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On the Feedback Control of Impulsive Dynamic Systems

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***Abstract*— An Euler solution concept and weak invariance conditions are presented for impulsive control systems. In this paper, we model impulsive systems by measure driven differential inclusions. The Euler solution is defined in the original time frame and does not require any reparameterization technique. As a consequence, we present a constructive result giving conditions for weak invariance. The proof of this result will provide a control synthesis in feedback form, which, in turn, will permit its implementation on-line.**

I. INTRODUCTION

In this article, we examine a number of issues concerning the feedback control of impulsive dynamic systems. More specifically, we consider impulsive control systems whose dynamics are given by measure driven differential equations. This class of systems can be regarded as a proper extension of conventional dynamic control systems whereby the control space of measurable functions is enlarged in order to also include measures. The state trajectory is obtained by a Stieltjes integration of the dynamics, and, therefore, they belong to the class of functions of bounded variation, i.e., they may exhibit discontinuities.

This article follows along the lines of the previous work [8], [11] in which the solution concept plays a critical role. In particular, for a trajectory to be well defined, any jump must be such that there should exist a path joining the two jump endpoints that satisfies the “singular” dynamics. By this, we mean that this path has to satisfy a certain differential equation in which the state variable derivative is taken with respect to a certain parameter that depends on the control measure.

There is a substantial motivation fueling the research in this area: **1)** The general interest of developing a control theory generalizing the conventional one for systems with absolutely continuous trajectories; **2)** Many important classes of applications are better addressed by considering state trajectories that may exhibit discontinuities: space navigation, resources management, investment policies, impact dynamics, composition between systems with fast and slow dynamics and, in general, dynamical systems admitting control actions during the singular phases of their motion [1]; **3)** The impulsive control systems has also been recognized as a good framework to model hybrid dynamic control systems [6], i.e., systems whose state evolution combine time-driven (intrinsic to natural systems) and event-driven dynamics (typical of artificial systems) [2].

In most of the previous work, the control measure was considered as an open loop control law and the derivation of optimality and stability conditions are based on reparameterization techniques [9], [3], [8], [7]. However, there are many problems for which is of interest to derive a constructive method for the feedback control measure. For that purpose we develop an extension of the Euler solution concept presented in [5] to the impulsive case. This impulsive Euler solution concept is defined in the original time frame, does not require the usage of reparameterization techniques, allows the definition of the measure in a feedback form and does not require any regularity assumptions of selections of the measure driven differential inclusion. These are the main novelties in relation to [12]. This solution concept will play a major role in the constructive result on weak invariance presented in this article. The proof of this result will provide a control synthesis in feedback form, which, in turn, will permit its implementation on-line.

The article is organized in five sections. In section II, we introduce the impulsive control framework and discuss some key concepts. Then, in section III, we present the impulsive Euler solution concept. In section IV we provide a constructive result on weak invariance. We close with some concluding remarks.

II. THE IMPULSIVE CONTROL FRAMEWORK

We consider the following class of measure driven differential inclusion:

$$\begin{cases} dx(t) \in F(t, x(t))dt + G(t, x(t))\mu(dt), \ t \in [0, \infty) \\ x(0) = x_0, \end{cases} \tag{1}$$

where the multifunctions F and G maps $[0, \infty) \times \mathbb{R}^n$ into subsets of $\mathbb{R}^n$ and $\mathbb{R}^{n\times q}$, respectively, and the measure μ belongs to a positive convex pointed cone K in $\mathbb{R}^q$. By $\mu \in K$ it is meant that $\mu \in C^*([0, +\infty); K)$, where $C^*([0, +\infty); K)$ denotes the set in the dual space of continuous functions from $[0, +\infty)$ to $\mathbb{R}^q$ with values in K.

In what follows $AC([0, +\infty); \mathbb{R}^n)$ means the space of absolutely continuous $\mathbb{R}^n$-valued functions on $[0, +\infty)$ and $BV^+([0, +\infty); \mathbb{R}^n)$ represents the space of $\mathbb{R}^n$-valued functions on $[0, +\infty)$ of bounded variation and which are continuous from the right on $(0, +\infty)$. The measure $\bar{\mu}$ denotes the total variation of the measure μ : $\bar{\mu}(dt) := \sum_{i=1}^{q} \mu^i(dt)$ where μ^i represents each component of the vector valued measure μ.

The data is assumed to satisfy the following standing hypothesis:

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(h1) For every $x \in \mathbb{R}^n$, $F(t,x)$ and $G(t,x)$ are nonempty compact convex sets and upper semicontinuous.
(h2) Linear growth: there are positive constants γ and c such that for all (t,x) $v \in F(t,x) \Longrightarrow \|v\| \leq \gamma\|x\| + c$ and $v \in G(t,x) \Longrightarrow \|v\| \leq \gamma\|x\| + c$.
(h3) The total variation measure $\bar{\mu}$ is bounded: $\|\bar{\mu}\|_{TV} \leq M < +\infty$.

