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Optimal Filtering in Multiple Channel Networked Control Systems with Multiple Packet Dropout*

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Abstract— **This article studies the problem of optimal filtering in multiple channel networked control systems (NCSs) with multiple packet dropouts. A generalized formulation is employed to model the multiple packet dropout in multiple channel case, the random dropout rates are transformed into stochastic parameters in the system's representation. By generalization of the definition of the $\mathscr{H}_2$-norm, generalized relations for the stochastic norm of a linear discrete-time stochastic parameter system represented in the state space form are studied. The stochastic norm of the estimation error is used as a criteria for filter design in the NCS framework. A set of linear matrix inequalities (LMIs) is given to solve the corresponding filter design problem. A simulation example is presented for clarification.**

I. INTRODUCTION

State feedback is the most widely used methodology in modern control systems. State feedback control implicitly assumes that all state variables are measurable. However, in practice, some state variables may not be directly accessible or the corresponding sensing devices may be very expensive or unavailable. In such cases, state filters are used to give an estimate of unavailable states.

In the last two decades, advances in computer technology, communication and control introduced a modern control system architecture termed as networked control systems (NCSs). In an NCS, sensors and actuators exchange information with a controller through a shared communication medium. Compared to conventional point-to-point system interconnection, using an NCS has advantages such as easy installation and reduced set-up, wiring and maintenance costs, and increased system flexibility.

In a classical configuration, a real-time communication between all components of the control system is assumed. However, in a large and complex control system, point-to-point wiring becomes costly, inflexible, unreliable and even impractical due to limited input-output capacity. In the other hand, in an NCS, the controller and all of the sensors and actuators are connected to a shared medium. Because of the shared medium, only a limited number of connections can be implemented simultaneously. In another word, the basic assumption of real time data exchange is not valid in an NCS. This limitation introduces some new challenges in an NCS. The network can be modeled as a switch that opens and closes in a random manner as shown in Figure 1. When a switch is open, its output is held at the previous value and the data packet is lost. Data packet dropout can occur due to node failures or network congestion and is a common problem in networked systems. In real-time feedback control systems, it is normally advantageous to discard the old packets and consider the new ones so that the controller always receives fresh data for control calculation. Packet dropouts usually occur randomly. Because of random dropout, classical estimation and control methods cannot be used directly in NCS systems. Dropouts degrade system performance and make the filtering and estimation more difficult and challenging.

Networked control systems (NCSs) and the random packet dropouts have gained high attention in research studies in last few years (e.g., see [7], [8], [13], [15]–[17] and references therein). Specifically, the problem of stochastic packet dropout has been studied in single channel sensor delay systems, [15], and single channel NCSs in the $\mathscr{H}_2$ and $\mathscr{H}_\infty$ framework [16], [17]. Also, the problem of stabilization and control has been studied recently in these systems (e.g., see [9], [10], [20]–[22] and references therein).

In all of the NCS studies so far, a single channel system has been considered. It means that all of the sensor information or controller command information pass through a single channel. But it should be noted that in practice, especially in complex systems with a large number of input-outputs, multiple channel modeling should be considered. For example, sensor networks can be modeled by this method where each sensor measurement should be independently encapsulated in one single packet and should be transmitted to the controller through its own channel. If multiple channels are available, then the probability of successful transmission will greatly increase. To the best of our knowledge, due to their complexity, multiple channel NCSs with multiple packet dropouts have not been studied yet.

In this article, we consider the problem of optimal filtering in a multiple channel NCS with multiple packet dropouts. The random dropout rates are transformed into stochastic parameters in the system's representation. By generalization of the stochastic $\mathscr{H}_2$-norm [16], generalized relations for the stochastic norm of a linear discrete-time stochastic parameter system represented in the state space form are studied. The stochastic norm of the estimation error is used as a criteria for filter design in the NCS framework. Set of linear matrix inequalities (LMIs) [1] is given to solve the corresponding

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filter design problem. Simulation example is presented for clarification.

The remainder of this article is organized as follows. In Section 2, the formulation of a multiple channel NCS with multiple packet dropouts, and the filter relations are given. Section 3 introduces the generalized stochastic $\mathscr{H}_2$-norm of a system represented in the state-space form with several stochastic parameters. The LMI formulation of optimal filtering based on the stochastic norm of a stochastic parameter system is given in Section 4. Simulation example is presented in Section 5 to show the applicability and effectiveness of the proposed theory followed by concluding remarks in Section 6.

