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Set-membership Filtering of Uncertain Discrete-Time Rational Systems through Recursive Algebraic Representations and LMIs

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Abstract— **This paper is about robust filtering/prediction of nonlinear discrete-time systems with rational dependence on the state or uncertainty vectors. The problem is dealt with in a set-membership context, in that the system initial condition, the uncertainties, as well as the noises affecting the dynamics and the measurements are unknown but bounded. A new recursive approach to the prediction of confidence ellipsoids enclosing the state vector at each sampling time is proposed, based on the reformulation of the system model as a Recursive Algebraic Representation. The solution is expressed as a convex optimization problem under Linear Matrix Inequality (LMI) constraints. The method is assessed on some examples.**
Keywords: Set-membership filtering/prediction; Nonlinear rational systems; Recursive Algebraic Representations; Linear Matrix Inequalities.

I. INTRODUCTION

Two main paradigms underly the state estimation of a dynamic system in the presence of disturbances. One of them consists in formulating the problem in a stochastic context. Given a known system, a probabilistic description of its initial condition and of the noises affecting its dynamics and measurements, Bayesian filters aim at computing the posterior probability density function of the state vector conditioned on the measured output trajectory. Within this class of estimators, the celebrated Kalman filter [1] provides a closed-form solution for linear Gaussian systems. The problem gets significantly more difficult in the nonlinear/nonGaussian cases, and approximations are sought for. Linearizing the system model around its estimated trajectory, as done in the extended Kalman filter algorithm, can lead to biased and/or unconsistent estimates [2]. As an alternative, the Unscented Kalman filter [3] provides an approximation of the true posterior mean and covariance with a sound theoretical analysis of performance. Nevertheless, for high nonlinearities and/or multimodal posterior pdfs, one must return to Monte Carlo methods [4].

While H_2 (Kalman) filters, based on a deterministic or stochastic least-squares criterion, have led to real-time implementations in a wide range of applications—ranging from navigation systems to economics, signal processing applications, robotics, etc.—they assume a perfect model of the system and a complete characterization of disturbances. To obtain estimators with guaranteed behaviors even if disturbances are not perfectly known, the “worst-case” H_∞ framework has been developed [5]. Robust H_2 and H_∞ filtering for linear sytems affected by parametric uncertainty and for some classes of uncertain nonlinear systems is still an active field of research, see for instance [6][7][8].

An alternative paradigm to filtering relies on *guaranteed state estimation*. Given a prior *set-membership*—or *unknown-but-bounded*—description of the system initial condition and of the disturbances affecting it, the aim is to propagate over time *confidence sets* enclosing the realizations of the state vector which are consistent with the measurements [9]. Results have been obtained considering several shapes of confidence sets, e.g. based on interval analysis [10] or ellipsoidal calculus [11]. Guaranteed state filtering presents the advantage of being effective even if stochastic assumptions cannot be strictly justified. Its potential pitfalls are the complexity of the involved computations and the conservativeness of the computed confidence sets. The applications are also many, e.g. signal processing applications, robots localization, fault detection, etc.

Significant work has been done towards the guaranteed state estimation of uncertain systems. Among the most advanced solutions, ellipsoidal confidence ellipsoids have been obtained in [12][13] for discrete-time linear systems with linear fractional parametric uncertainty. Thanks to a quadratic outer approximation of the uncertainty, the authors turn the problem into a convex optimization program under Linear Matrix Inequalities (LMI) constraints [14], which enjoys nice tractability properties. After embedding nonlinearities or Taylor expansion residuals into uncertainty, this strategy has been successfully applied to practical problems entailing the state estimation of nonlinear uncertain systems, e.g. the visual-based pose estimation of a camera in [15] and the localization of mobile robots in [16]. In [15], the visual interactions are modeled through state and output equations with rational dependence on the state. The versatility of such so-called nonlinear rational models has been acknowledged for long in order to tackle the control of many physical devices, e.g. power generators [17], visual servos [18] or chemical reactors [19], to cite few.

This paper attempts to propose a new LMI approach to the prediction of confidence ellipsoids for uncertain nonlinear rational systems. It is organized as follows. The problem is first formulated in Section II. A recursive approach is then developed in Sections III and IV, based on the reformulation of the system model as a Recursive Algebraic Representation. Also, some remarks and further extensions are discussed. Numerical experiments on two case studies constitute Section V. Some conclusions and prospects end the paper.

II. PROBLEM STATEMENT

A. Notation

The notation is standard. A column vector is denoted by an underlined lowercase letter, e.g. $\underline{v}$, and its entries have the form $v_1, v_2, \ldots$. Capital letters are used for matrices, e.g. M. The zero vector is termed $\underline{0}$, while $\mathbb{I}$ and $\mathbb{O}$ stand for the identity and zero matrices. These vectors/matrices may be subscripted by their dimensions—e.g. $\underline{0}_n$, $\mathbb{I}_n$, $\mathbb{O}_{n\times m}$—but these are omitted whenever they can be inferred from the context. The transpose operator is represented by $'$. The symbol $\underline{e}_i$ denotes the canonical column vector of appropriate dimensions whose i^{th} entry is 1 and other entries are 0. Last, the notation $M > 0$ (resp. $M \geq 0$) means that M is symmetric and positive definite (resp. positive semidefinite.)

As for sets, the notation $\mathscr{V}(\mathscr{X}_k)$ stands for the set of vertices of a polytope $\mathscr{X}_k$. The operator $\times$ performs the Cartesian product of two sets.

