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Power Series Solution of the Hamilton-Jacobi-Bellman Equation for DAE Models with a Discounted Cost

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Abstract— **This paper considers infinite horizon optimal feedback control of nonlinear models with discounted cost. The paper includes two extensions of existing results about optimal feedback control. First, it is proven that for real analytic state-space models, a time-invariant real analytic feedback solution exists, even when the cost function includes a discount factor, provided certain regularity conditions. Second, the result is generalized to nonlinear DAE models. The feedback solution is valid in a neighborhood of the origin. In both cases, explicit formulas for the series expansions of the cost function and control law are given.**

I. Introduction

The standard approach to compute the optimal solution to an optimal feedback control problem is to use the Hamilton-Jacobi-Bellman equation (HJB). The problem is that this equation involves a nonlinear partial differential equation, which for most problems is very hard to solve analytically. The idea is to instead compute the optimal solution, *i.e.*, the optimal performance criterion and the corresponding feedback law, expressed as power series expansions. For state-space models this idea was first considered by [1]. The results in that paper show that the terms in the power series expansions can be obtained sequentially, by first solving a quadratic optimal control problem for the linearized system and then a series of linear partial differential equations. Further, a formal proof of the convergence of the power series is presented in the case when the input signal is scalar and the system has the form $\dot{x} = f(x) + Bu$. In [2], these results are extended to general state-space systems, $\dot{x} = f(x, u)$, and this work is extended even more in [3]. In the earlier works [1], [2], the functions involved are required to be analytic functions around the origin. In [3], this requirement is relaxed to twice differentiability. An alternative proof to the one presented in [3] is given in [4], where the requirements on the cost function are relaxed. [5] studied the case when the dynamics of an external signal generator are included, and in [6] the case when the system is not stabilizable or not detectable is investigated. Finally, the method above is extended to nonlinear DAE models in [7].

In some cases, it is interesting to have a bound on the convergence rate for the states of a system as the time goes to infinity. For linear systems, this fact was developed already in [8] and has later been developed in numerous publications. For other problems, it can be interesting to reduce the penalty when the time increases, see [9]. The main motivation for this choice is often that the most important parts of the state trajectory are those in the beginning, while what happens far into the future is less interesting. The feature above is obtained using a discount factor in the cost function. In this paper, the power series method for solving optimal control problems will be extended to deal with cost functions involvning such a discount factor both for nonlinear state-space models and DAE models.

In practice, the series solution needs to be truncated and the result is an approximative solution. Therefore, this kind of methods is often denoted approximative methods even though the complete power series expansions of the performance criterion and feedback law yield the true optimal solution. There are other methods which theoretically describe the exact optimal solution but in practice are truncated, see [10] and references therein.

Notation: The notation in this paper is fairly standard. The Jacobian matrix $\frac{\partial h}{\partial x}$ will be denoted h_x and $(\cdot)^{[i]}$ will be used to denote the terms of order i in a power series expansion. $Q \succ 0$ means that Q is a real positive definite matrix. $\lfloor m \rfloor$ will denote the integer part of m.

II. Problem Formulation

The considered optimal control problem is to minimize the integral criterion

$$V(x_{1,0}) = \inf_{u(\cdot)} \int_0^\infty L(x_1, x_3, u) e^{\lambda t} \, dt \tag{1}$$

subject to the differential-algebraic system

$$\hat{F}_1(\dot{x}_1, x_1, x_3, u) = 0 \tag{2a}$$

$$\hat{F}_2(x_1, x_3, u) = 0 \tag{2b}$$

with some initial condition $x_1(0) = x_{1,0}$. The term $\lambda \in \mathbb{R}$ is constant and can take arbitrary values both positive and negative. It will be denoted the discount factor. The initial condition is assumed to be consistent, *i.e.*, to satisfy $\hat{F}_2\big(x_{1,0}, x_3(0), u(0)\big) = 0$. Furthermore, it is assumed that following assumption is satisfied.