Clearly, upper semicontinuity plays the same role of continuity when instead of having multifunctions we have just functions.

At this point, we should note that the measure driven differential inclusion problem has an apparent ill-definition due to the discontinuity of x at the support of singular atomic atoms. Thus, it is not clear a priori which value of x should be plugged in the argument of G when an atom (singular atomic component) of the measure is present.This issue is addressed by extending the solution concept introduced by several authors [4], [11], [8]. We adapt the solution concept to encompass the unbounded interval $[0, +\infty)$.

Definition 2.1: A trajectory x of bounded variation, with $x(0) = x_0$, is admissible for problem (1) if $x(t) = x_{ac}(t) + x_s(t)$ $\forall t \in [0, \infty)$, where

$$\dot{x}_{ac}(t) \in F(t, x(t)) + G(t, x(t))w_{ac}(t) \text{ a.e.}$$
$$x_s(t) = \int_{[0,t]} g_{sc}(\tau)w_{sc}(\tau)d\bar{\mu}_{sc}(\tau) + \int_{[0,t]} g_{sa}(\tau)\bar{\mu}_{sa}(d\tau).$$

Here, $\bar{\mu}$ is the total variation measure associated with μ while μ_{sc}, μ_{sa} and μ_{ac} represents, respectively, the singular continuous, the singular atomic and the absolutely continuous components of μ (Lebesgue decomposition). The variable w_{ac} represents the Lebesgue time derivative of μ_{ac} while w_{sc} is the Radon-Nicodym derivative of μ_{sc} with respect to $\bar{\mu}_{sc}$. The solution x_{ac} is an absolutely continuous function and the differential inclusion is to be interpreted in the selector sense where the selections has to obey some regularity properties, namely, measurably and continuity on time and on the state, respectively. The function $g_{sc}(\cdot)$ is a μ_{sc} measurable selection of $G(\cdot, x(\cdot))$ and $g_{sa}(\cdot)$ is a $\bar{\mu}_{sa}$ measurable selection of the multifunction

$$\tilde{G}(t, x(t^-); \mu(\{t\})) : [0, \infty) \times \mathbb{R}^n \times K \hookrightarrow \mathcal{P}(\mathbb{R}^n), \quad (2)$$

which specifies the reachable set of the singular dynamics at $(t, x(t^-))$ when the control measure has an atom $\mu\{t\}$. When $\mu\{t\} = 0$ then $\tilde{G} = 0$ and when $\mu\{t\} > 0$ is given by:

$$\tilde{G}(t, z; \alpha) := \Big\{ |\alpha|^{-1} [\xi(s) - \xi(0)] : \dot{\xi}(s) \in G(t, \xi(s))v(s) \; \bar{\eta}\text{–a.e.}, \ \xi(0) = z, \ \xi \in AC([0, |\alpha|]; \mathbb{R}^n), \; v(s) \in K \cap \bar{B}, \ \int_0^{|\alpha|} v(s)ds = \alpha \Big\}.$$

where $|\alpha| := \sum_{i=1}^q \alpha^i$.

We remark the emergence of the "additional control" v is due to the non-uniqueness of the integral of the matrix-valued function G in relation to the measure μ_{sa} in the absence of the commutativity property of the vector fields defined by its columns. In other words, the control v completely specifies how the measure components contribute to the total variation of $\bar{\mu}$. Without this additional control, it would be ambiguous how the jump on x was generated since more than one possibility would be possible due to the non commutativity property of G. Note that if the control measure μ is scalar-valued [10], [11], then the definition of the jump becomes simpler.

The solutions of system (1) are robust in the sense that the set of solutions has desirable closure properties with respect to perturbations of the driving measure μ and the initial state. This robustness property is essential to be consistent with the interpretation of μ_{sa} as an idealization of conventional controls taking large values during short periods.

III. Euler solution concept

In this section, we present a solution concept for the impulsive control problem that does not require any regularity assumptions on selections of F and G, namely continuity. This solution concept is based in the Euler solution already available for conventional systems [5] and provides an extension for the impulsive case. We will denominate it by impulsive Euler solution concept. Besides the non regularity assumption it will be derived in the original time frame and no reparameterization technique is used. This solution concept provides a formalism enabling the definition of the measure in feedback, which will be given in the next section.

