - K_\infty H) \otimes I - I \otimes F']$ has eigenvalues $\lambda_i(A - K_\infty H) + \lambda_j(F) \neq 0$. We note that the observability of the full system requires full rankness of

$$\begin{bmatrix} sI - A & -B \\ 0 & sI - F \\ H & C \end{bmatrix}$$

for all $s \in \mathbb{C}$ by the PBH-test [12, p. 762]. Observability guarantees that the error covariance matrix of the combined system state and the cryptodeterministic state are bounded.

Example

In Figure 2 we show an observation for a system with state space parameters

$$A = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, G = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, H = [1, 0] \text{ and } Q = R = 1.$$

It is known that an additive harmonic signal (a position, say) of frequency $\omega = 1/\sqrt{10}$ corrupts the output on top of a unit variance white noise. The crypto-deterministic filter output ('xe' and 'be') are shown in Figure 3. A comparison is given with the unperturbed output x and the actual perturbing bias signal b. The continuous time system was integrated with stepsize $\Delta_t = 0.005$, using the sochastic differential equation numerical algorithm in [10]. Hence the real time at step 5000 is 25 sec. After about 5 seconds the cryptodeterministic signal is completely captured.

VI. Observability

More generally, consider a quasi periodic bias signal formed by a superposition of k oscillations. The system is augmented with $2k$ additional states (for the bias). In the worst case, no information is initially available for this bias, and we set $P_b(0) = 0$. The matrices V_x and V_b have respectively the dimension $n \times 2k$ and $2k \times 2k$, so that since $V_x = LV_b$, we get $L \in \mathbb{R}^{n \times 2k}$. Since this yields a set of $(n \times 2k)$ equations in $(n \times 2k)$ unknowns, the system is solvable. Letting then $W = M^{-1}$, gives

$$\dot{W} = \Phi'(L'H' + C')R^{-1}(HL + C)\Phi \tag{51}$$

it follows that $W(T) = \int_0^T \Phi'(L'H' + C')R^{-1}(HL + C)\Phi \mathrm{d}t$ is the observability Gramian, $= \mathscr{O}_{(F, R^{-1/2}(HL+C))}$, of the pair $(F, R^{-1/2}(HL + C))$. Hence observability of the system $(F, R^{-1/2}(HL + C))$ is required for the inversion to be possible. For nonsingular noise, this is equivalent to the observability of the pair $(F, HL + C)$ over the interval $(0, T)$. By the PBH test, this is again equivalent to

$$\mathrm{rank} \begin{bmatrix} sI - F \\ HL + C \end{bmatrix} = 2k, \quad \forall s \in \mathbb{C}.$$

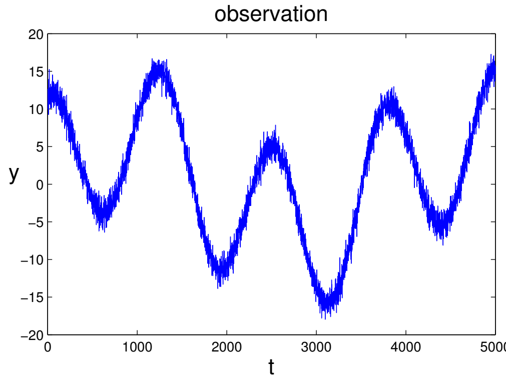

Fig. 2. Corrupted system output

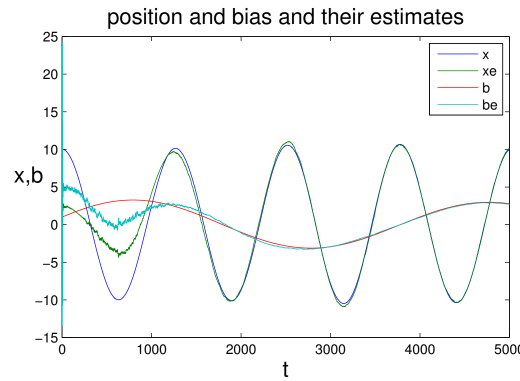

Fig. 3. Actual signals and their estimates

VII. Delay Systems as Crypto-Deterministic Systems

Let the bias be driven by an autonomous functional differential equation, say of the form

$$\dot{b}(t) = F_0 b(t) + F_1 b(t - \tau). \tag{52}$$

A sufficient condition for asymptotic stability is the existence of a triple (P_0, P_1, P_2) of positive definite matrices satisfying a Riccati equation $F_0'P_0 + P_0F_0 + P_1 + P_0F_1P_1^{-1}F_1'P_0 + P_2 =$

0 [13]. Such a system has a countably infinite number of eigenvalues, and with each corresponds an unknown initial condition for the bias. One way to represented the dynamics of an autonomous delay system with a crisp delay is by a multi-mode multi-dimensional system [15] with increasing dimension as time evolves.