Assumption 1: It holds that $\hat{F}(0,0,0,0) = 0$. Furthermore, $\hat{F}_{1;\dot{x}_1}(0,0,0,0)$ and $\hat{F}_{2;x_3}(0,0,0)$ are nonsingular.
From the implicit function theorem, it then follows that there exists a neighborhood Ω of the origin, such that for $(\dot{x}_1, x_1, x_3, u) \in \Omega$, the DAE model can be written as

$$\dot{x}_1 = \mathcal{L}(x_1, u) \tag{3a}$$

$$x_3 = \mathcal{R}(x_1, u) \tag{3b}$$

If the original DAE model not satisfies Assumption 1, it is shown in [11] that using so-called index reduction rather general models can be rewritten in this form.

The assumptions above only guarantee that the DAE model has an underlying state-space model. However, the functions $\mathcal{L}$ and $\mathcal{R}$ may be hard or even impossible to express in closed form. Therefore, the derived method will rely on the Taylor series of $\hat{F}_1$, $\hat{F}_2$ and $\hat{L}$. It means that the following assumption is needed.

Assumption 2: The functions $\hat{F}_1$ and $\hat{F}_2$ in (2), and L in (1) are real analytic in $\mathcal{W}$, which is a neighborhood of the origin $(\dot{x}_1, x_1, x_3, u) = 0$.

Analyticity of the functions involved makes it possible to write them as power series

$$\hat{F}_1(\dot{x}_1, x_1, x_3, u) = \\ - E_1\dot{x}_1 + A_{11}x_1 + A_{12}x_3 + B_1u + \hat{F}_{1h}(\dot{x}_1, x_1, x_3, u) \tag{4a}$$

$$\hat{F}_2(x_1, x_3, u) = A_{21}x_1 + A_{22}x_3 + B_2u + \hat{F}_{2h}(x_1, x_3, u) \tag{4b}$$

$$L(x, u) = x^TQx + u^TRu + 2x^TSu + L_h(x, u) \tag{4c}$$

that are convergent in $\mathcal{W}$. The functions $\hat{F}_{1h}$ and $\hat{F}_{2h}$ include terms beginning with order two, while the higher order terms of the performance criterion, that is L_h, is of order three at least.

The linearization of the system (2) is easily be found as

$$\dot{x}_1 = \hat{A}x_1 + \hat{B}u \tag{5a}$$

$$x_3 = -A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}B_2u \tag{5b}$$

where

$$\hat{A} = E_1^{-1}\big(A_{11} - A_{12}A_{22}^{-1}A_{21}\big) \tag{5c}$$

$$\hat{B} = E_1^{-1}\big(B_1 - A_{12}A_{22}^{-1}B_2\big) \tag{5d}$$

The objective is to find the optimal feedback control locally around the origin. However, because of the infinite horizon, the considered class of feedback laws needs to satisfy some extra conditions.

Assumption 3: The considered feedback laws are described by uniformly convergent power series

$$u(x_1) = Dx_1 + u_h(x_1) \tag{6}$$

where $u_h(x_1)$ are terms of at least order two. Furthermore,

$$\mathrm{Re}\,\mathrm{eig}(\hat{A} + \hat{B}D) < \min(0, -\tfrac{\lambda}{2})$$

where $\hat{A}$ and $\hat{B}$ are given in (5).

The last part of the assumption is introduced for two reasons. First, it is necessary for the proof to have a feedback law that stabilizes the system, and thereby makes a neighborhood of the origin invariant. Second, the control law needs to ensure convergence of the integral criterion, locally.

As can be seen above, the discount factor is used to obtain a controller for which the linearization of the closed loop gets a prescribed degree of stability. That is, the poles are placed to the left of some specific limit, see [12]. It can also be used to reduce the penalty for large t. This is for example used in [9]. One interesting fact, that in many cases probably can be rather important, is that despite that the cost function is explicitly time-varying, the optimal solution will still be time-invariant. That the optimal feedback law becomes time-invariant simplifies the implementation of it. In [12] it is shown that a discount factor term $e^{\lambda t}$, in principle, is the only time-varying element allowed in order to have this property.