B. The considered problem

Consider the following discrete-time uncertain nonlinear system

$$\begin{aligned}\underline{x}_{k+1} &= A(\underline{x}_k,\underline{\delta}_k)\underline{x}_k + B_{\underline{w}}(\underline{x}_k,\underline{\delta}_k)\underline{w}_k + B_{\underline{u}}(\underline{x}_k,\underline{\delta}_k)\underline{u}_k\\ \underline{y}_k &= C(\underline{x}_k,\underline{\delta}_k)\underline{x}_k + D_{\underline{w}}(\underline{x}_k,\underline{\delta}_k)\underline{w}_k + D_{\underline{u}}(\underline{x}_k,\underline{\delta}_k)\underline{u}_k\end{aligned} \tag{1}$$

where k terms the time index, and the vectors $\underline{x} \in \mathbb{R}^{n_{\underline{x}}}$, $\underline{\delta} \in \mathbb{R}^{n_{\underline{\delta}}}$, $\underline{w} \in \mathbb{R}^{n_{\underline{w}}}$, $\underline{y} \in \mathbb{R}^{n_{\underline{y}}}$, $\underline{u} \in \mathbb{R}^{n_{\underline{u}}}$ respectively gather the states, uncertainties, noises, outputs and control inputs. The system (1) is said rational iff the matrix functions $A(.,.)$, $B_{\underline{w}}(.,.)$, $B_{\underline{u}}(.,.)$, $C(.,.)$, $D_{\underline{w}}(.,.)$, $D_{\underline{u}}(.,.)$ are rational in their arguments. The uncertainty vector is assumed to be time-varying and to lie in a given convex polytope $\Delta_k \subset \mathbb{R}^{n_{\underline{\delta}}}$ at time k. The matrix functions involved in (1) are assumed well defined on $\mathscr{X}_k \times \Delta_k$, with $\mathscr{X}_k$ a given convex polytope of $\mathbb{R}^{n_{\underline{x}}}$.

A confidence ellipsoid $\mathscr{E}_k$ enclosing the state vector at time k being given, together with a set-valued description of the noise vector $\underline{w}_k$ and the knowledge of $\underline{u}_k$ and $\underline{y}_k$, the aim is to compute an optimized confidence ellipsoid $\mathscr{E}_{k+1}$ surrounding the state vector $\underline{x}_{k+1}$ at next time $k+1$. This is a nonlinear prediction problem in a set-membership context. More precisely, $\mathscr{E}_k$ inside $\mathscr{X}_k$ is assumed to be described by

$$\underline{x}_k \in \mathscr{E}_k \triangleq \{\underline{x} : \underline{x} = \hat{\underline{x}}_k + E_k\underline{z},\ \underline{z} \in \mathscr{Z}_k\} \tag{2}$$

$$\text{with } \mathscr{Z}_k \triangleq \{\underline{z} \in \mathbb{R}^{n_{\underline{x}}},\ ||\underline{z}|| \leq 1\}, \tag{3}$$

where $\hat{\underline{x}}_k$ and $E_k \in \mathbb{R}^{n_{\underline{x}}\times n_{\underline{x}}}$ respectively term $\mathscr{E}_k$'s center and "shape matrix". E_k is supposed full-rank, or, equivalently, the "squared shape matrix" $P_k = E_kE_k'$ satisfies $P_k > 0$, so that (2)–(3) are equivalent to

$$\underline{x}_k \in \mathscr{E}_k \triangleq \{\underline{x} : (\underline{x} - \hat{\underline{x}}_k)'P_k^{-1}(\underline{x} - \hat{\underline{x}}_k) \leq 1\}. \tag{4}$$

A "minimum size" ellipsoid

$$\mathscr{E}_{k+1} \triangleq \{\underline{x} : (\underline{x} - \hat{\underline{x}}_{k+1})'P_{k+1}^{-1}(\underline{x} - \hat{\underline{x}}_{k+1}) \leq 1\} \tag{5}$$

is then sought for, enclosing the values of $\underline{x}_{k+1}$ which satisfy (1) upon the data of $\underline{u}_k$ and $\underline{y}_k$, whatever $\underline{x}_k$ in $\mathscr{E}_k$, $\underline{\delta}_k$ in Δ_k, and the noise vector $\underline{w}_k$ in

$$\mathscr{W}_k \triangleq \{\underline{w} : \underline{w}'Q_m\underline{w} \leq 1,\ m = 1,\ldots,m_{\underline{w}}\}. \tag{6}$$

The "size" of $\mathscr{E}_{k+1}$, henceforth denoted by $f(P_{k+1})$, is related to its shape matrix. In the following, $f(P_{k+1})$ will be either $\operatorname{trace}(P_{k+1})$ or $\log\det P_{k+1}$, depending on whether the sum of the squared semi-axes lengths or the volume of $\mathscr{E}_{k+1}$ must be minimized [14].

III. PRELIMINARY RESULTS

A. Recursive algebraic representations

In the vein of [20], which deals with the control of continuous-time rational systems, the starting point of the method proposed hereafter is to turn (1) into the following *Recursive Algebraic Representation*

$$\begin{pmatrix}\underline{x}_{k+1}\\ \underline{y}_k\\ \underline{0}\end{pmatrix} = \begin{pmatrix} A_1(\underline{x}_k,\underline{\delta}_k) & A_2(\underline{x}_k,\underline{\delta}_k) & A_3(\underline{x}_k,\underline{\delta}_k) & A_4(\underline{x}_k,\underline{\delta}_k)\\ C_1(\underline{x}_k,\underline{\delta}_k) & C_2(\underline{x}_k,\underline{\delta}_k) & C_3(\underline{x}_k,\underline{\delta}_k) & C_4(\underline{x}_k,\underline{\delta}_k)\\ \Omega_1(\underline{x}_k,\underline{\delta}_k) & \Omega_2(\underline{x}_k,\underline{\delta}_k) & \Omega_3(\underline{x}_k,\underline{\delta}_k) & \Omega_4(\underline{x}_k,\underline{\delta}_k)\end{pmatrix}\begin{pmatrix}\underline{x}_k\\ \underline{\pi}_k(\underline{x}_k,\underline{\delta}_k)\\ \underline{w}_k\\ \underline{u}_k\end{pmatrix} \tag{7}$$

