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# Optimal Control of a Parabolic PDE System Arising in Plasma Transport via Diffusivity–Interior–Boundary Actuation

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***Abstract*— In this paper, we study an optimal control problem arising in plasma transport which is governed by a singularly perturbed system. Time-scale separation allows us to focus on an uncoupled parabolic PDE with diffusivity–interior–boundary actuation. We prove the existence of the optimal control solution and carry out numerical experiments using sequential quadratic programming (SQP).**

## I. Introduction

Plasma, typically an ionized gas with free electrons, is considered to be a distinct state of matter because of its unique properties. The free electrical charges, without attachment to any atom, make the plasma electrically conductive so that it responds strongly to electromagnetic fields. Plasma transport studies the behaviors of physical variables such as plasma density, temperature and current (which is related to the magnetic flux), where multi–scale dynamics are common phenomena (see, e.g., [1], [2]).

We consider a simplified plasma transport model with 1D Eulerian geometry. The variations of the plasma temperature $T$ and magnetic flux $\psi$ are governed by the following system of coupled partial differential equations ($t > 0$ and $0 < x < 1$) (see, e.g., Chapter VI and Chapter VII in [3]):

$$\varepsilon \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left\{ D\left(T, \psi\right) \frac{\partial T}{\partial x} \right\} + \Gamma(T, \psi) + S_T(x, t), \qquad (1)$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial x}\left( \eta(T) \frac{\partial \psi}{\partial x} \right) + f(x) v(t), \qquad (2)$$

where $\varepsilon$ is a small scale constant. $D(T, \psi)$ is the energy transport coefficient, the nonlinear function $\Gamma(T, \psi)$ represents Joule heating, and $S_T(x, t)$ represents any external heating source, which can be used to shape the temperature profile. The nonlinear function $\eta(T)$ is the magnetic diffusion coefficient and is dependent on the temperature profile, $f(x)$ is the (positive) input function with respect to the interior control $v(t)$. In this simplified transport model (1)–(2), the contributions due to electron–ion energy exchange, radiation and excitation losses have been neglected.

The initial and boundary conditions for the transport model (1)–(2) are specified by

$$T(x, 0) = T_0(x), \quad \psi(x, 0) = \psi_0(x), \quad x \in [0, 1],$$
$$\frac{\partial T(0, t)}{\partial x} = \frac{\partial \psi(0, t)}{\partial x} = 0, \; T(1, t) = \varsigma, \; \frac{\partial \psi(1, t)}{\partial x} = w(t),$$

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where $T_0$ and $\psi_0$ are known positive continuous functions defined on $[0, 1]$ which are compatible with the boundary conditions, $\varsigma$ is a small known constant, and $w(t)$ is a boundary control of the system through the right–end Neumann condition.

For some plasma transport processes in fusion tokamaks, it is possible to assume that the temperature takes an spatial–temporal separation form, i.e., $\eta\left(x, t\right) = \gamma(x) u(t)$, where $\gamma(x)$ is a known spatial function which is identified from experimental data, and $u(t)$ is a continuous temporal function which depends on the dynamics of the temperature. Dynamic systems with two distinct time scales are referred to as singularly perturbed systems [4]. This is the case for system (1)-(2), where the scale parameter is very small, i.e., $\varepsilon \ll 1$. Therefore, since the dynamics of the temperature is much faster than that of the magnetic flux, we can consider the function $u(t)$ as a diffusivity control that could be used to shape the magnetic flux profile.

Taking into account the multiple scales of the problem, we can decouple the transport equations (1)–(2) using a singular perturbation approach. Thus, we define a control problem for a parabolic PDE (magnetic flux transport) with diffusivity–interior–boundary actuation. In controlled fusion experiments, it has been proved very important to achieve specific magnetic flux profiles to enhance confinement and steady-state operation. By using physical actuation mechanisms such as external heating sources, non-inductive current drives (neutral beams or radiofrequency waves), and total plasma current, we can indeed achieve independent diffusivity, interior and boundary actuation. This is a novel control problem arising in the field of plasma physics and controlled fusion.

In this paper, we consider an optimal control problem for the decoupled magnetic flux transport dynamics and study the existence of its solution as well as its numerical computation. We organize this paper as follows. In Section II, we present the mathematical model and formulate the optimal control problem. In Section III, we give the functional setting and necessary technical lemmas which are used for the proofs in this paper. In Section IV, we study the solution bound estimates which are used to show the existence of the optimal control in Section V. In Section VI, we summarize the foundation of PDE-based optimization and sequential quadratic programming (SQP), which is a powerful method to find the numerical solution for the optimal control problem proposed in the paper. We carry out numerical experiments and show the results in Section VII. We close the paper by stating the conclusions and research issues in Section VIII.