III. The State-Space Model Case

If the DAE model has no constraints and is explicit in the states, the optimal control problem can be written as

$$\begin{aligned} V(x_0) = \inf_{u(\cdot)} &\int_0^\infty L(x,u)e^{\lambda t}\, dt \\ \text{s.t.} \quad &\dot{x} = F(x,u) \\ &x(0) = x_0 \end{aligned} \tag{7}$$

The solution to this optimal control problem is given by the following Hamilton-Jacobi-Bellman equation (HJB), see [9],

$$0 = \min_u L(x,u) + \lambda V(x) + V_x(x)F(x,u)$$

and the optimal feedback law $u_*(x)$ must solve the equations

$$0 = L\big(x, u_*(x)\big) + \lambda V(x) + V_x(x)F\big(x, u_*(x)\big) \tag{8a}$$

$$0 = L_u\big(x, u_*(x)\big) + V_x(x)F_u\big(x, u_*(x)\big) \tag{8b}$$

In this case, only stabilizing feedback laws are considered and by just evaluate the cost criterion, it follows that the optimal return function $V(x)$ must have the structure

$$V(x) = x^TPx + V_h(x) \tag{9}$$

where P is a symmetrical matrix and $V_h(x)$ contains the terms of order three and higher, see [3], [7]. If the expressions for V and u are substituted into (8), the result are two polynomial equations in x with coefficients including the unknowns parameters in V and u_*. The following theorem states when an analytic solution to these equations exists and how it can be computed for state-space models.

Theorem 1: Consider the optimal control problem (7), satisfying Assumption 2. Furthermore, assume that the quadratic part of the cost function (4c) satisfies $\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \succ 0$. Then there exists an optimal feedback law $u_*(x)$ satisfying Assumption 3 if the ARE

$$0 = (A+\tfrac{\lambda}{2}I)^TP + P(A+\tfrac{\lambda}{2}I) - (PB+S)R^{-1}(PB+S)^T + Q \tag{10a}$$

has a unique positive-semidefinite solution such that the matrix $A + BD_*$ with D_* given by

$$D_* = -R^{-1}(S^T + B^TP) \tag{10b}$$

satisfies

$$\mathrm{Re}\,\mathrm{eig}(A + BD_*) < \min(0, -\tfrac{\lambda}{2}) \tag{11}$$

The equations (10) also determine the lowest order terms in $V(x)$ and $u_*(x)$, respectively. The higher order terms in

$V(x)$ and $u_*(x)$ can be computed recursively by solving

$$V_x^{[m]}(x)A_c x + \lambda V^{[m]}(x) = -\sum_{k=3}^{m-1} V_x^{[k]}(x)Bu_*^{[m-k+1]}(x) - \sum_{k=2}^{m-1} V_x^{[k]}(x)F_h^{[m-k+1]}(x,u_*) - L_h^{[m]}(x,u_*) - 2\sum_{k=2}^{\lfloor \frac{m-1}{2} \rfloor} u_*^{[k]}(x)^T R u_*^{[m-k]}(x) - u_*^{[m/2]}(x)^T R u_*^{[m/2]}(x) \tag{12a}$$

where $m = 3, 4, \ldots$ and $A_c = A + BD_*$, and

$$u_*^{[k]}(x) = -\frac{1}{2}R^{-1}\Big(V_x^{[k+1]}(x)B + \sum_{i=1}^{k-1} V_x^{[k-i+1]}(x)F_{h;u}^{[i]}(x,u_*) + L_{h;u}^{[k]}(x,u_*)\Big) \tag{12b}$$

for $k = 2, 3, \ldots$.
In the equations above, $F^{[i]}$ denotes the i:th order terms of F and $\lfloor i \rfloor$ denotes the floor function, which gives the largest integer less than or equal to i. Moreover, in (12) we use the conventions that $\sum_k^l = 0$ for $l < k$ and that the terms $u^{[m/2]}$ are to be omitted if m is odd.
Proof: The full version of the proof of the case with $\lambda = 0$ can be found in [3], while the general proof for arbitrary λ can be found in [7]. Below, a sketch of the most important steps is presented.