II. Problem Formulation

The schematic of the multiple channel NCS under study is depicted in Figure 1. We suppose that the controller is already designed. $\tilde{\Omega}$ is the exogenous noise input, and z is the signal to be estimated, and $\hat{z}$ is its estimate. The filtering error, $\tilde{z}$ is defined as $\tilde{z} = z - \hat{z}$. The plant is a multiple-

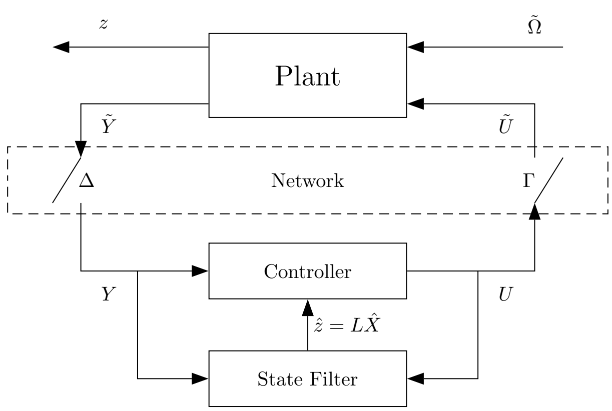

Fig. 1. Multiple channel NCS schematic with packet dropouts

input, multiple-output (MIMO) discrete-time linear time-invariant (LTI) one subject to random disturbances. Also, the sensor data are contaminated with noise. The plant can be represented by the following equations:

$$\begin{cases} \tilde{X}_{k+1} &= \mathbf{a}\tilde{X}_k + \mathbf{b}_1\tilde{U}_k + \mathbf{b}_2\tilde{\Omega}_k \\ \tilde{Y}_k &= \mathbf{c}\tilde{X}_k + \mathbf{d}_1\tilde{U}_k + \mathbf{d}_2\tilde{\Omega}_k, \end{cases} \tag{1}$$

where $\tilde{X}_k \in \mathbb{R}^n$ is the plant state vector, and $\mathbf{a}$, $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{c}$, $\mathbf{d}_1$ and $\mathbf{d}_2$ are system parameter matrices with appropriate dimensions. $\tilde{Y}_k \in \mathbb{R}^p$ is the system output vector contaminated with noise, $\tilde{\Omega}_k$, and $\tilde{U}_k \in \mathbb{R}^m$ is the system command input vector:

$$\tilde{Y}_k = \begin{bmatrix} \tilde{y}_{1k} \\ \vdots \\ \tilde{y}_{pk} \end{bmatrix}, \quad \tilde{U}_k = \begin{bmatrix} \tilde{u}_{1k} \\ \vdots \\ \tilde{u}_{mk} \end{bmatrix}, \quad \tilde{\Omega}_k = \begin{bmatrix} \tilde{\omega}_{1k} \\ \vdots \\ \tilde{\omega}_{mk} \end{bmatrix}. \tag{2}$$

We consider the case where there are m input channels and p output channels in the network. In another words, it is assumed that each input actuator or output sensor has its own channel.

Consider the system described by (1). The system outputs, $\tilde{y}_{ik}$s, are passed through the network and there may be random dropouts, only the probability of the dropouts, α_is, are known. Thus, the current observation at channel i, y_{ik}, is the current system output, $\tilde{y}_{ik}$, with the probability of α_i. In the case of no new data, previous data will be used, so the previous data, $y_{i,k-1}$, will be used with the probability of $(1-\alpha_i)$. The filter has knowledge of the current control command, but the system input, $\tilde{u}_{jk}$, is the current controller output, u_{jk}, with the probability of β_j or the previous one, $\tilde{u}_{j,k-1}$, with the probability of $(1-\beta_j)$. These expressions can be represented by the following relations:

$$\begin{cases} y_{ik} &= \delta_{ik}\tilde{y}_{ik} + \bar{\delta}_{ik}y_{i,k-1}, & i = 1,\cdots,p \\ \tilde{u}_{jk} &= \gamma_{jk}u_{jk} + \bar{\gamma}_{jk}\tilde{u}_{j,k-1}, & j = 1,\cdots,m, \end{cases} \tag{3}$$

with

$$\bar{\delta}_{ik} = 1 - \delta_{ik}, \quad \bar{\gamma}_{jk} = 1 - \gamma_{jk} \tag{4}$$

where y_{ik} and u_{jk} are the controller inputs and outputs, repectively, and the stochastic parameters δ_{ik}s and γ_{jk}s are Bernoulli distributed white sequences taking the values of 0 or 1 with

$$\mathrm{prob}\{\delta_{ik} = 1\} = \mathscr{E}\{\delta_{ik}\} = \alpha_i, \ \ 0 \le \alpha_i \le 1, \ \ i = 1,\cdots,p \tag{5}$$

and

$$\mathrm{prob}\{\gamma_{jk} = 1\} = \mathscr{E}\{\gamma_{jk}\} = \beta_j, \ \ 0 \le \beta_j \le 1, \ \ j = 1,\cdots,m \tag{6}$$

where α_is and β_js are known constants. Also suppose that δ_{ik}s and γ_{jk}s are uncorrelated with each other, $\tilde{\omega}_{ik}$s, and the initial state values, so

$$\mathrm{prob}\{\delta_{ik} = 0\} = 1 - \alpha_i, \quad \mathrm{var}\{\delta_{ik}\} = \alpha_i(1-\alpha_i) = q_i^2, \tag{7}$$

and

$$\mathrm{prob}\{\gamma_{jk} = 0\} = 1 - \beta_j, \quad \mathrm{var}\{\gamma_{jk}\} = \beta_j(1-\beta_j) = r_j^2, \tag{8}$$