Hereafter, we assume that the multifunctions F and G do not depend on time. Also, we will assume the measure to be scalar valued. The extension for the vector valued case would be possible but, due to space limitations, we do not give further details. Hence, we study the Euler solution of the following problem:

$$\begin{cases} dx(t) \in F(x(t))dt + G(x(t))d\mu(t), \ t \in [a, b] \\ x(0) = x_0, \end{cases} \quad (3)$$

when μ is scalar valued.

Assume we have selections f, g (possibly non regular) and a given bounded scalar measure μ defined on $[a, b]$ such that $|\mu([a, b])| \leq M < +\infty$ ($\int_{[a,b]} d\mu(t) \leq M$). Thus, we obtain the following initial-value problem:

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t))d\mu(t), \ t \in [a, b] \\ x(0) = x_0, \end{cases} \quad (4)$$

In order to sample this system and obtain the Euler solution we need first to define the following function:

$$\eta(t) := t + \int_{[a,b]} d\mu(t), \ t \in [a, b]. \quad (5)$$

We define a partition on the range of function η as follows:

$$\pi = \{s_0, s_1, \cdots, s_N\}, \quad (6)$$

where $s_0 = \eta(a^-)$ and $s_N = \eta(b^+)$. We do not require a uniform partition but, to simplify the presentation, we assume that when the measure is singular atomic we have to define node points precisely on $\eta(t^-)$ and on $\eta(t^+)$. Having made this partition on the range of $\eta(t)$ we can compute the corresponding points in the t-domain. Let $t_0, t_1, \cdots, t_N$ be such points. For each node on the range of η there is only one corresponding point in t-domain since $\eta(t)$ is a strictly increasing function. As a remark, note that if at a given instant t the measure is singular atomic then there are partition nodes in the t-domain having the same value. In figure 1 we present a schematic view of this sampling scheme.

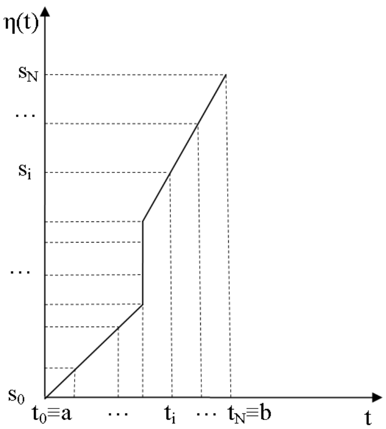

Fig. 1. The sampling scheme

Now we are in conditions to present an Euler polygonal arc x_π (the subscript π means that the Euler polygonal arc depends on the particular partition π made on the range of η). First, we consider the interval $t \in [t_0, t_1]$ and define the following dynamical system:

$$dx(t) = f(x_0)dt + g(x_0)d\mu(t),\ x(a) = x_0.$$

We define the node point $x_1 := x(t_1)$. Note that the preceding dynamical system has a unique solution since the selections f and g are set to be constant. Also note that the interval $[t_0, t_1]$ can be a singleton, which would mean that the measure at t_0 would have an atom. In this case, to compute the node point x_1 we would apply a singular atomic measure with total variation equal to $s_1 - s_0$. Next, we proceed and now we consider the interval $[t_1, t_2]$ and the following dynamical system:

$$dx(t) = f(x_1)dt + g(x_1)d\mu(t),\ x(t_1) = x_1.$$

The next node point will be defined as $x_2 := x(t_2)$. We proceed like this until we have covered all elements of the partition and an arc x_π is obtained on $[a, b]$.

The diameter of the partition π is given by:

$$\alpha_\pi := \max\{s_i - s_{i-1} : 1 \leq i \leq N\}.$$

An Euler solution to the initial-value problem (4) means any function x of bounded variation which is the uniform limit of the Euler polygonal arcs x_{π_j} corresponding to some sequence $\pi_j \to 0$. In this context, $\pi_j \to 0$ means that $\alpha_{\pi_j} \downarrow 0$. Clearly, as the diameters of the partition π tends to zero then the number of node points tends to infinity. The following theorem presents some important properties of the Euler solution concept.

Theorem 3.1: Suppose we are given selections f and g of F and G, respectively, such that for positive constants γ and c and for all $(t, x) \in [a, b] \times \mathbb{R}^n$ we have $\|f(x)\| \leq \gamma\|x\| + c$ and $\|g(x)\| \leq \gamma\|x\| + c$. Moreover, assume also that we are given a measure μ such that $|\mu([a, b])| \leq M < +\infty$. Then: **a)** at least one Euler solution x exists for problem (4) on $[a, b]$ and any Euler solution is of bounded variation; **b)** any Euler arc x on $t \in [a, b]$ satisfies

$$\begin{aligned}&\|x(t) - x(a)\| \\ &\leq (t - a + |\mu([a, t])|)e^{\gamma(t-a+|\mu([a,t])|)}(\gamma\|x_0\| + c).\end{aligned}$$