The first step is to evaluate the performance criterion for arbitrary $u(x)$ such that $A + BD$ satisfies the eigenvalue condition (11). The closed loop system will, because of Assumption 2 and 3, have the form

$$\dot{x} = F\big(x, u(x)\big) = (A + BD)x + F_h\big(x, u(x)\big) \tag{13}$$

where $F_h(\cdot)$ is a uniformly convergent power series around the origin, beginning with terms of order two. The solution to the closed loop system will be denoted $x(t, x_0)$ where $x_0 \in \mathbb{R}^n$ is the initial value, that is, $x(0, x_0) = x_0$. In a neighborhood of the origin, the following inequality is ensured by Assumption 3

$$|x(t, x_0)| \leq C_0 e^{\mu t}|x_0|$$

for some μ that satisfies $\operatorname{Re}\operatorname{eig}(A + BD) < \mu < \min(0, -\frac{\lambda}{2})$. Using Assumption 2, it then follows that

$$J\big(x_0, u\big) = \int_0^\infty L\big(x(t,x_0), u(x(t,x_0))\big)e^{\lambda t}\, dt$$

is uniformly convergent locally around the origin. It can also be shown that for all such $u(x)$, $J(x_0, u)$ will satisfy

$$0 = \lambda J(x_0, u) + J_{x_0}(x_0, u)F\big(x_0, u(x_0)\big) + L\big(x_0, u(x_0)\big) \tag{14}$$

for x_0 in a neighborhood of the origin.

In the next step, it is shown that for an arbitrary p, the following equation

$$0 = L_u(x, u) + F_u(x, u)p \tag{15}$$

has a unique analytic solution $u_*(x, p)$ near the origin in $\mathbb{R}^{2n}$ for which $u_*(0, 0) = 0$. Furthermore,

$$u_*(x, p) = -\tfrac{1}{2}R^{-1}(2S^T x + B^T p) + u_{*,h}(x, p) \tag{16}$$

where $u_{*,h}(x, p)$ is a convergent power series in a neighborhood of the origin beginning with second order terms.

The major step is then to show that there exists a $u_*(x) = u_*\big(x, p(x)\big)$ that satisfies Assumption 3 and solves (15) with $p(x) = V_x(x)$, since the optimal feedback law have to satisfy (8b). Here $V(x)$ denotes $J(x, u)$ with $u_*(x)$ as feedback law. This $u_*(x)$ is then the optimal control, which can be shown rather forwardly from the fact that R in (4c) is assumed to be positive definite, see [7].

To show the existence of such a u_*, study the nonlinear Hamiltonian system

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = H \begin{pmatrix} x \\ p \end{pmatrix} + r(x, p) \tag{17}$$

where H is

$$H = \begin{pmatrix} A - BR^{-1}S^T & -\frac{1}{2}BR^{-1}B^T \\ -2(Q - SR^{-1}S^T) & -(A - BR^{-1}S^T + \lambda I)^T \end{pmatrix}$$

and

$$r(x,p) = \begin{pmatrix} Bu_{*,h} + F_h\big(x, u_*(x,p)\big) \\ -2Su_{*,h} - L_{h;x}\big(x, u_*(x,p)\big) - F_{h;x}\big(x, u_*(x,p)\big) \end{pmatrix}$$

It can be shown that by using the nonsingular real linear transformation

$$\begin{pmatrix} y \\ q \end{pmatrix} = M \begin{pmatrix} x \\ p \end{pmatrix}$$

where

$$M = \begin{pmatrix} I - 2Q_*P & Q_* \\ 2P & -I \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} I & Q_* \\ 2P & 2PQ_* - I \end{pmatrix}$$

the system (17) is transformed into

$$\begin{pmatrix} \dot{y} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} A_c & 0 \\ 0 & -(A_c + \lambda I)^T \end{pmatrix} \begin{pmatrix} y \\ q \end{pmatrix} + r_M(y, q) \tag{18}$$

where

$$r_M(y, q) = Mr\big(M^{-1}(y, q)\big)$$

The expression for A_c is given by

$$A_c = A + BD_*, \quad D_* = -R^{-1}(S^T + B^T P)$$

where P and Q_* are the positive definite solutions to (10a) and

$$0 = (A_c + \tfrac{1}{2}\lambda)Q_* + Q_*(A_c + \tfrac{1}{2}\lambda)^T + \tfrac{1}{2}BR^{-1}B^T$$

respectively. This means that if the ARE (10a) has a solution that satisfies condition (11) with D given by (10b), the nonlinear Hamiltonian system can divided into two parts corresponding to y and q in (18). The linearization of the y-part has eigenvalues that satisfy condition (11), while the q-part has all eigenvalues to the right of $-\lambda/2$ in the complex plane. Then, it can be shown that there exists an n-dimensional stable manifold, described by a function $p_*(x)$, on which the required convergence rate is obtained given that the eigenvalues of A_c, *i.e.*, the system matrix for the closed loop system, satisfies the condition (11). In the other n-dimensions, the solution will not have this property, which means that the performance criterion will diverge for those directions.