Now, Equations (1) and (3) can be put together to have the multiple channel NCS formulation with multiple packet dropouts as follows:

$$\begin{cases} \tilde{X}_{k+1} &= \mathbf{a}\tilde{X}_k + \mathbf{b}_1\tilde{U}_k + \mathbf{b}_2\tilde{\Omega}_k \\ \tilde{Y}_k &= \mathbf{c}\tilde{X}_k + \mathbf{d}_1\tilde{U}_k + \mathbf{d}_2\tilde{\Omega}_k \\ Y_k &= \Delta_k\tilde{Y}_k + \bar{\Delta}_kY_{k-1} \\ \tilde{U}_k &= \Gamma_kU_k + \bar{\Gamma}_k\tilde{U}_{k-1}, \end{cases} \tag{9}$$

where

$$\Delta_k = \mathrm{diag}(\delta_{1k},\cdots,\delta_{pk}), \quad \Gamma_k = \mathrm{diag}(\gamma_{1k},\cdots,\gamma_{mk}), \tag{10}$$

and

$$\bar{\Delta}_k = I_p - \Delta_k, \quad \bar{\Gamma}_k = I_m - \Gamma_k, \tag{11}$$

and Y_k and U_k can be defined similar to $\tilde{Y}_k$ and $\tilde{U}_k$ in (2). In order to get a compact representation, we augment the system states, measurement and the system input:

$$X_{k+1} = \begin{bmatrix} \tilde{X}_{k+1} \\ Y_k \\ \tilde{U}_k \end{bmatrix}, \tag{12}$$

thus,

$$\begin{cases} X_{k+1} &= \mathbf{a}_k X_k + \mathbf{b}_{1k} U_k + \mathbf{b}_{2k} \tilde{\Omega}_k \\ Y_k &= \mathbf{c}_k X_k + \mathbf{d}_{1k} U_k + \mathbf{d}_{2k} \tilde{\Omega}_k \\ z_k &= L X_k \end{cases} \tag{13}$$

where z_k is the signal to be estimated and

$$\mathbf{a}_k = \begin{bmatrix} \mathbf{a} & 0 & \mathbf{b}_1 \bar{\Gamma}_k \\ \Delta_k \mathbf{c} & \bar{\Delta}_k & \Delta_k \mathbf{d}_1 \bar{\Gamma}_k \\ 0 & 0 & \bar{\Gamma}_k \end{bmatrix}, \ \mathbf{b}_{1k} = \begin{bmatrix} \mathbf{b}_1 \Gamma_k \\ \Delta_k \mathbf{d}_1 \Gamma_k \\ \Gamma_k \end{bmatrix}, \ \mathbf{b}_{2k} = \begin{bmatrix} \mathbf{b}_2 \\ \Delta_k \mathbf{d}_2 \\ 0 \end{bmatrix}$$
$$\mathbf{c}_k = \begin{bmatrix} \Delta_k \mathbf{c} & \bar{\Delta}_k & \Delta_k \mathbf{d}_1 \bar{\Gamma}_k \end{bmatrix}, \quad \mathbf{d}_{1k} = \Delta_k \mathbf{d}_1 \Gamma_k, \qquad \mathbf{d}_{2k} = \Delta_k \mathbf{d}_2. \tag{14}$$

Note that $\mathbf{a}_k$, $\mathbf{b}_{1k}$, $\mathbf{b}_{2k}$, $\mathbf{c}_k$, $\mathbf{d}_{1k}$ and $\mathbf{d}_{2k}$ are functions of Δ_k and Γ_k, but for simplicity, this dependency haven't been shown explicitly.