## II. STATEMENT OF CONTROL PROBLEM

Taking into account the spatial–temporal separation form $\eta(x,t) = \gamma(x)u(t)$ and the singularity ($\varepsilon \to 0$) of the energy transport equation (1), we rewrite the PDE (2) over the domain $Q_T = \{(x,t)|x \in \Omega = [0,1]; 0 \le t \le T\}$ as

$$\begin{cases} \dfrac{\partial \psi}{\partial t} = \dfrac{\partial}{\partial x}\left(\gamma(x)u(t)\dfrac{\partial \psi}{\partial x}\right) + f(x)v(t), \\ \dfrac{\partial \psi}{\partial x}(0,t) = 0, \quad \dfrac{\partial \psi}{\partial x}(1,t) = w(t), \\ \psi(x,0) = \psi_0(x), \end{cases} \tag{3}$$

where $\psi(x,t)$ is the state variable, and $\gamma(x)$, $f(x)$ are positive geometry parameters. The initial distribution of the state is denoted by $\psi_0(x)$. The three control functions $u(t)$, $v(t)$ and $w(t)$ represent the diffusivity, interior and (Neumann) boundary control, respectively, which satisfy the following constraints:

$$\mathcal{U} = \{u(t)|0 < L_u \le u(t) \le U_u,\ u \in C^1[0,T]\}, \tag{4}$$

$$\mathcal{V} = \{v(t)|0 < L_v \le v(t) \le U_v,\ v \in C^1[0,T]\}, \tag{5}$$

$$\mathcal{W} = \{w(t)|0 < L_w \le w(t) \le U_w,\ w \in C^1[0,T]\}, \tag{6}$$

where $L_{(\cdot)}$ and $U_{(\cdot)}$ are physical lower and upper bounds, respectively. The control goal can be stated as the following optimization problem:

$$\begin{aligned} &\min_{u\in\mathcal{U},v\in\mathcal{V},w\in\mathcal{W}} \quad J(u,v,w) \\ =&\frac{1}{2}\int_0^T \left[\theta_u u^2(t) + \theta_v v^2(t) + \theta_w w^2(t)\right] dt \\ &+ \frac{1}{2}\int_\Omega \theta_\psi \left|\psi(x,T) - \psi^d(x)\right|^2 dx, \end{aligned} \tag{7}$$

where $\theta_{(\cdot)}$ are weighting factors and $\psi^d(x)$ is the desired profile at $t = T$.

## III. FUNCTIONAL SETTINGS AND TECHNICAL LEMMAS

We define the following functional spaces

$$L^2(\Omega) = \left\{ f \,\middle|\, \int_\Omega |f|^2 dx < \infty \right\} \tag{8}$$

$$H^2(\Omega) = \{f|f \in L^2(\Omega) \text{ and } f' \in L^2(\Omega)\} \tag{9}$$

and denote their dual spaces [5] as $\left(L^2(\Omega)\right)'$ and $\left(H^2(\Omega)\right)'$, respectively. It is known in functional analysis that $L^2(\Omega)$ is self reflexive, i.e., $L^2(\Omega) = \left(L^2(\Omega)\right)'$. In the definitions (8)–(9), we can find that for any $f \in H^2(\Omega)$, it satisfies $f \in L^2(\Omega)$. Therefore, we have the embedding (inclusion) relation $H^2(\Omega) \subset L^2(\Omega)$. The dual representation of this embedding (inclusion) relation is $\left(L^2(\Omega)\right)' \subset \left(H^2(\Omega)\right)'$. Then we use the self reflexivity property $L^2(\Omega) = \left(L^2(\Omega)\right)'$, to connect these two inclusions and obtain the famous Gelfand triple [5]: $H^2(\Omega) \hookrightarrow L^2(\Omega) = \left(L^2(\Omega)\right)' \hookrightarrow \left(H^2(\Omega)\right)'$, where the notation $\hookrightarrow$ represents embedding (it roughly means inclusion). All the embeddings in the Gelfand triple are continuous, dense and compact. We introduce the functional space

$$\Xi = \left\{ \xi \in L^2(0,T;H^2(\Omega)); \frac{\partial \xi}{\partial t} \in L^2(0,T;\left(H^2(\Omega)\right)') \right\} \tag{10}$$

endowed with the norm

$$\|\xi\|_\Xi = \|\xi\|_{L^2(0,T;H^2(\Omega))} + \|\dot{\xi}\|_{L^2(0,T;(H^2(\Omega))')}. \tag{11}$$

*Lemma 1:* $\Xi$ is a Banach space. Every $\xi \in \Xi$ is continuous almost everywhere (a.e.) on $[0,T]$ with values in $L^2(\Omega)$. The embedding $\Xi \hookrightarrow L^2(0,T;L^2(\Omega))$ is compact.