The step that closes the existence proof is then that if the control law is chosen as $u_*(x_0, p_*(x_0))$, it can be shown that $V_x(x_0) = p_*(x_0)$ for x_0 in a neighborhood of the origin. It means that since $u_*(x_0, p_*(x_0))$ is the solution to (15) with $p = p_*(x_0)$, it follows that

$$0 = F_u\big(x_0, u_*(x_0)\big)V_x(x_0) + L_u\big(x_0, u_*(x_0)\big)$$

and since $u_*(x_0, p_*(x_0))$ satisfies the condition in Assumption 3, it follows from (14) that

$$0 = \lambda V(x_0) + F\big(x_0, u_*(x_0)\big)V_{x_0}(x_0) + L\big(x_0, u(x_0)\big)$$

where we have used the definition that $V(x_0) = J(x_0, u_*)$.

The equations (10) and (12) used to compute the solution are then obtained by analyzing the terms of order m and $m-1$ of (8a) and (8b), respectively. These two equations are polynomial in x and have to be satisfied for x in a neighborhood of the origin. Therefore, the coefficients for different orders in x all have to be zero. Then if $m = 2$ is considered, the obtained equations become (10) while for $m \geq 3$, the outcome are the equations in (12). ■

The theorem above is formulated in terms of the ARE (10a). The following lemma shows some typical situations when the ARE has a solution satisfying the conditions.

Lemma 1: Consider the ARE (10a). Assume the assumptions in Theorem 1 are satisfied. Then there exists a unique positive semi-definite solution such that the eigenvalues of $A + BD$ satisfies condition (11) if

- $\lambda = 0$: (A, B) is stabilizable.
- $\lambda > 0$: (A, B) is controllable or $(A + \frac{1}{2}\lambda I, B)$ is stabilizable.
- $\lambda < 0$: $(A + \frac{1}{2}\lambda I, B)$ is stabilizable or (A, B) controllable, and the solution yields $\text{eig}(A + BD) < 0$.

Proof: See for example [13] and [12]. ■

Note that under the assumptions about stabilizability or controllability made above, it is ensured that there exists a unique positive semi-definite solution such that with D in (10b), the eigenvalues of $A + BD$ will have real parts less than $-\lambda/2$ (which is necessary in order to obtain a convergent performance criterion). However, in the case when $\lambda < 0$, it means that the eigenvalues need not satisfy condition (11) and an extra condition is therefore added in the lemma. Since the extra condition is included, it is not guaranteed from the problem data that a solution exists. However, at least in randomly generated problems, it actually seems to happen quite often. For these cases the optimal feedback law is found.

The optimal solution can be computed recursively order by order. First the lowest order terms are obtained by solving (10) as

$$u_*^{[1]}(x) = D_* x, \quad V^{[2]}(x) = x^T P x$$

and having these, the higher order terms in $V(x)$ and $u_*(x)$ are obtained uniquely from (12), in the sequence

$$V^{[3]}(x), u_*^{[2]}, V^{[4]}(x), u_*^{[3]}, \ldots$$

to any order, see [3] or [7]. Furthermore, as can be seen the optimal solution becomes time-invariant.

Note that the equations obtained from the higher order terms in (12) are linear in the coefficients from $V^{[m]}$ and $u_*^{[m-1]}$, both separately and simultaneously. If solved recursively, rather high orders can be computed. However, the size of the set of equations grows rather fast with the number of states which limits the size of the problems that can be handled. Another small note is that it is not necessary to have a system that is analytic. If the model is $\mathcal{C}^r$ it has been shown in [6] that the optimal return function up to $\mathcal{C}^{r-2}$ exists and can be computed as above.