Considering the linear stochastic discrete-time system as in (13), we want to find the estimate $\hat{z}_k$ of z_k such that the $\mathscr{H}_2$-norm of the filtering error is minimized. Now, consider the following filter:

$$F : \begin{cases} \hat{X}_{k+1} &= \mathbf{a}_f \hat{X}_k + \mathbf{b}_f U_k + \mathbf{c}_f Y_k \\ \hat{z}_k &= L_f \hat{X}_k, \end{cases} \tag{15}$$

where $\hat{X}_k$ is an estimate of the state, and $\mathbf{a}_f$, $\mathbf{b}_f$, $\mathbf{c}_f$ and L_f are the filter parameters to be designed. The filtering error is defined as $\tilde{z}_k = z_k - \hat{z}_k$. Now, the system states, X_k, and the filter states, $\hat{X}_k$, can be augmented to get the following augmented system:

$$G : \begin{cases} \zeta_{k+1} &= A_k \zeta_k + B_{1k} U_k + B_{2k} \tilde{\Omega}_k \\ \tilde{z}_k &= C \zeta_k, \end{cases} \tag{16}$$

where

$$A_k = \begin{bmatrix} \mathbf{a}_k & 0 \\ \mathbf{c}_f \mathbf{c}_k & \mathbf{a}_f \end{bmatrix}, \quad B_{1k} = \begin{bmatrix} \mathbf{b}_{1k} \\ \mathbf{b}_f + \mathbf{c}_f \mathbf{d}_{1k} \end{bmatrix}, \quad B_{2k} = \begin{bmatrix} \mathbf{b}_{2k} \\ \mathbf{c}_f \mathbf{d}_{2k} \end{bmatrix},$$
$$C = \begin{bmatrix} L & -L_f \end{bmatrix}, \qquad \zeta_k = \begin{bmatrix} X_k \\ \hat{X}_k \end{bmatrix}. \tag{17}$$

In the next section, to find the optimal filter gains, first the generalized relations for the $\mathscr{H}_2$-norm of systems with stochastic parameters is found.

Remark. The formulation given in this section is for the case when number of channels equals the number of sensors or actuators. This modeling can be easily modified to be used in the case when several sensors or actuators share the same channel. The derivation is straightforward and omitted.

III. $\mathscr{H}_2$-NORM OF SYSTEMS WITH STOCHASTIC PARAMETERS

As was shown in the previous section, the formulation of state estimation in the multiple channel NCSs with random packet dropouts leads to the state space representation of a system with several stochastic parameters. The problem of state filtering for systems with stochastic parameters has been studied before [16], [17]. In this section, we generalize our previous work to extend the problem to a more general case with several stochastic parameters. The stochastic $\mathscr{H}_2$-norm ($\mathscr{H}_{2s}$-norm) of the filtering error dynamics can be used as a performance index for the filter design. The LMI formulation of the performance index and corresponding constraints are presented in the next section.

For a deterministic stable discrete-time linear time-invariant (LTI) system, we have the following two facts:

Fact 1: *If the input is standard (unit variance) white noise, then the root-mean-square value of the output equals the $\mathscr{H}_2$-norm of the system [2].*

Fact 2: *An immediate consequence of Parseval's equality is that if the input is the unit impulse, then the 2-norm of the output equals the $\mathscr{H}_2$-norm of the system [2].*

As the NCS under consideration is reformulated as a time-varying stochastic system, the classical norm definition needs to be modified to be applicable in this case. Consider a stable time-varying stochastic system G with both deterministic input vector, U_k, and a noise input vector, $\tilde{\Omega}_k$, with each element as an unit variance white noise as in (16) where A_k, B_{1k} and B_{2k} are stochastic time dependent matrices. In the following, for simplicity, we replace $\tilde{\Omega}_k$ with Ω_k.

To handle the problem of both the deterministic and stochastic inputs, the linearity property of the system is used to write

$$\tilde{z}_k = \tilde{z}_{1k} + \tilde{z}_{2k} = G_1 U_k + G_2 \Omega_k, \tag{18}$$

where

$$G1 : \begin{cases} \zeta_{1,k+1} &= A_k \zeta_{1,k} + B_{1k} U_k \\ \tilde{z}_{1k} &= C \zeta_{1,k} \end{cases} \tag{19}$$

and

$$G2 : \begin{cases} \zeta_{2,k+1} &= A_k \zeta_{2,k} + B_{2k} \Omega_k \\ \tilde{z}_{2k} &= C \zeta_{2,k}. \end{cases} \tag{20}$$

Following the general definition of the $\mathscr{H}_2$-norm of a time-invariant system, the $\mathscr{H}_2$-norm of the stable stochastic time varying system G_1 is defined as

$$\|G_1\|_{2s}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \text{trace}\{\mathscr{E}\{\tilde{z}_{1k} \tilde{z}_{1k}'\}\}, \tag{21}$$

where the input U_k is a vector of unit impulses. Similarly, we can define

$$\|G_2\|_{2s}^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \text{trace}\{\mathscr{E}\{\tilde{z}_{2k} \tilde{z}_{2k}'\}\}, \tag{22}$$

where the input Ω_k is a vector of standard (unit variance) white noises.

In the following, relations are derived in a closed form for the $\mathscr{H}_2$-norm of G_2. The derivations for G_1 will be very similar and will be discussed later. A discussion of the $\mathscr{H}_2$-norm of G follows.