*Lemma 2 (Poincare Inequality [6]):* For all $\xi \in C^1(\Omega)$, the following inequality holds for any subset $[0,r] = B_r \subset \Omega$:

$$\int_{B_r} (\xi - \overline{\xi})^2 dx \le C \int_{B_r} |\nabla \xi|^2 dx, \tag{12}$$

where $C$ is a positive constant and

$$\overline{\xi} = \frac{1}{\mathrm{Vol}(B_r)} \int_{B_r} \xi(x) dx, \tag{13}$$

where $\mathrm{Vol}(B_r)$ represents the volume of $B_r$.

*Lemma 3 (Fatou's Lemma [6]):* If $\{\xi_n\}$ is a sequence of nonnegative measurable functions on $\Omega$, then

$$\int_\Omega \liminf_{n\to\infty} \xi_n dx \le \liminf_{n\to\infty} \int_\Omega \xi_n dx. \tag{14}$$

*Lemma 4 (Cauchy's inequality [6]):* Given functions $f, g \in L^2(0,1)$ and $\mu > 0$, then we have the following inequality:

$$\int_0^1 fg dx \le \frac{1}{2\mu}\int_0^1 f^2 dx + \frac{\mu}{2}\int_0^1 g^2 dx. \tag{15}$$

## IV. A PRIORI ESTIMATES

We note that the solution of the PDE (3) depends on all the given control functions $u, v, w$ and also on the initial distribution $\psi_0(x)$. In this section, we will give some bounds estimates (a priori estimates) for the solution of the PDE system (3). Noting that the a priori bound estimate problem is different from a control design problem where the control functions are to be determined, we assume that the control functions are given to study the dynamics of the solutions. We first propose the following homogenization transform, where $w(t)$ is given such that $w \in \mathcal{W}$:

$$\Psi(x,t) = \psi(x,t) - \frac{1}{2}x^2 w(t), \tag{16}$$

which satisfies the homogeneous boundary conditions:

$$\frac{\partial \Psi}{\partial x}(0,t) = \frac{\partial \psi}{\partial x}(0,t) = 0, \tag{17}$$

$$\frac{\partial \Psi}{\partial x}(1,t) = \frac{\partial \psi}{\partial x}(1,t) - w(t) = 0. \tag{18}$$

Then, using (3) and (16)–(18), it is readily to obtain the following PDE for $\Psi$:

$$\begin{cases} \dfrac{\partial \Psi}{\partial t} = \dfrac{\partial}{\partial x}\left(\gamma(x)u(t)\dfrac{\partial \Psi}{\partial x}\right) - \dfrac{1}{2}x^2\dfrac{dw}{dt} \\ \qquad + \left(x\dfrac{d\gamma(x)}{dx} + \gamma(x)\right)u(t)w(t) + f(x)v(t), \\ \dfrac{\partial \Psi}{\partial x}(0,t) = \dfrac{\partial \Psi}{\partial x}(1,t) = 0, \\ \Psi(x,0) = \psi_0(x) - \dfrac{1}{2}x^2 w(0) = \Psi_0(x). \end{cases} \tag{19}$$

*Theorem 5:* We assume $\gamma \in C^1(\Omega) \cap L^2(\Omega)$, and $f \in L^2(\Omega)$, then for any $u \in \mathcal{U}, v \in \mathcal{V}$ and $w \in \mathcal{W}$, the solution $\Psi(x,t;u,v,w)$ of (19) exists and satisfies the bound estimate

$$\int_{Q_T} \left( |\Psi|^2 + \left|\frac{\partial \Psi}{\partial x}\right|^2 + \left|\frac{\partial \Psi}{\partial t}\right|^2 \right) dxdt \leq K_1, \tag{20}$$

where $K_1$ is constant and independent of the control functions $u, v, w$.

*Proof:* Existence and uniqueness of solution can be proved by following the literature, e.g., [6]. It remains to us to give a bound estimate.

Step 1. We multiply both sides of the PDE (19) by $\Psi$ and integrate over $Q_t = \{(x,\tau)|x \in \Omega = [0,1]; 0 \leq \tau \leq t \leq T\}$,

$$\begin{aligned}
&\frac{1}{2}\int_{Q_t} \frac{\partial \Psi^2}{\partial \tau} dxd\tau = \frac{1}{2}\int_\Omega \Psi^2(x,t)dx - \frac{1}{2}\int_\Omega \Psi^2(x,0)dx \\
&= -\int_{Q_t} \gamma(x)u(\tau)\left|\frac{\partial \Psi}{\partial x}\right|^2 dxd\tau + \int_{Q_t} f(x)v(\tau)\Psi dxd\tau \\
&+ \int_{Q_t} \left[ \left( x\frac{d\gamma(x)}{dx} + \gamma(x) \right) u(\tau)w(\tau) - \frac{x^2}{2}\frac{dw}{d\tau} \right] \Psi dxd\tau.
\end{aligned} \tag{21}$$