IV. The DAE Model Case

To solve the optimal control problem described by (1) and (2), the problem is rewritten as the following equivalent optimal control problem

$$V(x_{1,0}) = \inf_{u(\cdot)} \int_0^\infty \hat{L}(x_1, u)e^{\lambda t}\, dt$$

subject to the dynamics

$$\dot{x}_1 = \mathcal{L}(x_1, u)$$

where $\mathcal{L}$ is given by (3a) and

$$\hat{L}(x_1, u) = L\big(x_1, \mathcal{R}(x_1, u), u\big) \tag{19}$$

This reformulation can always be done because of Assumption 1. Then, in principle, the optimal control problem is a standard problem in the state variables x_1 and state-space theory is applicable. However, as mentioned earlier, there is a major computational barrier, namely that $\mathcal{L}$ and $\mathcal{R}$ are usually not explicit. However, Assumptions 1 and 2 ensure the existence of convergent power series of $\mathcal{L}(x_1, u)$ and $\mathcal{R}(x_1, u)$, and using the method in the following section, these power series can be computed to any order.

A. Power Series Expansion of the Reduced Problem

A keystone in the derived method for solving optimal control problems for nonlinear DAE models is that the power series expansions of $\mathcal{R}(x_1)$ and $\mathcal{L}(x_1, u)$ can be computed recursively. Let

$$\dot{x}_1 = \mathcal{L}(x_1, u) = \mathcal{L}^{[1]}(x_1, u) + \mathcal{L}_h(x_1, u) \tag{20a}$$

$$x_3 = \mathcal{R}(t, x_1, u) = \mathcal{R}^{[1]}(x_1, u) + \mathcal{R}_h(x_1, u) \tag{20b}$$

where both $\mathcal{L}_h(x_1, u)$ and $\mathcal{R}_h(x_1, u)$ contain terms in x_1 and u beginning with order two.

From (4b) the series expansion of $\hat{F}_2$ is given by

$$\hat{F}_2(x_1, x_3, u) = A_{21}x_1 + A_{22}x_3 + B_2 u + \hat{F}_{2h}(x_1, x_3, u)$$

By combining this equation with (20b), it follows that

$$\begin{aligned} 0 = A_{21}x_1 + A_{22}\big\{\mathcal{R}^{[1]}(x_1, u) + \mathcal{R}_h(x_1, u)\big\} \\ + B_2 u + \hat{F}_{2h}\big(x_1, \mathcal{R}^{[1]}(x_1, u) + \mathcal{R}_h(x_1, u), u\big) \end{aligned}$$

The equation above has to be satisfied for all (x_1, u) in a neighborhood of the origin, which means that the first order term of $\mathcal{R}(x_1, u)$ will be given by

$$\mathcal{R}^{[1]}(x_1, u) = -A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}B_2 u \tag{21}$$

since all other terms are of higher order than one. Furthermore, since $\hat{F}_{2h}(x_1, x_3, u)$ has an order of at least two, it follows that

$$\begin{aligned} &\hat{F}_{2h}^{[m]}\big(x_1, \mathcal{R}(x_1, u), u\big) = \\ &\qquad \hat{F}_{2h}^{[m]}\big(x_1, \mathcal{R}^{[1]}(x_1, u) + \ldots + \mathcal{R}^{[m-1]}(x_1, u), u\big) \end{aligned}$$

This makes it is possible to derive a recursive expression for a general order term of $\mathcal{R}(x_1, u)$ as

$$\mathcal{R}^{[m]}(x_1,u) = -A_{22}^{-1}\hat{F}_{2h}^{[m]}\big(x_1, \mathcal{R}^{[1]}(x_1,u)+\ldots+\mathcal{R}^{[m-1]}, u\big)$$