By using the G_2 subsystem representations in (20),

$$\mathscr{E}\{\tilde{z}_{2k} \tilde{z}_{2k}'\} = \mathscr{E}\{(C\zeta_{2k})(\zeta_{2k}' C')\} = \mathscr{E}\{C L_{2k} C'\}, \tag{23}$$

where

$$L_{2k} = \mathscr{E}\{\zeta_{2k} \zeta_{2k}'\}. \tag{24}$$

Thus,

$$\begin{aligned} L_{2,k+1} &= \mathscr{E}\{(A_k \zeta_{2k} + B_{2k} \Omega_k)(A_k \zeta_{2k} + B_{2k} \Omega_k)'\} \\ &= \mathscr{E}\{A_k L_{2k} A_k' + B_{2k} B_{2k}'\}. \end{aligned} \tag{25}$$

Matrices A_k, B_{2k}, C_k and D_{2k} are dependent on stochastic vectors Δ_k and Γ_k. The δ_{ik}s and γ_{jk}s are Bernoulli distributed white sequences with known mean values of α_is and β_is and variances of q_i^2s and r_j^2s respectively.
The δ_{ik}s can be written as the sum of their mean value and the zero mean stochastic variables λ_{ik}s with the same variance:

$$\delta_{ik} = \alpha_i + \lambda_{ik}, \tag{26}$$

and

$$\mathscr{E}\{\lambda_{ik}\} = 0,\ \mathrm{var}\{\lambda_{ik}\} = q_i^2,\ \mathscr{E}\{\lambda_{ik}\lambda_{js}\} = 0,\ \forall\ k \neq s, \tag{27}$$

or,

$$\Delta_k = A_\alpha + \Lambda_k, \quad \mathrm{var}\{\Lambda\} = Q^2, \tag{28}$$

where

$$A_\alpha = \mathrm{diag}(\alpha_1, \cdots, \alpha_p), \quad Q = \mathrm{diag}(q_1, \cdots, q_p). \tag{29}$$

Similarly,

$$\gamma_{ik} = \beta_i + \theta_{ik}, \tag{30}$$

$$\mathscr{E}\{\theta_{ik}\} = 0,\ \mathrm{var}\{\theta_{ik}\} = r_i^2,\ \mathscr{E}\{\theta_{ik}\theta_{js}\} = 0,\ \forall\ k \neq s. \tag{31}$$

or,

$$\Gamma_k = B_\beta + \Theta_k, \quad \mathrm{var}\{\Theta\} = R^2, \tag{32}$$

where

$$B_\beta = \mathrm{diag}(\beta_1, \cdots, \beta_m), \quad R = \mathrm{diag}(r_1, \cdots, r_m). \tag{33}$$

Now, from (14) and (17), we can write

$$A_k = A + \tilde{A}_{1l}\Delta_k\tilde{A}_{1r} + \tilde{A}_{2l}\Gamma_k\tilde{A}_{2r} + \tilde{A}_{3l}\Delta_k\tilde{A}_{3r}\underline{\Gamma}_k, \tag{34}$$

where

$$\underline{\Gamma}_k = \mathrm{diag}\{0, 0, \Gamma_k, 0\} \tag{35}$$

and A, $\tilde{A}_{il}$ and $\tilde{A}_{ir}$, $\ i = 1, 2, 3$, are known constant matrices defined as follows with appropriate dimensions for 0s and Is in the matrices:

$$A = \begin{bmatrix} \mathbf{a} & 0 & \mathbf{b}_1 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & \mathbf{c}_f & 0 & \mathbf{a}_f \end{bmatrix}, \tag{36}$$

$$\tilde{A}_{1l} = \begin{bmatrix} 0 \\ -I \\ 0 \\ \mathbf{c}_f \end{bmatrix}, \quad \tilde{A}_{2l} = \begin{bmatrix} \mathbf{b}_1 \\ 0 \\ I \\ 0 \end{bmatrix}, \quad \tilde{A}_{3l} = \begin{bmatrix} 0 \\ -I \\ 0 \\ -\mathbf{c}_f \end{bmatrix}, \tag{37}$$

$$\begin{aligned} &\tilde{A}_{1r} = \begin{bmatrix} \mathbf{c} & I & \mathbf{d}_1 & 0 \end{bmatrix}, \quad ,\tilde{A}_{2r} = \begin{bmatrix} 0 & 0 & -I & 0 \end{bmatrix}, \\ &\tilde{A}_{3r} = \begin{bmatrix} 0 & 0 & \mathbf{d}_1 & 0 \end{bmatrix}. \end{aligned} \tag{38}$$