We use Cauchy inequality (Lemma 4) for the last two terms in (21), then we can rewrite (21) as

$$\begin{aligned}
&\int_\Omega \Psi^2(x,t)dx + 2\int_{Q_t} \gamma(x)u(\tau)\left|\frac{\partial \Psi}{\partial x}\right|^2 dxd\tau \\
&\leq C_0 + \frac{2}{\mu}\int_{Q_t} \Psi^2 dxd\tau,
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
C_0 = \mu \int_{Q_t} &\left| \left( x\frac{d\gamma(x)}{dx} + \gamma(x) \right) u(\tau)w(\tau) - \frac{x^2}{2}\frac{dw}{d\tau} \right|^2 dxd\tau \\
&+ \mu \int_{Q_t} (fv)^2 dxd\tau + \int_\Omega \Psi^2(x,0)dx.
\end{aligned}$$

Defining the average value ($|\Omega|$ = length of $\Omega$)

$$\overline{\Psi} = \frac{1}{|\Omega|T}\int_{Q_T} \Psi(x,t)dxdt, \tag{23}$$

where $T$ represents the length of the time interval $[0,T]$, we can use Poincare inequality (Lemma 2) to obtain

$$\begin{aligned}
\int_{Q_T} \Psi^2 dxdt &\leq \int_{Q_T} \left|\Psi - \overline{\Psi}\right|^2 dxdt + \int_{Q_T} \overline{\Psi}^2 dxdt \\
&\leq C_1 \int_{Q_T} \left|\frac{\partial \Psi}{\partial x}\right|^2 dxdt + \int_{Q_T} \overline{\Psi}^2 dxdt,
\end{aligned} \tag{24}$$

where $C_1$ is a positive constant.

Step 2. To obtain an estimate for the term $\int_{Q_T} \overline{\Psi}^2 dxdt$, we integrate the PDE (19) over $Q_t$,

$$\begin{aligned}
&\int_\Omega \Psi(x,t)dx - \int_\Omega \Psi(x,0)dx = -\int_{Q_t} \frac{1}{2}x^2\frac{dw}{d\tau}dxd\tau \\
&\quad + \int_{Q_t} \left[ \left( x\frac{d\gamma(x)}{dx} + \gamma(x) \right) u(\tau)w(\tau) + f(x)v(\tau) \right] dxdt \\
&\leq C_2,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
C_2 = \max_t &\left\{ \frac{w(0) - w(t)}{6}, 0 \right\} \\
&+ \left| \gamma(1) - \int_0^1 \gamma(x)dx \right| \int_0^T u(\tau)w(\tau)d\tau \\
&+ \int_\Omega \gamma(x)dx \int_0^T u(\tau)w(\tau)d\tau + \int_{Q_T} f(x)v(\tau)dxd\tau.
\end{aligned}$$

Then, we integrate $\int_\Omega \Psi(x,t)dx$ from 0 to $t$,

$$\int_{Q_t} \Psi(x,\tau)dxd\tau \leq \left( C_2 + \int_\Omega |\Psi_0(x)|dx \right) t \leq C_3 T, \tag{26}$$

where $C_3 := C_2 + \int_\Omega |\Psi_0(x)|dx$. Taking into account the definition of the mean value over $Q_T$ (23), we can rewrite (26) to obtain

$$\overline{\Psi} = \frac{1}{|\Omega T|}\int_{Q_T} \Psi(x,\tau)dxd\tau \leq \frac{C_3}{|\Omega|}, \tag{27}$$

which makes (24) become

$$\int_{Q_T} \Psi^2 dxdt \leq C_1 \int_{Q_T} \left|\frac{\partial \Psi}{\partial x}\right|^2 dxdt + \frac{C_3^2 T}{|\Omega|}. \tag{28}$$

Now we can use (28) to update the bound in (22)

$$\begin{aligned}
&\int_\Omega \Psi^2(x,t)dx + 2\int_{Q_T} \gamma(x)u(t)\left|\frac{\partial \Psi}{\partial x}\right|^2 dxdt \\
&\leq C_0 + \frac{2C_1}{\mu}\int_{Q_T} \left|\frac{\partial \Psi}{\partial x}\right|^2 dxdt + \frac{2C_3^2 T}{\mu|\Omega|}.
\end{aligned} \tag{29}$$

We note that the continuous coefficient $\gamma(x)u(t)$ in (29) can be bounded from below, i.e., $\inf_{x,t}[\gamma(x)u(t)] \leq \gamma(x)u(t)$. Then, (29) becomes

$$\begin{aligned}
&\sup_t \int_\Omega \Psi^2(x,t)dx \\
&\quad + 2\inf_{x,t}\left[ \gamma(x)u(t) - \frac{C_2}{\mu} \right] \int_{Q_T} \left|\frac{\partial \Psi}{\partial x}\right|^2 dxdt \leq C_4,
\end{aligned} \tag{30}$$

where $C_4 = C_0 + \frac{2C_3^2 T}{\mu|\Omega|}$. We can choose $\mu$ large enough in (22) when using Cauchy's inequality and make