—or RAR—where $\underline{\pi}_k(\underline{x}_k,\underline{\delta}_k)$ terms a nonlinear vector function of $(\underline{x}_k,\underline{\delta}_k)$, and $A_1(.,.)$, $A_2(.,.)$, $A_3(.,.)$, $A_4(.,.)$, $C_1(.,.)$, $C_2(.,.)$, $C_3(.,.)$, $C_4(.,.)$, $\Omega_1(.,.)$, $\Omega_2(.,.)$, $\Omega_3(.,.)$, $\Omega_4(.,.)$ are affine matrix functions. The genuine state space model (1) can be recovered from the RAR (7) by eliminating $\underline{\pi}_k(\underline{x}_k,\underline{\delta}_k)$ if $\Omega_2(\underline{x}_k,\underline{\delta}_k)$ is column full-rank for all $(\underline{x}_k,\underline{\delta}_k) \in \mathscr{X}_k \times \Delta_k$.

From the same arguments of continuous-time systems [21], we can show the RAR is equivalent to the Nonlinear Fractional Transformation (NFT) representation of [22] when the matrices $A_1(.),\ldots,\Omega_4(.)$ are only parameter dependent. Besides, the RAR can model the whole class of rational functions with no singularities in the domain of interest and, through suitable change of variables, some trigonometric nonlinearities can be also modeled via the RAR framework [21].

B. An equivalent problem

Let the function $\underline{z}_k(.) : \mathscr{X}_k \rightarrow \mathbb{R}^{n_{\underline{x}}}$ map $\underline{x}_k$ into the unique vector $\underline{z}_k(\underline{x}_k)$ such that $\underline{x}_k = \hat{\underline{x}}_k + E_k\underline{z}_k(\underline{x}_k)$, and $\underline{\eta}(.,.) : \mathscr{X}_k \times \Delta_k \rightarrow \mathbb{R}^{n_{\underline{\eta}}}$ the vector function such that $\underline{\eta}(\underline{x}_k,\underline{\delta}_k) = \begin{pmatrix}\underline{z}_k'(\underline{x}_k) & \underline{\pi}_k'(\underline{x}_k,\underline{\delta}_k) & \underline{w}_k' & 1\end{pmatrix}'$, with $n_{\underline{\eta}} = n_{\underline{x}} + n_{\underline{\pi}} + n_{\underline{w}} + 1$. Then, the RAR in (7) can be written in terms of $\underline{d}_{k+1} \triangleq \underline{x}_{k+1} - \hat{\underline{x}}_{k+1}$ as

$$\begin{pmatrix}\underline{d}_{k+1}\\ \underline{0}\end{pmatrix} = \begin{pmatrix}\Phi_{(\underline{x}_k,\underline{\delta}_k)}(\hat{\underline{x}}_{k+1})\\ \Psi_0(\underline{x}_k,\underline{\delta}_k)\end{pmatrix}\underline{\eta}(\underline{x}_k,\underline{\delta}_k) \tag{8}$$

where the matrix functions $\Phi_{(.,.)}(\hat{\underline{x}}_{k+1}), \Psi_0(.,.)$ satisfy

$$\begin{aligned}\Phi_{(\underline{x},\underline{\delta})}(\hat{\underline{x}}_{k+1}) &\triangleq \begin{pmatrix} A_1(\underline{x},\underline{\delta})E_k & A_2(\underline{x},\underline{\delta}) & A_3(\underline{x},\underline{\delta}) & (A_1(\underline{x},\underline{\delta})\hat{\underline{x}}_k - \hat{\underline{x}}_{k+1} + A_4(\underline{x},\underline{\delta})\underline{u}_k)\end{pmatrix}\\ \Psi_0(\underline{x},\underline{\delta}) &\triangleq \begin{pmatrix} C_1(\underline{x},\underline{\delta})E_k & C_2(\underline{x},\underline{\delta}) & C_3(\underline{x},\underline{\delta}) & (C_1(\underline{x},\underline{\delta})\hat{\underline{x}}_k - \underline{y}_k + C_4(\underline{x},\underline{\delta})\underline{u}_k)\\ \Omega_1(\underline{x},\underline{\delta})E_k & \Omega_2(\underline{x},\underline{\delta}) & \Omega_3(\underline{x},\underline{\delta}) & (\Omega_1(\underline{x},\underline{\delta})\hat{\underline{x}}_k + \Omega_4(\underline{x},\underline{\delta})\underline{u}_k)\end{pmatrix}.\end{aligned} \tag{9}$$

Further, set $N_1 \triangleq \underline{e}'_{n_{\underline{\eta}}} = \begin{pmatrix} \underline{0}'_{(n_{\underline{\eta}}-1)} & 1 \end{pmatrix}$, with $N_1 \in \mathbb{R}^{1\times n_{\underline{\eta}}}$, and

$$\mathsf{N}_{(\underline{x}_k,\underline{\delta}_k)}(\underline{\hat{x}}_{k+1},P_{k+1}) \triangleq \Phi'_{(\underline{x}_k,\underline{\delta}_k)}(\underline{\hat{x}}_{k+1})P^{-1}_{k+1}\Phi_{(\underline{x}_k,\underline{\delta}_k)}(\underline{\hat{x}}_{k+1}) - N'_1N_1. \tag{10}$$

The prediction problem at hand can then be turned into

$$\min_{\underline{\hat{x}}_{k+1},P_{k+1}} f(P_{k+1}) \tag{11}$$

subject to $P_{k+1} > 0$
and to $\underline{\eta}'(\underline{x}_k,\underline{\delta}_k)\mathsf{N}_{(\underline{x}_k,\underline{\delta}_k)}(\underline{\hat{x}}_{k+1},P_{k+1})\underline{\eta}(\underline{x}_k,\underline{\delta}_k) \le 0$
$\quad \forall(\underline{x}_k,\underline{\delta}_k,\underline{w}_k) \in \mathscr{E}_k\times\Delta_k\times\mathbb{R}^{n_{\underline{w}}}$ s.t. $\underline{w}_k \in \mathscr{W}_k$ and $\underline{z}_k(\underline{x}_k) \in \mathscr{Z}_k$.