Proof: Consider the partition π defined before and its correspondent partition π_t in the t domain on $[a, b]$. Consider also the Euler polygonal arc and its nodes $x_0, x_1, \cdots, x_N$. Then, we can deduce the following relationship between consecutive node points:

$$\begin{aligned}\|x_{i+1} - x_i\| &= \left\|(t_{i+1} - t_i)f(x_i) + \int_{[t_i,t_{i+1}]} g(x_i)d\mu(t)\right\| \\ &\leq (\gamma\|x_i\| + c).(t_{i+1} - t_i + \int_{[t_i,t_{i+1}]} d\mu(t)). \qquad (7)\end{aligned}$$

Note that t_{i+1} can be equal to t_i when the measure is singular atomic and, consequently, the interval $[t_i, t_{i+1}]$ can be a singleton and $\int_{[t_i,t_{i+1}]} d\mu(t) = s_{i+1} - s_i$. Thus, with this estimative at hand, we can deduce that:

$$\begin{aligned}&\|x_{i+1} - x_0\| \leq \|x_{i+1} - x_i\| + \|x_i - x_0\| \\ &\leq \left[\gamma.\left(t_{i+1} - t_i + \textstyle\int_{[t_i,t_{i+1}]} d\mu(t)\right) + 1\right].\|x_i - x_0\| \\ &+(\gamma\|x_0\| + c).\left(t_{i+1} - t_i + \textstyle\int_{[t_i,t_{i+1}]} d\mu(t)\right).\end{aligned}$$

Applying the exercise in induction 4.1.8 of [5] we are led to the conclusion that:

$$\begin{aligned}\|x_i - x_0\| &\leq \exp\left(\sum_{j=0}^{i-1} \gamma\left(t_{j+1} - t_j + \int_{[t_j,t_{j+1}]} d\mu\right)\right). \\ &.\sum_{j=0}^{i-1}\left(t_{i+1} - t_i + \int_{[t_j,t_{j+1}]} d\mu\right).(\gamma\|x_0\| + c)\end{aligned}$$

which implies that

$$\|x_i - x_0\| \leq (b - a + M).e^{\gamma.(b-a+M)}.(\gamma\|x_0\| + c) := L.$$

Hence, we can conclude that all nodes of an Euler polygonal arc lie in a closed ball $x_i \in \bar{B}(x_0, L)$. Since between node points the selections f and g are constant then this property is also true for all values of x_π. That the Euler polygonal arc is a function of bounded variation is immediately clear since we are assuming that $|\mu([a, b])| \leq M$.

Now let π_j be a sequence of partitions such that $\pi_j \downarrow 0$. The corresponding Euler polygonal arcs x_{π_j} all satisfy $\|x_{\pi_j} - x_0\|_\infty \leq L$ and they are all functions of bounded variation. Therefore, by the Helly's selection principle (a compactness theorem for sequences of bounded variation and uniformly bounded functions) we can conclude that the sequence of Euler polygonal arcs has a limit, which is a bounded variation function. This establish part a) of the theorem.

The inequality of part b) is inherited by the limit of the subsequence x_{π_j}. We only need to identify t with b and M by $|\mu([a,t])|$. ∎

After presenting some important properties of the Euler solution we know need to show that it is in fact a trajectory of system (3).

Theorem 3.2: Let f and g be any selections of F and G, respectively, and μ be a given measure. If x is an Euler solution on $[a,b]$ of $dx(t) = f(x(t))dt + g(x(t))d\mu(t)$, $x(a) = x_0$, then x is a trajectory of system (3).

Proof: Let $t \in (c,d) \subset [a,b]$, where (c,d) represents all the intervals between measure's atoms, and let x_i and t_i represent the node point of an Euler polygonal arc before t. Then, defining $y_j(t) := x_i - x_{\pi_j}(t)$, we have that

$$\begin{aligned}\dot{x}_{\pi_j}(t) &= f(x_i) + g(x_i)w_{ac}(t) \in F(x_i) + G(x_i)w_{ac}(t)\\ &= F(x_{\pi_j}(t) + y_j(t)) + G(x_{\pi_j}(t) + y_j(t))w_{ac}(t), \qquad (8)\end{aligned}$$

$\mathcal{L} - a.e.$, and that

$$\frac{dx_{\pi_j}(t)}{d\mu_{sc}} = g(x_i) \in G(x_i) = G(x_{\pi_j}(t) + y_j(t)),\ \mu_{sc} - a.e..$$