$$\inf_{x,t}\left[ \gamma(x)u(t) - \frac{C_2}{\mu} \right] \geq 0. \tag{31}$$

Step 3. We multiply both sides of the PDE (19) by $\frac{\partial \Psi}{\partial t}$, integrate over $Q_T$, and apply Cauchy inequality to obtain

$$\begin{aligned}
&\int_{Q_T} \left|\frac{\partial \Psi}{\partial t}\right|^2 dxdt + \frac{1}{2}\int_{Q_T} \gamma(x)u(t)\frac{\partial}{\partial t}\left(\frac{\partial \Psi}{\partial x}\right)^2 dxdt \\
&= \int_{Q_T} F(x,t)\frac{\partial \Psi}{\partial t}dxdt \\
&\leq \frac{\mu}{2}\int_{Q_T} F^2(x,t)dxdt + \frac{1}{2\mu}\int_{Q_T} \left|\frac{\partial \Psi}{\partial t}\right|^2 dxdt
\end{aligned} \tag{32}$$

where

$$F(x,t) = -\frac{1}{2}x^2\frac{dw}{dt} + \left( x\frac{d\gamma(x)}{dx} + \gamma(x) \right) u(t)w(t) + f(x)v(t).$$

Then (32) becomes

$$\left(2-\frac{1}{\mu}\right)\int_{Q_T}\left|\frac{\partial\Psi}{\partial t}\right|^2 dxdt + \inf_{x,t}[\gamma(x)u(t)]\int_\Omega\left|\frac{\partial\Psi(x,T)}{\partial x}\right|^2 dx \le C_5, \tag{33}$$

where

$$C_5=\mu\int_{Q_T}F^2(x,t)dxdt+\inf_{x,t}[\gamma(x)u(t)]\int_\Omega\left|\frac{\partial\Psi(x,0)}{\partial x}\right|^2 dx.$$

Combining estimates (30) and (33), we can find that $\|\Psi\|^2$, $\|\frac{\partial\Psi}{\partial x}\|^2$ and $\|\frac{\partial\Psi}{\partial t}\|^2$ can be bounded by certain positive numbers. Therefore, there must exist a positive constant $K_1$ to satisfy the following estimate

$$\int_{Q_T}\left(|\Psi|^2+\left|\frac{\partial\Psi}{\partial x}\right|^2+\left|\frac{\partial\Psi}{\partial t}\right|^2\right)dxdt \le K_1. \tag{34}$$

■

*Corollary 6:* We assume $\gamma \in C^1(\Omega)\cap L^2(\Omega)$, and $f \in L^2(\Omega)$, then for any $u\in\mathcal{U}, v\in\mathcal{V}$ and $w\in\mathcal{W}$, the solution $\psi(x,t;u,v,w)$ of (3) exists and satisfies the following bound estimate

$$\int_{Q_T}\left(|\psi|^2+\left|\frac{\partial\psi}{\partial x}\right|^2+\left|\frac{\partial\psi}{\partial t}\right|^2\right)dxdt \le K_2, \tag{35}$$

where $K_2$ is a constant.

*Proof:* Taking into account that

$$\psi(x,t)=\Psi(x,t)+\frac{1}{2}x^2w(t), \tag{36}$$

$$\frac{\partial\psi}{\partial x}(x,t)=\frac{\partial\Psi}{\partial x}(x,t)+xw(t), \tag{37}$$

$$\frac{\partial\psi}{\partial t}(x,t)=\frac{\partial\Psi}{\partial t}(x,t)+\frac{1}{2}x^2\frac{dw}{dt}(t), \tag{38}$$

existence and uniqueness of the solution of (19) can ensure those of the solution of (3). Then, we have

$$\begin{aligned}&\int_{Q_T}|\psi|^2dxdt\\ &=\int_{Q_T}|\Psi+\frac{1}{2}x^4w(t)|^2dxdt\\ &\le\int_{Q_T}|\Psi|^2dxdt+\frac{1}{4}\int_\Omega x^2dx\int_0^T w^2(t)dt\\ &=\int_{Q_T}|\Psi|^2dxdt+\frac{1}{20}\int_0^T w^2(t)dt := C_6,\end{aligned} \tag{39}$$

$$\begin{aligned}&\int_{Q_T}\left|\frac{\partial\psi}{\partial x}\right|^2dxdt\\ &\le\int_{Q_T}\left|\frac{\partial\Psi}{\partial x}\right|^2dxdt+\int_\Omega x^2dx\int_0^T w^2(t)dt\\ &=\int_{Q_T}|\Psi|^2dxdt+\frac{1}{3}\int_0^T w^2(t)dt := C_7,\end{aligned} \tag{40}$$

and

$$\begin{aligned}&\int_{Q_T}\left|\frac{\partial\psi}{\partial t}\right|^2dxdt\\ &\le\int_{Q_T}\left|\frac{\partial\Psi}{\partial t}\right|^2dxdt+\frac{1}{4}\int_\Omega x^4dx\int_0^T\left(\frac{dw}{dt}\right)^2(t)dt\\ &=\int_{Q_T}\left|\frac{\partial\Psi}{\partial t}\right|^2dxdt+\frac{1}{20}\int_0^T\left(\frac{dw}{dt}\right)^2(t)dt := C_8.\end{aligned} \tag{41}$$