Now, define

$$A_q = \tilde{A}_{1l}Q\tilde{A}_{1r}, \quad A_r = \tilde{A}_{2l}R\tilde{A}_{2r}, \quad A_{qr} = \tilde{A}_{3l}Q\tilde{A}_{3r}\underline{R}, \tag{39}$$

with

$$\underline{R} = \mathrm{diag}\{0, 0, R, 0\}. \tag{40}$$

Considering the independence of states and system matrices, we can write

$$\mathscr{E}\{A_kL_{2k}A_k'\} = AL_{2k}A' + A_qL_{2k}A_q' + A_rL_{2k}A_r' + A_{qr}L_{2k}A_{qr}'. \tag{41}$$

Similar relations can be found for $\mathscr{E}\{B_{2k}B_{2k}'\}$ as follows:

$$\mathscr{E}\{B_{2k}B_{2k}'\} = B_2B_2' + B_{2q}B_{2q}', \tag{42}$$

where

$$B_{2k} = B_2 + \tilde{B}_{2l}\Delta_k\tilde{B}_{2r}, \tag{43}$$

with

$$\tilde{B}_{2l} = \begin{bmatrix} 0 \\ I \\ 0 \\ \mathbf{c}_f \end{bmatrix}, \quad \tilde{B}_{2r} = \mathbf{d}_2, \quad B_2 = \begin{bmatrix} \mathbf{b}_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_{2q} = \tilde{B}_{2l}Q\tilde{B}_{2r}. \tag{44}$$

Putting all these relations into equation (22), we have the following theorem:

Theorem 1. **($\mathscr{H}_{2s}$-norm)**- *Consider G_2, the stable discrete-time linear stochastic parameter system represented in (20). The $\mathscr{H}_{2s}$-norm of the system defined by (22) is*

$$\|G_2\|_{2s}^2 = trace\{CL_cC'\} \tag{45}$$

with

$$\begin{aligned} L_c &= \mathscr{E}\{A_kL_cA_k' + B_{2k}B_{2k}'\} \\ &= B_2B_2' + B_{2q}B_{2q}' + AL_cA' + A_qL_cA_q' + A_rL_cA_r' + A_{qr}L_cA_{qr}'. \end{aligned} \tag{46}$$

So far, we have found the $\mathscr{H}_{2s}$-norm of system G_2 with stochastic input Ω_k as in (20). Following the same method, similar relations are obtained for system G_1 in (19) with a deterministic input. The results are given in the following corollary.

Corollary 1. *Consider G_1, the stable discrete-time linear stochastic parameter system represented in (19). The $\mathscr{H}_{2s}$-norm of the system defined by (21) is*

$$\|G_1\|_{2s}^2 = trace\{CL_cC'\}, \tag{47}$$

where

$$\begin{aligned} L_c &= \mathscr{E}\{A_kL_cA_k' + B_{1k}B_{1k}'\} \\ &= B_1B_1' + B_{1q}B_{1q}' + B_{1qr}B_{1qr}' + AL_cA' + A_qL_cA_q' + \\ &\quad + A_rL_cA_r' + A_{qr}L_cA_{qr}' \end{aligned} \tag{48}$$

with

$$\begin{aligned} &B_{1k} = B_1 + \tilde{B}_{1,1l}\Gamma_k + \tilde{B}_{1,2l}\Delta_k\tilde{B}_{1,2r}\Gamma_k \\ &\tilde{B}_{1,1l} = \begin{bmatrix} \mathbf{b} \\ 0 \\ I \\ 0 \end{bmatrix}, \quad \tilde{B}_{1,2l} = \begin{bmatrix} 0 \\ I \\ 0 \\ \mathbf{c}_f \end{bmatrix}, \quad \tilde{B}_{1,2r} = \mathbf{d}_1, \\ &B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathbf{b}_f \end{bmatrix}, \quad B_{1r} = \tilde{B}_{1,1l}R, \quad B_{1qr} = \tilde{B}_{1,2l}Q\tilde{B}_{1,2r}R. \end{aligned} \tag{49}$$

Now, to combine the stochastic and deterministic inputs, the weighted $\mathscr{H}_2$-norm of G is defined as follows:

$$\|G\|_{2s}^2 = \|G_1\|_{2s}^2 + \rho\|G_2\|_{2s}^2, \tag{50}$$

where $\rho \in \mathbb{R}$ ($\rho > 0$) is a weighting factor.

The following theorem gives the relations for the stochastic weighted $\mathscr{H}_2$-norm of the system G. The proof is straightforward and is omitted.

Theorem 2. *Consider G, the stable discrete-time linear stochastic parameter system represented in (16). The $\mathscr{H}_{2s}$-norm of the system defined by (50) is*

$$\|G\|_{2s}^2 = trace\{CL_cC'\}, \tag{51}$$

where

$$\begin{aligned} L_c =& B_1B_1' + B_{1qr}B_{1qr}' + \rho(B_2B_2' + B_{2q}B_{2q}') + AL_cA' + \\ &+ A_qL_cA_q' + A_rL_cA_r' + A_{qr}L_cA_{qr}'. \end{aligned} \tag{52}$$

IV. Optimal Filter Design

Now, we have the required tools to solve the optimal $\mathscr{H}_2$ filtering problem in the multiple channel NCS framework with multiple packet dropouts. We want to design a filter F as in (15) such that the estimation error variance is minimized. Based on the $\mathscr{H}_{2s}$-norm definition, it is needed to minimize the $\mathscr{H}_{2s}$-norm of the filtering error dynamics to solve the filtering problem.