By equation (7), the following estimative is possible to obtain:

$$\begin{aligned}&\|y_j(t)\| = \|x_{\pi_j}(t) - x_i\| \leq\\ &(\gamma\|x_i\| + c)\left(t - t_i + \int_{[t_i,t]} d\mu(t)\right) \leq \alpha_{\pi_j}.(\gamma\|x_i\| + c).\end{aligned}$$

Since $\|x_i\|$ is bounded (see proof of theorem 3.1) we can conclude that $\|y_j(t)\|_\infty \to 0$, as $\pi_j \downarrow 0$. We can also conclude that $\|\dot{x}_{\pi_j}(t)\|_2$ is bounded in the L_2 space associated with the Lebesgue measure as well as $\left\|\frac{dx_{\pi_j}(t)}{d\mu_{sc}(t)}\right\|_2$ associated with L_2 space relative to measure μ_{sc} because we have seen in theorem 3.1 that $\|x_{\pi_j}(t)\|_\infty$ is bounded. Then, invoking weak compactness in L_2, we conclude that we can extract a subsequence of $\dot{x}_{\pi_j}(t)$ and of $\frac{dx_{\pi_j}(t)}{d\mu_{sc}(t)}$ such that they converge weakly to v_0 and v_1, respectively. By construction, x_{π_j} is continuous and is given by:

$$x_{\pi_j}(t) = x_{\pi_j}(c) + \int_c^t \dot{x}_{\pi_j}(\tau)d\tau + \int_{[c,t]} \frac{dx_{\pi_j}(\tau)}{d\mu_{sc}} d\mu_{sc}(\tau),$$

Passing to the limit we obtain:

$$x(t) = x(c) + \int_c^t v_0(\tau)d\tau + \int_{[c,t]} v_1(\tau)d\mu_{sc}(\tau).$$

Then, clearly, x is an arc on intervals of the type $[c,d]$ with $\dot{x}(t) = v_0(t)$ $\mathcal{L}$-a.e. and $\frac{dx(t)}{d\mu_{sc}} = v_1$ μ_{sc}-a.e. Now we only need to show that $v_0(t) \in F(x(t)) + G(x(t))w_{ac}(t)$ a.e. and that $v_1(t) \in G(x(t))$ μ_{sc}-a.e.. To verify the property of v_0, we use the definition of $\dot{x}_{\pi_j}(t)$ (equation (8)) to decompose $v_0(t)$ as follows:

$$v_0(t) = v_F(t) + v_G(t)w_{ac}(t).$$

Then, by construction, we know that the subsequences $v_{F,j}(t)$ and $v_{G,j}(t)$ originating $v_F(t)$ and $v_G(t)$ are such that $v_{F,j}(t) \in F(x_{\pi_j}(t)+y_j(t))$ and $v_{G,j}(t) \in G(x_{\pi_j}(t)+y_j(t))$. Hence, applying theorem 3.5.24 of [5], we conclude that $v_F(t) \in F(x(t))$ and $v_G(t) \in G(x(t))$, which leads us to the required property of $v_0(t)$. In order to proof the required property of $v_1(t)$ we could adapt theorem 3.5.24 of [5] such that the L_2 space is associated with the μ_{sc} measure. Then, the result would follow.

To complete the proof we only need to verify that the singular Euler trajectory is in fact a trajectory of system (3). To do so consider an atom of the measure at t_a (the procedure would be the same for all atoms of the measure) and the Euler singular polygonal arc associated with it on $s \in [0, |\mu(\{t_a\})|]$. Let x_i be the node point immediately before s. Then, we conclude that:

$$\dot{x}_{\pi_j}(s) = g(x_i) \in G(x_i) = G(x_{\pi_j}(s) + y_j(s)),$$

with $y_j(s) = x_i - x_{\pi_j}(s)$. By equation (7) we know that:

$$\|y_j(s)\| = \|x_{\pi_j}(s) - x_{\pi_j}(s_i)\| \leq (s - s_i)\,(\gamma\|x_{\pi_j}(s_i)\| + c).$$

Thus, as $\pi_j \downarrow 0$ we can conclude that $\|y_j(s)\|_\infty \to 0$. Hence, by the application of theorem 4.1.11 of [5] (interpreting time with parameter s) we conclude that the Euler singular arc $x(s)$ is, in fact, a trajectory of G, which establishes the desired result. ∎

IV. Weak invariance

This section provides invariance results for the impulsive control system and constitutes an extension of the work developed for conventional systems [5]. We wish to find selections and a control measure such that the conventional and singular trajectory remain in a given set. We present results that make use of the Euler solution concept to compute feedback selections of F and G and a feedback measure such that the resulting trajectories have the desired properties. We will present sufficient conditions expressed in the original time frame.