Therefore, we can follow the same procedure as in the proof of Theorem 5 to finish this proof. ■

## V. Existence of Optimal Control

Assume three minimizing sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$, such that

$$\lim_{n\to\infty}J(u_n,v_n,w_n)=\inf_{u,v,w}J(u,v,w).$$

Let $\psi_n=\psi(u_n,v_n,w_n)$ be the corresponding solution of the PDE (3), then by the a priori estimates in Theorem 5 and Corollary 6, we can obtain

$$\int_{Q_T}\left(|\psi_n|^2+\left|\frac{\partial\psi_n}{\partial x}\right|^2+\left|\frac{\partial\psi_n}{\partial t}\right|^2\right)dxdt \le K_2, \tag{42}$$

where $K_2$ is a constant independent of $n$. By the weak convergence theory [5], we can extract weakly convergent sequences

$$\begin{aligned}\frac{\partial\psi_n}{\partial t}&\longrightarrow\frac{\partial\psi}{\partial t}, \text{ weakly in } L^2(0,T;\left(H^2(\Omega)\right)'),\\ \psi_n&\longrightarrow\psi, \text{ weakly in } L^2(0,T;H^2(\Omega)),\\ u_n&\longrightarrow u, \text{ weakly in } L^\infty(0,T),\\ v_n&\longrightarrow v, \text{ weakly in } L^\infty(0,T),\\ w_n&\longrightarrow w, \text{ weakly in } L^\infty(0,T).\end{aligned}$$

Additionally, we note that the embedding $\Xi\hookrightarrow L^2(L^2(\Omega))$ is compact (Lemma 1), then the sequence $\{\psi_n\}$ admits a subsequence which converges strongly in $L^2(L^2(\Omega))$. Therefore, we can show the existence of the optimal controls $(Q:=(u,v,w)^T)$,

$$\begin{aligned}J(Q^*)&\le\inf_n\int_\Omega\psi_n(x,T)dx+\inf_n\int_0^T\alpha u_n^2+\beta v_n^2+\gamma w_n^2dt\\ &\le\liminf_{n\to\infty}J(Q_n),\end{aligned} \tag{43}$$

where we used Fatou's lemma (Lemma 3) to change the order of the $\inf$ and $\lim$ operations.

## VI. PDE-based Optimization

In this section, we rewrite the PDE–based optimization problem into a large scale ODE-based optimization problem and summarize the foundation of the nonlinear programming using the sequential quadratic programming (SQP) method (see, e.g., [7]). We discretize the PDE on a given spatial grid, which generates a large scale ODE–based optimization problem:

$$\min_{y,p} \quad \Phi(y,p) \tag{44}$$

$$\frac{dy}{dt} = F(t,y,p), \quad y(0) = y_0 \tag{45}$$

$$g(t,y(t),p) \geq 0, \tag{46}$$

where $p$ is the parameterization vector of the control functions $u, v, w$; cost functional (44) is the discrete version of the cost functional (7); ODE system (45) for $y$ represents the space discrete version of the PDE system (3) for $\psi(x,t)$; and inequality (46) includes all the constraints in terms of the optimization problem stated in Section II.

Introducing a new vector $\mathbf{x} = (y,p)^T$, we can rewrite the ODE-based optimization problem (44)–(46) into the following standard nonlinear programming formulation:

$$\min_{\mathbf{x}} \quad \mathcal{F}(\mathbf{x}) := \Phi(y,p) \tag{47}$$

$$c_1(\mathbf{x}) := \frac{dy}{dt} - F(t,y,p) = 0, \tag{48}$$

$$c_2(\mathbf{x}) := g(t,y(t),p) \geq 0. \tag{49}$$

The Lagrangian multiplier method [8] can be used to solve this constrained optimization problem (47)–(49). One can define the Lagrangian $\mathcal{L}(\mathbf{x},\pi) := \mathcal{F}(\mathbf{x}) - \pi^T c(\mathbf{x})$, where $c(\mathbf{x}) = (c_1(x), c_2(x))^T$ represents the constraints and $\pi$ is the Lagrangian multiplier. Then, the nonlinear optimization problem can be reformulated as

$$\min_{\mathbf{x}} \mathcal{L}(\mathbf{x}), \quad c_1(\mathbf{x}) = 0, \, c_2(\mathbf{x}) \geq 0. \tag{50}$$