C. Towards a sufficient condition with limited conservativeness

Define the matrices $N_{\underline{z}} \triangleq \begin{pmatrix} \mathbb{I}_{n_{\underline{x}}} & \mathbb{O}_{n_{\underline{x}}\times(n_{\underline{\pi}}+n_{\underline{w}})} & 0 \\ \underline{0}'_{n_{\underline{x}}} & \underline{0}'_{(n_{\underline{\pi}}+n_{\underline{w}})} & 1 \end{pmatrix}$ and $N_{\underline{w}} \triangleq \begin{pmatrix} \mathbb{O}_{n_{\underline{w}}\times(n_{\underline{x}}+n_{\underline{\pi}})} & \mathbb{I}_{n_{\underline{w}}} & 0 \\ \underline{0}'_{(n_{\underline{x}}+n_{\underline{\pi}})} & \underline{0}'_{n_{\underline{w}}} & 1 \end{pmatrix}$, with $N_{\underline{z}} \in \mathbb{R}^{(n_{\underline{x}}+1)\times n_{\underline{\eta}}}$ and $N_{\underline{w}} \in \mathbb{R}^{(n_{\underline{w}}+1)\times n_{\underline{\eta}}}$. Consequently, for all $(\underline{x}_k,\underline{\delta}_k)$ in $\mathscr{X}_k\times\Delta_k$, one gets $N_1\underline{\eta}(\underline{x}_k,\underline{\delta}_k) = 1$, $N_{\underline{z}}\underline{\eta}(\underline{x}_k,\underline{\delta}_k) = \begin{pmatrix}\underline{z}'_k(\underline{x}_k) & 1\end{pmatrix}'$ and $N_{\underline{w}}\underline{\eta}(\underline{x}_k,\underline{\delta}_k) = \begin{pmatrix}\underline{w}'_k & 1\end{pmatrix}'$. Also introduce the following matrices Υ_m, $m = 0,\ldots,m_{\underline{w}}$:

$$\begin{aligned} \Upsilon_0 &\triangleq N'_{\underline{z}}\begin{pmatrix} \mathbb{I}_{n_{\underline{x}}} & \underline{0} \\ \underline{0}' & -1 \end{pmatrix} N_{\underline{z}}, \\ \Upsilon_m &\triangleq N'_{\underline{w}}\begin{pmatrix} Q_m & \underline{0} \\ \underline{0}' & -1 \end{pmatrix} N_{\underline{w}},\ m = 1,\ldots,m_{\underline{w}}. \end{aligned} \tag{12}$$

Then, $\underline{w}_k \in \mathscr{W}_k$ and $\underline{z}_k(\underline{x}_k) \in \mathscr{Z}_k$, with $\mathscr{W}_k$ and $\mathscr{Z}_k$ specified in (6) and (3) are equivalent to

$$\underline{\eta}'(\underline{x}_k,\underline{\delta}_k)\Upsilon_m\underline{\eta}(\underline{x}_k,\underline{\delta}_k) \le 0,\ m = 0,\ldots,m_{\underline{w}}. \tag{13}$$

By applying the $\mathscr{S}$-procedure [14], a sufficient condition for the decision variables $\underline{\hat{x}}_{k+1},P_{k+1}$ to satisfy the constraints of (11) is as follows:

$$\begin{aligned} & P_{k+1} > 0 \\ & \text{and } \exists\tau_0 \ge 0, \tau_1 \ge 0,\ldots,\tau_{m_{\underline{w}}} \ge 0 \text{ such that} \\ & \quad \forall(\underline{x}_k,\underline{\delta}_k) \in \mathscr{E}_k\times\Delta_k, \\ & \quad\quad \underline{\eta}'(\underline{x}_k,\underline{\delta}_k)\mathsf{M}_{(\underline{x}_k,\underline{\delta}_k)}(\underline{\hat{x}}_{k+1},P_{k+1},\tau_0,\ldots,\tau_{m_{\underline{w}}})\underline{\eta}(\underline{x}_k,\underline{\delta}_k) \le 0, \\ & \quad\quad \text{with } \mathsf{M}_{(\underline{x}_k,\underline{\delta}_k)}(\underline{\hat{x}}_{k+1},P_{k+1},\tau_0,\ldots,\tau_{m_{\underline{w}}}) \triangleq \\ & \quad\quad\quad \mathsf{N}_{(\underline{x}_k,\underline{\delta}_k)}(\underline{\hat{x}}_{k+1},P_{k+1}) - \textstyle\sum_{m=0}^{m_{\underline{w}}}\tau_m\Upsilon_m. \end{aligned} \tag{14}$$

Rather than requiring $\mathsf{M}_{(\underline{x},\underline{\delta})}(\underline{\hat{x}}_{k+1},P_{k+1},\tau_0,\ldots,\tau_{m_{\underline{w}}})$ to be negative semidefinite over $\mathscr{E}_k\times\Delta_k$, the bounding polytope $\mathscr{X}_k$ is considered, which satisfies $\mathscr{E}_k \subset \mathscr{X}_k$ by assumption, and the following lemma is used, from [20][17]:

Lemma 1: Define two vectors $\underline{x}$ and $\underline{\delta}$ such that $(\underline{x},\underline{\delta})$ belongs to $\mathscr{X}\times\Delta$, with $\mathscr{X} \subset \mathbb{R}^{n_{\underline{x}}}$ and $\Delta \subset \mathbb{R}^{n_{\underline{\delta}}}$ two given convex polytopes. Let $\Sigma_0(.,.) = \Sigma'_0(.,.)$ be an affine matrix function on $\mathscr{X}\times\Delta$ taking its values in $\mathbb{R}^{n_\sigma\times n_\sigma}$. A nonlinear vector function $\underline{\sigma}(.,.) : \mathscr{X}\times\Delta \longrightarrow \mathbb{R}^{n_\sigma}$ being prescribed, consider the following constraint

$$\forall(\underline{x},\underline{\delta}) \in \mathscr{X}\times\Delta,\ \underline{\sigma}'(\underline{x},\underline{\delta})\Sigma_0(\underline{x},\underline{\delta})\underline{\sigma}(\underline{x},\underline{\delta}) \le 0. \tag{15}$$

If an affine matrix function $\Sigma_1(.,.) \in \mathbb{R}^{m_\sigma\times n_\sigma}$ can be exhibited such that $\Sigma_1(\underline{x},\underline{\delta})\underline{\sigma}(\underline{x},\underline{\delta}) = \underline{0}$ always holds on $\mathscr{X}\times\Delta$, the following sufficient condition to (15)*—much less conservative than requiring $\Sigma_0(\underline{x},\underline{\delta}) \le 0$ over $\mathscr{X}\times\Delta$—can be drawn:*

$$\begin{aligned} & \exists L \in \mathbb{R}^{n_\sigma\times m_\sigma} \text{ such that } \forall(\underline{x},\underline{\delta}) \in \mathscr{V}(\mathscr{X}\times\Delta), \\ & \quad \Sigma(\underline{x},\underline{\delta}) \triangleq \Sigma_0(\underline{x},\underline{\delta}) + L\Sigma_1(\underline{x},\underline{\delta}) + \Sigma'_1(\underline{x},\underline{\delta})L' \le 0. \end{aligned} \tag{16}$$

This is a set of LMIs on $L \in \mathbb{R}^{n_\sigma\times m_\sigma}$, each one computed at an element of $\mathscr{V}(\mathscr{X}\times\Delta)$, i.e. at a vertex of $\mathscr{X}\times\Delta$. ■

Proof: By convexity, (16) holds for all $(\underline{x},\underline{\delta})$ in $\mathscr{X}\times\Delta$. Pre- and post- multiplying $\Sigma(\underline{x},\underline{\delta})$ by $\underline{\sigma}(\underline{x},\underline{\delta})$ then leads to (15). The sufficient condition (16) can be viewed as the application of the Finsler's lemma [14] to the problem of satisfying (15) for all $(\underline{x},\underline{\delta})$ such that $\Sigma_1(\underline{x},\underline{\delta})\underline{\sigma}(\underline{x},\underline{\delta}) = \underline{0}$, by adjoining this last constraint through a matrix multiplier L which is independent of $(\underline{x},\underline{\delta})$. ■

The matrix function $\Sigma_1(.,.)$ is termed a *linear annihilator* of $\underline{\sigma}(.,.)$, for it is affine in its arguments and because its post-multiplication by $\underline{\sigma}(.,.)$ gives $\underline{0}$. Importantly, the conservativeness of (16) is all the less important as extra independent lines are stacked to $\Sigma_1(.,.)$.

In view of the above arguments, a linear annihilator $\Psi_{\underline{\eta}}(.,.)$ of $\underline{\eta}(.,.)$, satisfying

$$\forall(\underline{x}_k,\underline{\delta}_k) \in \mathscr{X}_k\times\Delta_k,\ \Psi_{\underline{\eta}}(\underline{x}_k,\underline{\delta}_k)\underline{\eta}(\underline{x}_k,\underline{\delta}_k) = \underline{0} \tag{17}$$

must be determined so as to deduce a sufficient condition to (14) with reduced conservativeness through Lemma 1. Note that despite (14) entails a quadratic form on $\underline{\eta}(.,.)$ to be satisfied at each $(\underline{x}_k,\underline{\delta}_k) \in \mathscr{E}_k\times\Delta_k$, to obtain tractable conditions the outer polytope $\mathscr{X}_k$ is considered instead of $\mathscr{E}_k$. Indeed, from convexity arguments, if LMIs like (16)—which depend affinely on $(\underline{x}_k,\underline{\delta}_k)$—are satisfied for all $(\underline{x}_k,\underline{\delta}_k) \in \mathscr{V}(\mathscr{X}_k\times\Delta_k)$, then they also hold for all $(\underline{x}_k,\underline{\delta}_k) \in \mathscr{E}_k\times\Delta_k$. A straight choice is to set the matrix function $\Psi_{\underline{\eta}}(.,.)$ to $\bar{\Psi}_{\underline{\eta}}(.,.)$ defined herebelow:

$$\bar{\Psi}_{\underline{\eta}}(\underline{x}_k,\underline{\delta}_k) \triangleq \begin{pmatrix} \Psi_0(\underline{x}_k,\underline{\delta}_k) \\ E_k\ \ \mathbb{O}\ \ \mathbb{O}\ \ (\underline{\hat{x}}_k - \underline{x}) \end{pmatrix}. \tag{18}$$

Yet, less pessimistic conclusions can be drawn by selecting

$$\Psi_{\underline{\eta}}(\underline{x}_k,\underline{\delta}_k) = \begin{pmatrix} \bar{\Psi}_{\underline{\eta}}(\underline{x}_k,\underline{\delta}_k) \\ \Psi_{\underline{z}}(\underline{x}_k,\underline{\delta}_k)\ \ \mathbb{O}\ \ \mathbb{O}\ \ \underline{0} \end{pmatrix}, \tag{19}$$

where $\Psi_{\underline{z}}(\underline{x}_k,\underline{\delta}_k)$ is specified as a linear annihilator of $\underline{z}_k(\underline{x}_k,\underline{\delta}_k)$, e.g.