Before we present the main result, we first need to provide an auxiliary result on proximal aiming. Assume that selections f, g are given (possibly non-regular) as well as a measure. We wish to study if the Euler trajectories associated with the given selections and measure "move toward" a closed set. Sufficient conditions for this problem are presented in the next proposition.

Proposition 4.1: Let f, g satisfy the linear growth conditions $\|f(x)\| \leq \gamma\|x\| + c$ and $\|g(x)\| \leq \gamma\|x\| + c$, $\forall x$ and let x be an Euler arc for the system $dx = f(x)dt + g(x)d\mu(t)$ on $t \in [a,b]$. Assume also that a measure μ is given such that $\int_{[a,b]} d\mu(t) \leq M < +\infty$. Let Ω be an open set containing x $\forall t \in [a,b]$ (in the robust solution sense) and suppose that for every $z \in \Omega$ there is a $p \in \mathrm{proj}_S(z)$ such that:

$$\begin{aligned}&\langle f(z) + g(z)w_{ac}(t), z - p\rangle \leq 0,\ \mu_{ac}\text{-a.e. on } [a,b],\\ &\langle g(z), z - p\rangle \leq 0,\ \mu_{sa} \text{ and } \mu_{sc} \text{ a.e. on } [a,b],\end{aligned}$$

where $w_{ac}(t)$ represents the Lebesgue time derivative of μ_{ac} (this condition could also be expressed in a compact form as $\langle f(z)dt + g(z)d\mu, z - p\rangle \leq 0$ μ-a.e. on $[a, b]$). Then we have:

$$d_S(x(t)) \leq d_S(x(a^-)) \; \forall t \in [a, b]$$

and, for every singular arc,

$$d_S(x(s)) \leq d_S(x(\eta(t_a^-))) \; \forall s \in [\eta(t_a^-), \eta(t_a^+)],$$

where t_a represents the support of the singular atomic component and $x(\eta(t_a^-)) = x(t_a^-)$.

Proof: Let x_π be one Euler polygonal arc in the sequence converging uniformly to the Euler solution x. As before, consider the partition on the range of function η and the corresponding node points t_i and x_i, with $i = 0, 1, \cdots, N$, such that $x_0 = x(a^-)$. We can choose a set Ω such that the Euler arc x_π (in the robust sense) lies in it for all $t \in [a, b]$. For each node point we can compute its projection p_i on S such that $p_i \in proj_S(x_i)$. We know that $dx_\pi(t) = f(x_i)dt + g(x_i)d\mu(t)$ on $t \in [t_i, t_{i+1}]$ and, by the hypothesis of the theorem, that $\langle dx_\pi(t), x_i - p_i\rangle \leq 0$. Hence, the following estimative can be derived using equation (7), definition of $d_S(x)$ and noting that $x_1 - x_0 = \int_{[t_0,t_1]} dx_\pi(t)$:

$$\begin{aligned} &d_S^2(x_1) \leq \{ \text{ Since } p_0 \in S\} \leq \|x_1 - p_0\| \\ &= \|x_1 - x_0\|^2 + \|x_0 - p_0\|^2 + 2\langle x_1 - x_0, x_0 - p_0\rangle \\ &\leq (t_1 - t_0 + \textstyle\int_{[t_0,t_1]} d\mu(t))^2(\gamma\|x_0\| + c)^2 + d_S^2(x_0) \\ &\quad +2 \textstyle\int_{[t_0,t_1]} \langle dx_\pi(t), x_0 - p_0\rangle \\ &\leq (t_1 - t_0 + \textstyle\int_{[t_0,t_1]} d\mu(t))^2(\gamma\|x_0\| + c)^2 + d_S^2(x_0) \quad . \end{aligned}$$

We can follow the same procedure for all nodes of the Euler polygonal arc x_π, such that:

$$d_S^2(x_i) \leq (t_i - t_{i-1} + \int_{[t_{i-1},t_i]} d\mu(t))^2(\gamma\|x_{i-1}\| + c)^2 + d_S^2(x_{i-1}).$$

Note that, depending on the measure, it is possible that some time nodes t_i have the same value, which would imply the singularity of the arc in that segment. Proceeding iteratively and recalling, by theorem 3.1, that x_i $\forall i$ we can derive that:

$$\begin{aligned} &d_S^2(x_i) - d_S^2(x_0) \leq \\ &\leq (\gamma L + c)^2 . \textstyle\sum_{l=1}^{i} \left[t_l - t_{l-1} + \int_{[t_{l-1},t_l]} d\mu(t)\right]^2 \\ &\leq (\gamma L + c)^2 .\alpha_\pi \textstyle\sum_{l=1}^{i} \left[t_l - t_{l-1} + \int_{[t_{l-1},t_l]} d\mu(t)\right] \\ &\leq (\gamma L + c)^2 .\alpha_\pi(b - a + M). \end{aligned}$$