The SQP method can be used to solve (50) by generating a sequence of iteration points $(\mathbf{x}_k, \pi_k)$ which converge to a local minimum pair $(\mathbf{x}^*, \pi^*)$. Let $(\mathbf{x}_k, \pi_k)$ be the current estimate of $(\mathbf{x}^*, \pi^*)$, then the nonlinear optimization problem (50) can be linearized around $(\mathbf{x}_k, \pi_k)$ to obtain the following quadratic programming (QP) problem:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathcal{L}(\mathbf{x}_k) + \nabla \mathcal{L}(\mathbf{x}_k)^T (\mathbf{x} - \mathbf{x}_k) \\ & + \frac{1}{2} (\mathbf{x} - \mathbf{x}_k)^T \nabla^2 \mathcal{L}_k (\mathbf{x} - \mathbf{x}_k) \end{aligned} \tag{51}$$

$$c_1(\mathbf{x}_k) + \nabla c_1^T(\mathbf{x}_k)(\mathbf{x} - \mathbf{x}_k) = 0, \tag{52}$$

$$c_2(\mathbf{x}_k) + \nabla c_2^T(\mathbf{x}_k)(\mathbf{x} - \mathbf{x}_k) \geq 0. \tag{53}$$

The obtained QP problem (51)–(53) can be solved using the Newton's method and the current estimation $(\mathbf{x}_k, \pi_k)$ can be updated. An initial guess and an error tolerance condition are necessary to start and stop the SQP iteration, respectively. For more details on numerical optimization and SQP, please refer to [9]. Some commercially available software (such as Matlab [10], SNOPT [7]) can be used to implement the SQP algorithm.

## VII. Numerical Examples

We consider the following simplified system with $\gamma(x) = 1$ and $f(x) = \sin(\pi x)$:

$$\begin{cases} \dfrac{\partial \psi}{\partial t} = u(t) \dfrac{\partial^2 \psi}{\partial x^2} + \sin(\pi x) v(t), \\ \dfrac{\partial \psi}{\partial x}(0,t) = 0, \quad \dfrac{\partial \psi}{\partial x}(0,t) = w(t), \\ \psi(x,0) = 0.1. \end{cases} \tag{54}$$

The associated cost functional is given by

$$\begin{aligned} & \min_{u \in [1,2], v \in [0.1,10], w \in [0.1,10]} \quad J(u,v,w) \\ = & \frac{1}{2} \int_0^1 \left[ u^2(t) + v^2(t) + w^2(t) \right] dt \\ & + \frac{30}{2} \int_0^1 |\psi(x,1) - 1|^2 \, dx. \end{aligned} \tag{55}$$

We choose the values of the actuator trajectories at given equidistant points $0, 0.25, 0.5, 0.75, 1.0$ (unit: second). We use spline approximations to represent the control functions based on the values $\mathbf{u} = u(\mathbf{t}) \in \mathbb{R}^{1\times 5}$, $\mathbf{v} = v(\mathbf{t}) \in \mathbb{R}^{1\times 5}$ and $\mathbf{w} = w(\mathbf{t}) \in \mathbb{R}^{1\times 5}$ over $\mathbf{t} = [0, 0.25, 0.5, 0.75, 1.0]$. We discretize the temporal–spatial domain into the following equidistant grid

$$0 = x_1 < x_2 < \cdots < t_i < \cdots < x_M = 1, \quad M = 50, \tag{56}$$

$$0 = t_1 < t_2 < \cdots < x_j < \cdots < t_N = 1, \quad N = 20. \tag{57}$$

Then, we can write the discrete version of the cost functional (55) as

$$\begin{aligned} \min_{\mathbf{u},\mathbf{v},\mathbf{w}} \quad J_d(u,v,w) = & \frac{30 \Delta x}{2} \sum_{i=1}^{M} |\psi(x_i,1) - 1|^2 \\ & + \frac{\Delta t}{2} \sum_{j=1}^{N} \left[ u^2(t_j) + v^2(t_j) + w^2(t_j) \right]. \end{aligned} \tag{58}$$

We use the Matlab function `fmincon` to implement the numerical optimization in terms of the discrete cost functional (58). The Matlab function `pdepe` is used as the computing engine which is included in running the `fmincon` command. An initial guess of the control actuation ($\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$) is necessary to start the optimization process which is carried out by the Matlab function `rand`. The optimization results for the three controls are shown in Fig. 1– 3, while Fig. 4 shows the dynamic evolution of the PDE system with computed control functions. Fig. 5 extracts the final profile at $t = T$ which is close to the target $\psi^d(x) = 1$. We use Fig. 6 to show the change of the cost function value with respect to the optimization iterations.