$$\Psi_{\underline{z}}(\underline{x}_k,\underline{\delta}_k) = \begin{pmatrix} z_2 & -z_1 & 0 & \cdots & 0 & 0 \\ z_3 & 0 & -z_1 & \cdots & 0 & 0 \\ z_{n_{\underline{x}}} & 0 & 0 & \cdots & 0 & -z_1 \\ 0 & z_3 & -z_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & z_{n_{\underline{x}}} & 0 & \cdots & 0 & -z_2 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & z_{n_{\underline{x}}} & -z_{(n_{\underline{x}}-1)} \end{pmatrix},\ z_i = \underline{e}'_iE_k^{-1}(\underline{x}_k - \underline{\hat{x}}_k). \tag{20}$$

The main result can now be stated.

IV. AN LMI SOLUTION TO SET-MEMBERSHIP FILTERING

A. Main result

Theorem 1: The solutions $\underline{\hat{x}}_{k+1}$ *and* P_{k+1} *to the following problem also satisfy* (11)*:*

$$\min_{\underline{\hat{x}}_{k+1},P_{k+1},\tau_0,\ldots,\tau_{m_{\underline{w}}},L} f(P_{k+1}) \tag{21}$$

subject to $P_{k+1} > 0$,

$$\tau_0 \geq 0, \tau_1 \geq 0, \ldots, \tau_{m_{\underline{w}}} \geq 0$$

and

$$\left(\begin{array}{c|c} -P_{k+1} & \Phi_{(\underline{x},\underline{\delta})}(\underline{\hat{x}}_{k+1}) \\ \hline \Phi'_{(\underline{x},\underline{\delta})}(\underline{\hat{x}}_{k+1}) & -N_1'N_1 - \sum_{m=0}^{m_{\underline{w}}} \tau_m \Upsilon_m + L\Psi_{\underline{\eta}}(\underline{x},\underline{\delta}) + \Psi'_{\underline{\eta}}(\underline{x},\underline{\delta})L' \end{array}\right) \leq 0$$

for all $(\underline{x},\underline{\delta}) \in \mathscr{V}(\mathscr{X}_k \times \Delta_k)$,

with $\Phi_{(.,.)}(.)$, Υ_0, Υ_m *introduced in* (9)–(12)*,* $\Psi_{\underline{\eta}}(.,.)$ *defined in* (19)–(18)–(12)*, and L a matrix of appropriate dimensions. If* $f(P_{k+1}) = \mathrm{trace}(P_{k+1})$*, this is a convex problem.* ∎

Proof: For all $(\underline{x},\underline{\delta}) \in \mathscr{X}_k \times \Delta_k$, and thus for all $(\underline{x},\underline{\delta}) \in \mathscr{E}_k \times \Delta_k$, one has $\Psi_{\underline{\eta}}(\underline{x},\underline{\delta})\underline{\eta}(\underline{x},\underline{\delta}) = \underline{0}$. Then, by Lemma 1, a sufficient condition for $\underline{\eta}'(\underline{x},\underline{\delta})\mathsf{M}_{(\underline{x},\underline{\delta})}(\underline{\hat{x}}_{k+1},P_{k+1},\tau_0,\ldots,\tau_{m_{\underline{w}}})\underline{\eta}(\underline{x},\underline{\delta}) \leq 0$, appearing in (14), to hold for all $(\underline{x},\underline{\delta}) \in \mathscr{E}_k \times \Delta_k$ is that there exists a matrix multiplier L of appropriate dimensions such that

$$\begin{aligned} &\forall(\underline{x},\underline{\delta}) \in \mathscr{V}(\mathscr{X}_k \times \Delta_k), \\ &\mathsf{M}_{(\underline{x},\underline{\delta})}(\underline{\hat{x}}_{k+1},\ldots\ldots,\tau_{m_{\underline{w}}}) + L\Psi_{\underline{\eta}}(\underline{x},\underline{\delta}) + \Psi'_{\underline{\eta}}(\underline{x},\underline{\delta})L' \leq 0. \end{aligned}$$

The above matrix inequality expands into

$$\begin{aligned} &\Phi'_{(\underline{x},\underline{\delta})}(\underline{\hat{x}}_{k+1})P_{k+1}^{-1}\Phi_{(\underline{x},\underline{\delta})}(\underline{\hat{x}}_{k+1}) - N_1'N_1 \ldots \\ &\quad \ldots - \sum_{m=0}^{m_{\underline{w}}} \tau_m \Upsilon_m + L\Psi_{\underline{\eta}}(\underline{x},\underline{\delta}) + \Psi'_{\underline{\eta}}(\underline{x},\underline{\delta})L' \leq 0. \end{aligned} \tag{22}$$

This expression is turned into the last matrix inequality of Theorem 1 by applying the Schur lemma while taking account of the strict positive definiteness of P_{k+1}. ∎

B. Comments and Potentialities of the Approach

In the above main result, all the constraints have been expressed in terms of the vector function $\underline{\eta}(.,.)$. Through the incorporation by means of the $\mathscr{S}$-procedure of the quadratic functions on $\underline{\eta}(.,)$ under which $\underline{x}_{k+1} \in \mathscr{E}_{k+1}$ must hold, prior to the use of the Finsler's lemma so as to take account of the relationships uniting the entries of $\underline{\eta}(.,.)$, the proposed solution has been shown to handle Recursive Algebraic Representations involving affine matrix functions $A_1(.,),A_2(.,.),\ldots,\Omega_4(.,.)$. As the matrices $\Upsilon_0,\Upsilon_1,\ldots,\Upsilon_{m_{\underline{w}}}$ are constant, a straight extension to Theorem 1 can be derived by replacing the constant multipliers $\tau_0,\tau_1,\ldots,\tau_{m_{\underline{w}}}$ by affine scalar functions of $(\underline{x},\underline{\delta})$, i.e. by turning each τ_m, $m = 0,1,\ldots,m_{\underline{w}}$, into $\tau_m^0 + \underline{\tau}'_{m,1}\underline{x} + \underline{\tau}'_{m,2}\underline{\delta}$, with $\tau_m^0, \underline{\tau}'_{m,1}, \underline{\tau}'_{m,2}$ the new decision variables.