Hence, if we let $j \to \infty$ then the sequence of polygonal arcs x_{π_j} converges to x and $\alpha_\pi \downarrow 0$. Thus, we deduce that

$$d_S(x(t)) \leq d_S(x(a^-)) \; \forall t \in [a, b]$$

and, when the trajectory is singular, that

$$d_S(x(s)) \leq d_S(x(\eta(t_a^-))) \; \forall s \in [\eta(t_a^-), \eta(t_a^+)],$$

where $x(\eta(t_a^-) = x(t_a^-)$, as required. ■

Now we are in condition to present the weak invariance concept. Basically, we wish to know if there are Euler solutions of an impulsive system such that they remain in a given set. The topic of weak invariance not only encounters itself many practical control applications but also serves to study other problems like, for example, stability and optimality.

Definition 4.1: The system (S, F, G) is said to be weakly invariant if for all $x(a^-) \in S$, there are selections of $F(x)$ and $G(x)$ and a measure μ such that the associated trajectory $x(t)$ on $t \in [a, b]$ and $x(s)$ on $s \in [0, |\mu(\{t_a\})|]$ is such that:

$$x(a^-) = x_0, \; x(t) \in S \; \text{ and } x(s) \in S.$$

Remark 4.1: In this context, $x(t)$ stands for the trajectory when the measure is non atomic while $x(s)$ stands for the singular arcs at t_a, which represents the support of the singular atomic component.

Before we proceed and present sufficiency conditions for weak invariance we would like to recall the definition of lower Hamiltonian h associated with a multifunction F. The function h maps $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ and is defined as $h_F(t, x, p) := \min_{v \in F(t,x)} \langle p, v\rangle$.

Theorem 4.1: Suppose that for every $x \in S$ we have

$$\min\{h_F(x, N_S^P(x)), h_G(x, N_S^P(x))\} \leq 0.$$

Then (S, F, G) is weakly invariant on $[a, b]$.

Proof: For each $x \in \mathbb{R}^n$ compute $p = p(x) \in proj_S(x)$ and note that $x - p \in N_S^P(p)$. Then minimize the following functions:

$$v \to \langle v, x - p\rangle, \; w \to \langle w, x - p\rangle,$$

with $v \in F(p)$ and $w \in G(p)$. We define functions f_P and g_P by setting $f_P(x) = v$ and $g_P(x) = w$. By the hypothesis of the theorem, we know that at least one of the quantities $\langle f_P(x), x - p\rangle$ and $\langle g_P(x), x - p\rangle$ is nonpositive. If p_0 is any point in S then we can deduce that:

$$\begin{aligned} \|f_P(x)\| &= \|v\| \leq \gamma\|p\| + c \leq \gamma\|p - x\| + \gamma\|x\| + c \\ &= \gamma d_S(x) + \gamma\|x\| + c \leq \gamma\|x - p_0\| + \gamma\|x\| + c \\ &\leq 2\gamma\|x\| + \|\gamma p_0\| + c. \end{aligned}$$

A similar result could also be applied to g_P. Hence, both f_P and g_P satisfy the linear growth condition.

By the hypothesis of the theorem we can conclude that for every $x \in \mathbb{R}^n$ we have:

$$\min\{\langle f_P(x), x - p\rangle, \langle g_P(x), x - p\rangle\} \leq 0.$$

Hence, we can always compute a measure μ such that any Euler solution $x(t)$ of

$$dx = f_P(x)dt + g_P(x)d\mu(t), \tag{9}$$

on $t \in [a, b]$, satisfies μ-a.e. on $t \in [a, b]$:

$$\langle f_P(x(t))dt + g_P(x(t))d\mu(t), x(t) - p(t)\rangle \leq 0. \tag{10}$$

The construction of such measure is state dependent and can be computed as follows. Consider that we make a partition π of the interval $[0, b - a + M]$, which represents the maximum range for $\eta(t)$. The measure is constructed at each node point i, with $i = 0, \cdots, N$. If we choose a singular atomic measure (only possible if $\langle g_P(x_i), x_i - p_i\rangle \leq 0$) then