The problem of $\mathscr{H}_2$ filtering for deterministic discrete-time systems has been studied in the literature (see, e.g., [4], [14] and references therein). The problem has also been considered in stochastic domain in single channel networks (see, e.g., [15]–[17] and references therein). In the following, we try to adapt the filter design problem to the multiple channel cases by using the tools developed in the previous section.

As a generalization of the $\mathscr{H}_{2s}$ filtering in single channel cases ([15], [16]), the $\mathscr{H}_{2s}$ filtering in the multiple channel NCS can be formulated as follows:

$$\min_{\mathbf{a}_f,\mathbf{b}_f,\mathbf{c}_f,L_f,P} \quad \text{trace}(J) \tag{53}$$

$s.t.$

$$\begin{bmatrix} P & PC' \\ CP & J \end{bmatrix} > 0 \tag{54}$$

$$\begin{bmatrix} P & \Xi_A & \Xi_{B_1} & \Xi_{B_2} \\ * & \Xi_P & 0 & 0 \\ * & * & \Xi_{I3} & 0 \\ * & * & * & \Xi_{I2} \end{bmatrix} > 0 \tag{55}$$

where the matrix variables J and P and the matrix inequalities are symmetric, ρ is known, and

$$\begin{aligned} \Xi_A &= \begin{bmatrix} AP & A_qP & A_rP & A_{qr}P \end{bmatrix} \\ \Xi_{B_1} &= \begin{bmatrix} B_1 & B_{1r} & B_{1qr} \end{bmatrix} \\ \Xi_{B_2} &= \rho \begin{bmatrix} B_2 & B_{2q} \end{bmatrix} \\ \Xi_P &= \text{diag}(P,P,P,P) \\ \Xi_{I2} &= \text{diag}(I,I) \\ \Xi_{I3} &= \text{diag}(I,I,I). \end{aligned} \tag{56}$$

Now, it is desirable to convert the two matrix inequalities in (54) and (55) into LMIs. Then, the filter design problem turns out into a convex programming problem that can be solved efficiently by the numerical methods available.

Let us partition P and its inverse as

$$P = \begin{bmatrix} X & U \\ U' & X_2 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} Y & V \\ V' & Y_2 \end{bmatrix}, \tag{57}$$

where X, Y, X_2 and Y_2 are symmetric and positive definite matrices. Now, we define the following nonsingular matrices:

$$\begin{aligned} &\bar{T} = \begin{bmatrix} Z & Y \\ 0 & V' \end{bmatrix}, \quad T_1 = \begin{bmatrix} \bar{T} & 0 \\ 0 & I \end{bmatrix}, \\ &T_2 = \text{diag}(\bar{T},\bar{T},\bar{T},\bar{T},\bar{T},I,I,I,I,I), \end{aligned} \tag{58}$$

where $Z = X^{-1}$, and I is the identity matrix with appropriate dimension. By applying the congruence transformation with T_1 to (54), we get the following LMI:

$$T_1' \begin{bmatrix} P & PC' \\ * & J \end{bmatrix} T_1 = \begin{bmatrix} Z & Z & L' - G' \\ * & Y & L' \\ * & * & J \end{bmatrix} > 0, \tag{59}$$

where $G = L_fU'Z$. We can also get an LMI by applying the congruence transformation with T_2 to (55):

$$T_2' \; [55] \; T_2 > 0, \tag{60}$$

where it is easy to see that

$$\begin{aligned} \bar{T}'P\bar{T} &= \begin{bmatrix} Z & Z \\ Z & Y \end{bmatrix}, \\ \bar{T}'AP\bar{T} &= \begin{bmatrix} Z\mathbf{a} & Z\mathbf{a} \\ Y\mathbf{a} + F\mathbf{c} + Q & Y\mathbf{a} + F\mathbf{c} \end{bmatrix} \\ \bar{T}'A_*P\bar{T} &= \begin{bmatrix} Z\mathbf{a}_* & Z\mathbf{a}_* \\ Y\mathbf{a}_* + F\mathbf{c}_* & Y\mathbf{a}_* + F\mathbf{c}_* \end{bmatrix} \\ \bar{T}'B_1 &= \begin{bmatrix} Z\mathbf{b}_1 \\ Y\mathbf{b}_1 + M + F\mathbf{d}_1 \end{bmatrix}, \quad \bar{T}'B_2 = \begin{bmatrix} Z\mathbf{b}_2 \\ Y\mathbf{b}_2 + F\mathbf{d}_2 \end{bmatrix} \\ \bar{T}'B_{1,*} &= \begin{bmatrix} Z\mathbf{b}_{1*} \\ Y\mathbf{b}_{1*} + F\mathbf{d}_{1*} \end{bmatrix}, \quad \bar{T}'B_{2,*} \begin{bmatrix} Z\mathbf{b}_{2*} \\ Y\mathbf{b}_{2*} + F\mathbf{d}_{2*} \end{bmatrix}, \end{aligned} \tag{61}$$