First partition the positive time axis into the intervals $((k-1)\tau, k\tau]$, for $k = 1,2,\ldots$. Then define for $t = k\tau + \theta$,

$$b(t) = b(k\tau+\theta) \stackrel{\text{def}}{=} b_k(\theta)$$

where $\theta \in [-\tau, 0]$. We denote these subintervals as *couplets* [14]. In order to retrieve the bias signal, note that if we define in the the k-couplet the output

$$\beta(\theta) = b_k(\theta), \tag{53}$$

then indeed $\beta(\theta) = b(k\tau+\theta)$.

This establishes an equivalence between the delay system for $t < k\tau$ and a *finite* dimensional system in the interval $[-\tau, 0]$.

More precisely, if the initial data is ζ, the method of steps yields for the first couplet ($0 \le t < \tau$) the equation

$$\begin{aligned} \dot b_1(\theta) &= F_0 b_1(\theta) + F_1 \zeta(\theta) \\ \beta(\theta) &= b_1(\theta). \end{aligned}$$

In the second couplet, we get the couplet-form

$$\begin{aligned} \dot b_2(\theta) &= F_0 b_2(\theta) + F_1 b_1(\theta) \\ \dot b_1(\theta) &= F_0 b_1(\theta) + F_1 \zeta(\theta) \\ \beta(\theta) &= b_2(\theta), \end{aligned}$$

and so on. The first use of such a representation is attributed to Olbrot's PhD thesis (in Polish).

In general, we get for the autonomous mode, the k-couplet equation, describing the system and its output up to time $k\tau$ as

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix} b_k \\ b_{k-1} \\ \vdots \\ b_1 \end{bmatrix} = \begin{bmatrix} F_0 & F_1 & & \\ & F_0 & F_1 & \\ & & \ddots & F_1 \\ & & & F_0 \end{bmatrix} \begin{bmatrix} b_k \\ b_{k-1} \\ \vdots \\ b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ I \end{bmatrix} \zeta.$$

with $\beta = b_k$, Represent these equations in compact form by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\theta}\chi_k &= \mathscr{A}_k \chi_k + \Gamma_k \zeta & (54) \\ \beta &= \mathscr{C}_k \chi_k, & (55) \end{aligned}$$

where $\chi_k' = [b_k', b_{k-1}', \ldots, b_1']$, $\Gamma_k = e_k \otimes I$ and $\mathscr{C}_k = e_1' \otimes I$, of appropriate size. This leads then to the multi-mode multi-dimensional system (M^3D). Since the initial data is completely unknown, we set $\zeta = 0$, all initial uncertainty then be mapped to the initial uncertainty of the finite dimensional (but of increasing dimension) bias state.

$$\begin{aligned} \dot x &= Ax + B[I, 0, \ldots, 0]\chi_k + w & (56) \\ y &= Hx + C(I, 0, \ldots, 0)\chi_k + v. & (57) \end{aligned}$$

The equation for S assumes here the form (in the k-th couplet)

$$\begin{aligned} \dot S = (A - \widetilde P_x H' R^{-1} H) S - S\mathscr{A}_k + [B, 0, \ldots, 0] + \\ -\widetilde P_x H' R^{-1}[C, 0, \ldots, 0]. \end{aligned}$$

These equations decouple very nicely to

$$\begin{aligned} \dot S_1 &= (A - \widetilde P_x H' R^{-1} H) S_1 - S_1 F_0 + B - \widetilde P_x H' R^{-1} C \\ \dot S_2 &= (A - \widetilde P_x H' R^{-1} H) S_2 - S_1 F_1 - S_2 F_0 \\ &\vdots \\ \dot S_k &= (A - \widetilde P_x H' R^{-1} H) S_k - S_{k-1} F_1 - S_k F_0. \end{aligned}$$