## VIII. Conclusions

In this paper, we prove the existence of optimal controls of a parabolic PDE arising in plasma transport. For real–time tracking of the obtained optimal trajectories, we can linearize the original PDE with multiplicative control (bi-linearity) around the optimal trajectories to obtain a standard linear parabolic PDE with both the boundary control and interior

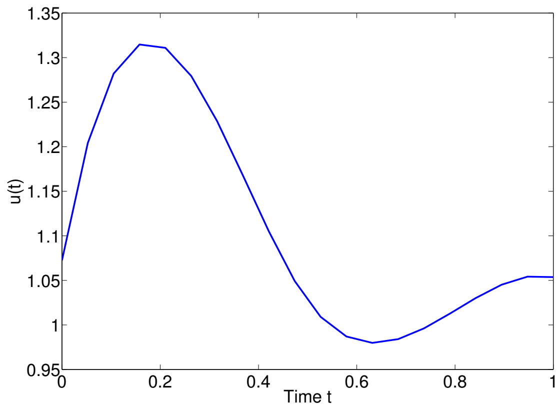


Fig. 1. Diffusivity control $u(t)$.

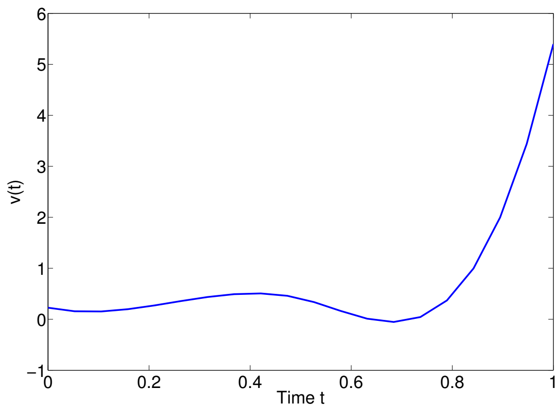


Fig. 2. Interior control $v(t)$.

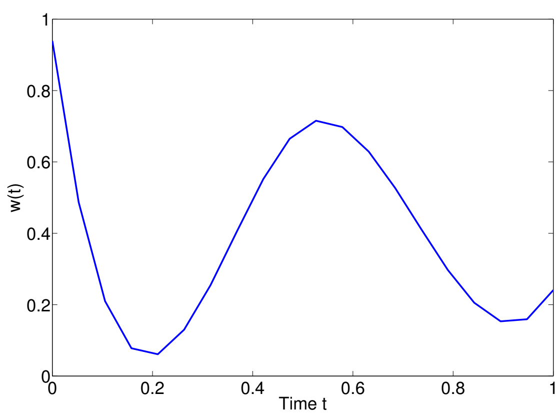


Fig. 3. Boundary control $w(t)$.

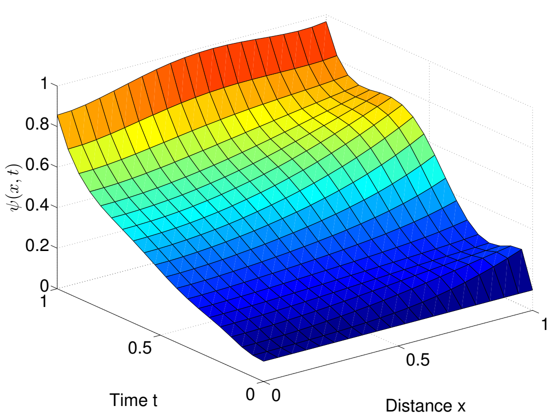


Fig. 4. Dynamic evolution of the controlled system.

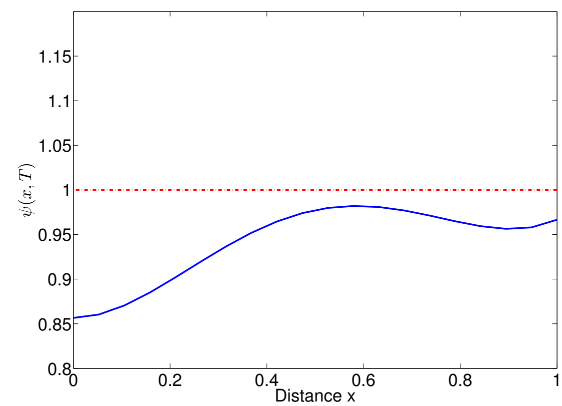


Fig. 5. Final profile at $t = T$.

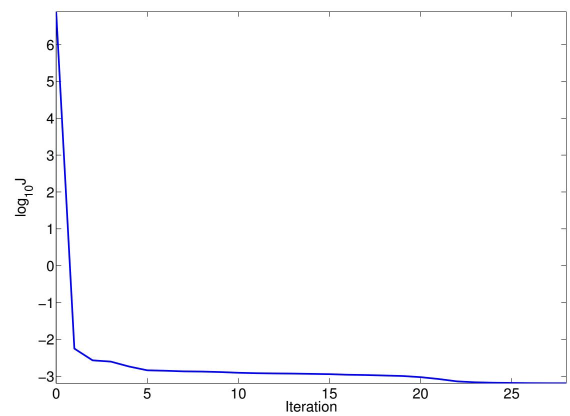


Fig. 6. Change of the cost function.

controls (the diffusivity control is included into the interior control vector after linearization). Many control design techniques for linear systems (either distributed parameter systems or lumped parameter systems) are then available to obtain feedback laws.

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