$t_{i+1} = t_i$ and $|\mu(\{t_i\})| = s_{i+1} - s_i$. Otherwise, if the option is an absolutely continuous or singular continuous measure (always possible except when $\langle f_P(x_i), x_i - p_i\rangle > 0$ and $\langle g_P(x_i), x_i - p_i\rangle = 0$) then we should choose μ_{ac} and μ_{sc} such that (10) is verified. We define $w_{ac}(t)$ to be constant along $[t_i, t_{i+1})$ and the measure μ_{sc} is selected in a way that the function $F_{sc}(t) = \int_{[t_i,t]} d\mu_{sc}(t)$ has constant increase rate k_{sc} on $t \in [t_i, t_{i+1})$ such that $F_{sc}(t_{i+1}) = k_{sc}(t_i)(t_{i+1} - t_i)$. In this case, the node point t_{i+1} is computed as follows:

$$\begin{aligned} s_{i+1} - s_i &= t_{i+1} - t_i + \int_{[t_i,t_{i+1}]} (w_{ac}(t)dt + d\mu_{sc}) \\ &= (t_{i+1} - t_i)(1 + w_{ac}(t_i) + k_{sc}(t_i)) \Rightarrow \\ & \quad t_{i+1} = t_i + \frac{s_{i+1} - s_i}{1 + w_{ac}(t_i) + k_{sc}(t_i)}. \end{aligned} \tag{11}$$

Recall that s_i represents the i^{th} node of partition π in the range of η. From the previous construction, we can conclude that x_π and η_π are computed simultaneously. The node points of function $\eta_\pi(t)$ are given by (t_i, s_i), with $i = 0, \cdots, N$, and

$$\eta_\pi(t) = \eta_\pi(t_i) + t - t_i + \int_{[t_i,t]} d\mu_\pi(t), \ t \in [t_i, t_{i+1})$$

with $\eta_\pi(a) = a$ and μ_π being the measure constructed as before. Now let π_j be a sequence of partitions such that $\pi_j \downarrow 0$. The corresponding functions η_{π_j} in $[a, b]$ are of bounded variation and uniformly bounded since, by construction, $|\mu_{\pi_j}| < M$. Hence, by the Helly's selection principle, we can conclude that there is a subsequence of the family $\{\eta_{\pi_j}\}$ having a limit η that is a function of bounded variation. Consequently, we can also conclude that there is a subsequence of $\{\mu_{\pi_j}\}$ such that $\mu_{\pi_j} \to^* \mu$. Simultaneously, as $\pi_j \downarrow 0$, we also obtain an Euler solution x for problem (9) associated with the measure computed before.

Then, we can use proposition 4.1 to conclude that for any $x_0 \in S$ the resulting trajectory of system (9) lies in set S. However, we should note that neither f_P nor g_P are selections of F and G, respectively. Then, we cannot use theorem 3.2 to show that the Euler arc x is a trajectory of system (3). Hence, to show the desired result we follow a similar procedure as done in theorem 4.2.4 of [5] and, consequently, we define the following multifunctions $F_S(x) := \text{co}\{F(p) : p \in \text{proj}_S(x)\}$ and $G_S(x) := \text{co}\{G(p) : p \in \text{proj}_S(x)\}$. Clearly, F_S and G_S are closed and convex and both satisfy $F_S(x) = F(x)$ and $G_S(x) = G(x)$ if $x \in S$. By construction, we conclude that f_P and g_P are selections of F_S and G_S and, by theorem 3.2, we conclude that the Euler solution x presented before is a trajectory of $dx \in F_S(x)dt + G_S(x)d\mu$ a.e. on $[a, b]$. Since $x(t) \in S and x(s) \in S$, we reach to the conclusion that x is also a trajectory of $dx \in F(x)dt + G(x)d\mu$ a.e. on $[a, b]$ because $F_S = F$ and $G_S = G$ on S. ■

Remark 4.2: During the construction of the Euler polygonal arcs x_{π_j} it can happen that we reach $t = b$ before we have covered all the elements of the partition π. This can happen because the partition π was defined assuming the scenario where $|\mu([a, b])| = M$. However, for a given problem it is possible that we do not reach the maximum value. In these cases we only use the portion of the partition π corresponding to the used total variation.

Remark 4.3: Since we have a constraint in the total variation of the measure and it was constructed in feedback, in some circumstances it can happen that the total variation is reached before the fixed final time b. In these situations we have to redefine the final time to be precisely the instant where the total variation was reached. As an alternative to this situation we can add a new condition on the preceding theorem, which would be: "If the maximum total variation for the measure is reached then for every $x \in S$ we have $h_F(x, N_S^P(x)) \leq 0$". Note that in the construction of the measure we have extra degree of freedom for the choice of μ. Hence, we can choose μ such that it minimizes its total variation.

V. Conclusions

In the research reported here, we extended some of the concepts of nonsmooth control of conventional dynamic systems to impulsive control systems modeled by measure driven differential equations. We presented an impulsive Euler solution where the partition nodes were dependent on the range of the measure. This solution concept was crucial to derive feedback synthesis procedures to the impulsive control problem.

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