For each k, a minimum-size convex polytope $\mathscr{X}_k$ enclosing the hypothesized ellipsoid $\mathscr{E}_k$ can be selected which edges are aligned with the principal axes of $\mathscr{E}_k$. Noticeably, $\mathscr{X}_k$ can be easily deduced from the eigendecomposition of P_k.

A first potentiality is left for impending research. Theorem 1 is a convex optimization program when the size criterion is the trace of the shape matrix. Although the nonlinearities and uncertainties are dealt with in a different way compared to [12][13], the approach developed in these last references entails an LMI constraint which is fairly similar to the ones to be computed at $\mathscr{V}(\mathscr{X}_k \times \Delta_k)$ in (21). It will be checked if, in nearly the same vein as [12][13], P_{k+1} and $\underline{\hat{x}}_{k+1}$ can be computed through decoupled recursions. The aim would be to also get a problem with reduced complexity, where both the trace and the log-determinant of the shape matrix can be used as size criteria.

A second very interesting potentiality is to propagate over time non-ellipsoidal confidence sets of the form

$$\underline{x}_k \in \mathscr{C}_k \triangleq \left\{\underline{x} : \left[\underline{\theta}(\underline{x}-\underline{\hat{x}}_k)\right]' P_k^{-1} \left[\underline{\theta}(\underline{x}-\underline{\hat{x}}_k)\right] \leq 1\right\}, \tag{23}$$

with $\underline{\theta}(\cdot)$ a prescribed rational vector function of its arguments. Defining non-quadratic Lyapunov functions in a control context was indeed proved to be workable in [20][21], by handling quadratic forms into higher-dimensional vectors and defining linear annihilators accordingly. The aim would be to adapt this to the considered filtering problem.

V. CASE STUDIES

The proposed approach has first been assessed in the simple numerical example used as a benchmark in [12] and in some references cited therein. So, the following uncertain discrete-time system has been considered

$$\begin{aligned} \underline{x}_{k+1} &= \begin{pmatrix} 0 & -0.5 \\ 1 & 1+0.3\underline{\delta}_k \end{pmatrix} \underline{x}_k + 0.02 \begin{pmatrix} -6 & 0 \\ 1 & 0 \end{pmatrix} \underline{w}_k \\ \underline{y}_k &= \begin{pmatrix} -100 & 10 \end{pmatrix} \underline{x}_k + 0.02 \begin{pmatrix} 0 & 1 \end{pmatrix} \underline{w}_k, \end{aligned} \tag{24}$$

where $\underline{\delta}_k$ lies in $\Delta_k \triangleq [-1;+1]$ and $\underline{w}_k$ is defined by (6) with $m_{\underline{w}} = 2$, $Q_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $Q_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The given confidence ellipsoid $\mathscr{E}_0$ at initial time $k = 0$ is centered on $\underline{x}_0 = \begin{pmatrix} 0 & 0 \end{pmatrix}'$ and its shape matrix is set to $E_0 = 3\mathbb{I}_2$.

Theorem 1 has been recursively applied so as to predict confidence ellipsoids at time $k+1$ upon their knowledge at time k and given the measurement $\underline{y}_k$. The results, reported on Figure 1 for the estimated first entry of the state vector, perfectly match the conclusions which would be obtained through the method developed by El Ghaoui and Calafiore in [12]. Importantly, the quadratic embedding of the uncertainty underlying their approach is tight in this case, so that the $\mathscr{S}$-procedure is their only source of conservativeness. Getting similar confidence sets is thus a good point.

Further, the proposed approach has been assessed on a second example of significantly higher complexity. It is declined into two cases, depending on the output equation.

$$\begin{aligned} \underline{x}_{k+1} &= \begin{pmatrix} \alpha_1 & 0 \\ 1-\alpha_1 & \alpha_2 \end{pmatrix} \underline{x}_k + \begin{pmatrix} 20 & 0 \\ 0 & 10 \end{pmatrix} \underline{w}_k + \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} \underline{u}_k \\ \underline{y}_k &= \begin{pmatrix} 1 & 1 \end{pmatrix} \underline{x}_k \qquad \text{(Case A)} \\ \text{or } \underline{y}_k &= \begin{pmatrix} 0 & 1-\alpha_2 \end{pmatrix} \underline{x}_k \qquad \text{(Case B)} \end{aligned} \tag{25}$$

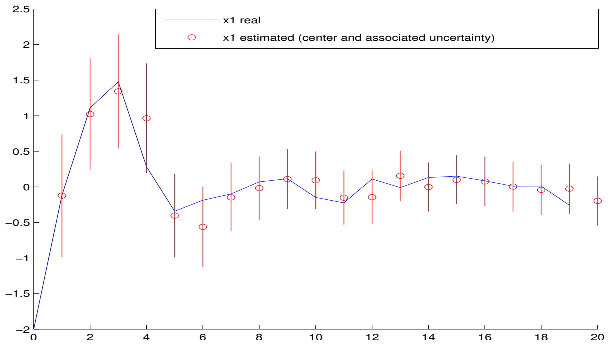


Fig. 1. Assessment of the method on the benchmark cited in [12] in a random "limiting" case (see $k=4$): projection of the predicted confidence ellipsoids on the $\underline{x}_1$ axis over time.