Since the initial conditions are zero, once S_1 is found, we obtain sequentially S_2, S_3, etc. A simple suboptimal filter (for time invariant systems and stationary noise) is obtained by taking the Wiener solution i.e., if $\overline P_x = \lim_{t\to\infty} \widetilde P_x(t)$, then $A_{cl} = A - \overline P_x H' R^{-1} H$, is the closed loop with dynamic matrix and we can iteratively integrate

$$S_k = \int_0^t \mathrm{e}^{-A_{cl}\theta} S_{k-1}(t-\theta) F_1 \mathrm{e}^{F_0\theta}\, \mathrm{d}\theta. \tag{58}$$

VIII. Conclusions and Further Work

We extended an old result by Friedland for efficient filtering in the presence of a static bias to the case where the bias signals are given by the response of persistent autonomous systems with random initial conditions. It has been shown that the optimal filter again decouples into a bias free and a bias error correction filter. We also derived the filter when the bias is generated by an autonomous delay equation, by applying a technique of representing the delay system in couplet form. It was shown that the Riccati equation for S may be solved as an iteration of the lower dimensional system.

It is believed that this work can be extended to include the case where the crypto-deterministic is nonlinear and possibly chaotic, by incorporating the theory of polynomial filters developed in [7], [8], [9]. This has potential applications in secure communication, synchronization and network control systems [2].

References

[1] R.W. Batterman, "Randomness and Probability in Dynamical Theories: On the Proposals of the Prigogine School," *Philosophy of Science*, Vol. 58, No. 2, pp. 241-263, 1991.

[2] R. Blind, U. Münz, and F. Allgöwer, "Almost Sure Stability and Transient Behavior of Stochastic Nonlinear Jump Systems Motivated by Networked Control Systems," *Proc. 46-th IEEE CDC*, New Orleans, LA, Dec 2007, 3327-3332

[3] S. Čelikovský and G. Chen, "Secure Synchronization of a Class of Chaotic Systems from a Nonlinear Observer Approach," *IEEE Transactions on Automatic Control*, Vol. 50, No. 1, pp. 76-82, 2005.

[4] P.M. Clarkson, *Optimal and Adaptive Signal Processing*, CRC Press, 1991.

[5] G.J. Fodjuouong, H.B. Fotsin and P. Woafo, "Synchronizing modified van der Pol - Duffing oscillators with offset terms using observer design: application to secure communication," *Physica Scripta*, Vol. 75, pp. 638-644, 2007.

[6] B. Friedland, "Treatment of Bias in Recursive Filtering," *IEEE Transactions on Automatic Control*, Vol. AC-14, No. 4, pp. 359-367, August 1969.

[7] A. Germani, C. Manes and P. Palumbo, "Polynomial extended Kalman filter," *IEEE Transactions on Automatic Control,* Volume 50, Issue 12, Dec. 2005, pp. 2059 - 2064.
[8] A. Germani, C. Manes and P. Palumbo, "Filtering of Stochastic Nonlinear Differential Systems via a Carleman Approximation Approach," *IEEE Transactions on Automatic Control,* Volume 52, Issue 11, Nov. 2007 pp. 2166 - 2172.
[9] A. Germani, C.Manes and P. Palumbo, "Simultaneous system identification and channel estimation: a hybrid system approach," *Proc. 46-th IEEE CDC*, New Orleans, LA, Dec 2007, 1764-1769.
[10] D.J. Higham, "An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations, *SIAM Review*, Vol. 43, No. 3, pp.525-546.
[11] H.J.C. Huijberts and H. Nijmeijer, "An Observer View on Synchronization," in *Nonlinear control in the year 2000*, A. Isidori, F. Lamnabhi-Lagarrigue and W. (eds.), Springer-Verlag, Lecture Notes in Control and Information Sciences, Vol. 258, pp. 509, 2001. Publisher: London ; New York : Springer, c2001-
[12] T. Kailath, A.H. Sayed and B. Hassibi, *Linear Estimation*, Prentice-Hall, 2000.
[13] E.I. Verriest, "Riccati Stability" in: *Unsolved Problems in Mathematical Systems and Control Theory*, V. Blondel and A. Megretski, (eds.) Princeton University Press, pp. 49-53, 2004.
[14] E.I. Verriest, "Finite Observation in Delay Systems," *Proceedings, 16th International Symposium on the Mathematical Theory of Networks and Systems*, Leuven, Belgium, 2004 isbn: 90-5682-517-8, MP-6.4.1-11, p#36.
[15] E.I. Verriest, "Multi-mode Multi-dimensional Systems," *Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems*, Kyoto, Japan, (July 24-28, 2006), pp. 1268-1274.