where $F = V\mathbf{c}_f$, $Q = V\mathbf{a}_fU'Z$ and $M = V\mathbf{b}_f$. Also, $*$ stands for q, r or qr. Thus, the results can be summarized as follows.

Theorem 3. **($\mathscr{H}_{2s}$-filtering)** *The filter design problem of (53)-(55) is equivalent to the following convex programming problem:*

$$\begin{aligned} &\min_{Z,Y,Q,F,G,M} \quad trace(J) \\ &s.t. \\ &\qquad (59) \text{ and } (60). \end{aligned} \tag{62}$$

To find the filter parameters, $\mathbf{a}_f$, $\mathbf{b}_f$, $\mathbf{c}_f$ and L_f, we need to know U and V, which do not appear in the LMIs. One of the matrices U or V can be defined freely. Different choices give us different filter state-space realizations. One logical choice is to set $L_f = L$ that can come from setting $V = V' = -Y$, leading to $U = U' = Z^{-1} - Y^{-1}$.

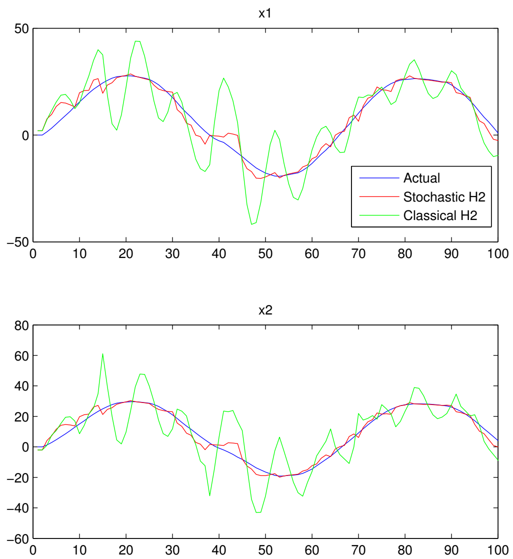

Fig. 2. Actual and estimated states for the classical and stochastic $\mathscr{H}_2$ filtering, $\alpha_1 = 0.2$, $\alpha_2 = 0.48$, $\beta_1 = 0.5$, $\beta_2 = 0.4$ and $\rho = 1$

V. Example

A simulation example is given in this section to support the developed theory. Consider a discrete-time LTI system represented by (13) with the following matrix values:

$$\mathbf{a} = \begin{bmatrix} 1.7240 & -0.7788 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 1 & 1 \\ 0.5 & 1 \end{bmatrix},$$
$$\mathbf{b}_2 = \begin{bmatrix} 0.05 & 0.05 \\ 0.1 & 0.01 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 0.0286 & 0.0264 \\ 0.286 & 0.0264 \end{bmatrix}, \tag{63}$$
$$\mathbf{d}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0.5 \end{bmatrix} \quad \mathbf{d}_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0.05 & 0.1 \end{bmatrix}, \quad L = I_2,$$

where I_2 is an identity matrix with the size of 2. The initial state values are $\tilde{x}(0) = [0\ \ 0]'$ and $\hat{\tilde{x}}(0) = [2\ \ -2]'$. The system states and their estimates due to sinusoidal input are plotted in Figure 2. Note that the controller is not designed here. It is assumed that it simply sends some sinusoidal commands. This figure shows the simulation results for the case when the average sensor to the controller and the controller to the actuator dropout rate are $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta_1 = 0.5$ and $\beta_2 = 0.4$, respectively, with a weighting factor of $\rho = 1$. This result shows the superiority of the proposed $\mathscr{H}_{2s}$ filtering over the classical one. In the classical method, no compensation is made for the dropouts; in the case of no new information, simply the previous ones are used.

VI. Conclusions

In this paper, the problem of optimal $\mathscr{H}_2$ filtering in the multiple channel NCS environment with multiple packet dropouts has been studied. The stochastic $\mathscr{H}_2$-norm of systems containing stochastic parameters was defined, and the relations were developed. A weighted $\mathscr{H}_2$-norm was generalized to be used in systems with both deterministic and stochastic inputs. Based on the new derivations, the problem was transformed into a set of LMIs that can be easily solved by existing software packages. A simulation example showed the effectiveness and applicability of the proposed method.

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