with

$$\alpha_1 = \frac{1}{200}(0.2+\frac{e_1'\underline{\delta}_k}{20})(100+\underline{e}_1'\underline{x}_k) \quad \beta_1 = \frac{200+\underline{e}_1'\underline{x}_k}{400}$$
$$\alpha_2 = \frac{1}{200}(0.2+\frac{e_2'\underline{\delta}_k}{20})(100+\underline{e}_2'\underline{x}_k) \quad \beta_2 = \frac{200+\underline{e}_2'\underline{x}_k}{400}. \tag{26}$$

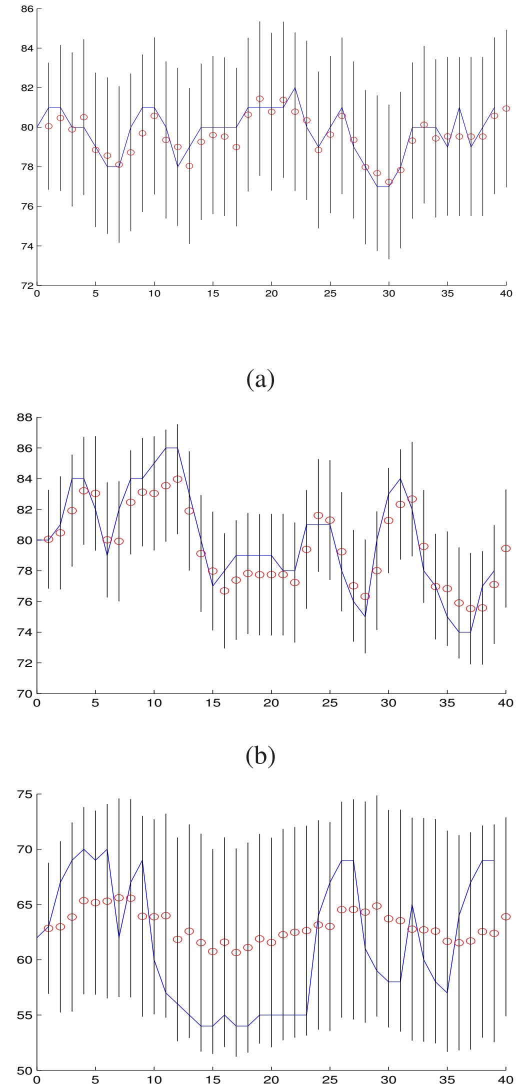
Here, $\underline{\delta}_k$ lies in $\Delta_k \triangleq [-0.1;+0.1]\times[-0.1;+0.1]$ and $\underline{w}_k$ is defined by (6) with $m_{\underline{w}}=2$, $Q_1 = \begin{pmatrix}100 & 0\\ 0 & 0\end{pmatrix}$, $Q_2 = \begin{pmatrix}0 & 0\\ 0 & 100\end{pmatrix}$.

In (Case A) the initial confidence ellipsoid $\mathscr{E}_0$ is defined by $\underline{x}_0 = \begin{pmatrix}90 & 80\end{pmatrix}'$ and $E_0 = 2\mathbb{I}_2$. The input signal is constant, with $\underline{u}_k = \begin{pmatrix}80 & -20\end{pmatrix}'$. Figures 2(a)-(b) show the evolution of the second entry of $\underline{x}$ along time, as well as the corresponding guaranteed prediction. In Figure 2(a), random values of the noises and uncertainties are simulated, and the prediction looks somewhat pessimistic. Yet, in Figure 2(b), the noises and uncertainties switch between the boundaries of their admissible sets, so as to get a less helpful scenario. In spite of this, the quality of the estimation looks fairly unaltered—i.e. the predicted values remain into reasonable limits—and the conservatism seems acceptable.

In (Case B) the initial confidence ellipsoid $\mathscr{E}_0$ is defined by $\underline{x}_0 = \begin{pmatrix}62 & 38\end{pmatrix}'$ and $E_0 = \mathbb{I}_2$. The input signal has the form $\underline{u}_k = \begin{pmatrix}80+\sin(2\pi * k/20) & -30\end{pmatrix}'$. Figure 2(c) displays the time history of the first entry of the state vector and of its prediction. It can be observed that the nonlinear mapping between the state and output vectors degrades the quality of the prediction. Nevertheless, the true state lies into the computed confidence sets. The pessimism seems fairly small, as at some times the true state entry reaches the vicinity of its guaranteed predicted boundaries (about 5% of these).

VI. CONCLUSION

This paper has proposed a technique to robust filtering/prediction of nonlinear discrete-time systems with rational dependence on the state and uncertain parameters. The proposed approach is devised in a set-membership context, in that the system initial condition, the uncertainties, as well as the noise affecting the dynamics and the measurements are unknown but bounded by given sets. The prediction is based on a recursive estimation of confidence ellipsoids enclosing the state vector at each sampling time, based on the reformulation of the system model as a Recursive Algebraic Representation. The solution is expressed as a convex optimization problem under Linear Matrix Inequality (LMI) constraints through the $\mathscr{S}$-Procedure and Finsler's lemma. Numerical experiments have demonstrated the potential of the proposed approach as a tool for state estimation/prediction of uncertain nonlinear discrete-time systems. Future research is concentrated on extending the approach to confidence sets defined by means of non-quadratic functions of the state vector.



(a) (b) (c)

Fig. 2. Assessment of the method on another nonlinear uncertain case study: projection of the confidence ellipsoids on the $\underline{x}_2$ axis over time. The red circles represent the center of the predicted ellipsoids.

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