In the same way, (4a), *i.e.*

$$\begin{aligned}&\hat{F}_1(\dot{x}_1, x_1, x_3, u) = \\ &-E_1\dot{x}_1 + A_{11}x_1 + A_{12}x_3 + B_1u + \hat{F}_{1h}(\dot{x}_1, x_1, x_3, u)\end{aligned}$$

can be combined with (20) yielding the equation

$$\begin{aligned}0 = &-E_1\mathcal{L}^{[1]}(x_1,u) - E_1\mathcal{L}_h(x_1,u) + A_{11}x_1 \\ &+ A_{12}\big(\mathcal{R}^{[1]}(x_1,u) + \mathcal{R}_h(x_1,u)\big) + B_1u \\ &+ \hat{F}_{1h}\big(\mathcal{L}^{[1]}(x_1,u) + \mathcal{L}_h(x_1,u), x_1, \mathcal{R}^{[1]}(x_1,u) \\ &+ \mathcal{R}_h(x_1,u), u\big)\end{aligned}$$

By assumption E_1 is nonsingular and for notational reasons, it will in the sequel of this paper, be assumed that it is an identity matrix. The first term in $\mathcal{L}(x_1, u)$ is obtained as

$$\mathcal{L}^{[1]} = A_{11}x_1 + A_{12}\mathcal{R}^{[1]}(x_1,u) + B_1u = \hat{A}x_1 + \hat{B}u$$

where

$$\hat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \hat{B} = B_1 - A_{12}A_{22}^{-1}B_2$$

and the second equality is obtained using (21). Since $\hat{F}_{1h}(\dot{x}_1, x_1, x_3, u)$ contains terms of at least order two it follows that

$$\begin{aligned}&\hat{F}_{1h}^{[m]}\big(\mathcal{L}(x_1,u), x_1, \mathcal{R}(x_1,u), ux\big) = \\ &\quad\hat{F}_{1h}^{[m]}\big(\mathcal{L}^{[1]}(x_1,u) + \ldots + \mathcal{L}^{[m-1]}(x_1,u), x_1, \\ &\qquad\mathcal{R}^{[1]}(x_1,u) + \ldots + \mathcal{R}^{[m-1]}(x_1,u), u\big)\end{aligned}$$

which shows that higher order terms in $\mathcal{L}(x_1, u)$ can be computed recursively using the expression

$$\begin{aligned}&\mathcal{L}^{[m]}(x_1,u) = A_{12}\mathcal{R}^{[m]}(x_1,u) + \\ &\quad\hat{F}_1^{[m]}\big(\mathcal{L}^{[1]}(x_1,u) + \ldots + \mathcal{L}^{[m-1]}(x_1,u), x_1, \\ &\qquad\mathcal{R}^{[1]}(x_1,u) + \ldots + \mathcal{R}^{[m-1]}(x_1,u), u\big)\end{aligned}$$

The equations to find the coefficients of $\mathcal{R}$ and $\mathcal{L}$ will be linear in the m:th order coefficients. It means that if the equations are solved recursively, the computation can be carried out rather fast. However, if the number of variables in either x_1 or x_3 are large, the number of equations will grow rapidly. For physical systems, the DAE model is often semi-explicit and can be written as

$$\dot{x}_1 = \hat{F}_1(x_1, x_3, u), \qquad 0 = \hat{F}_2(x_1, x_3, u)$$

The computations above can then be simplified substantially, since the power series of $\mathcal{L}(x_1, u)$ is obtained, without solving any equations, as the composition of the power series of $\hat{F}_1$ and $\mathcal{R}$.

Having the power series expansions of $\mathcal{R}(x_1, u)$, the series expansion of (19) can be computed as

$$\begin{aligned}\hat{L}(x_1,u) &= \begin{pmatrix} x_1 \\ u\end{pmatrix}^T \Pi^T \begin{pmatrix} Q & S \\ S^T & R\end{pmatrix} \Pi \begin{pmatrix} x_1 \\ u\end{pmatrix} + \hat{L}_h(x_1,u) \\ &= \begin{pmatrix} x_1 \\ u\end{pmatrix}^T \begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R}\end{pmatrix} \begin{pmatrix} x_1 \\ u\end{pmatrix} + \hat{L}_h(x_1,u)\end{aligned} \tag{22a}$$

where

$$\Pi = \begin{pmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & -A_{22}^{-1}B_2 \\ 0 & I\end{pmatrix} \tag{22b}$$

and

$$\begin{aligned}&\hat{L}_h(x_1,u) = \\ &= L_h(x_1, \mathcal{R}(x_1,u), u) + 2x_1^TQ_{12}\mathcal{R}_h(x_1,u) \\ &+ 2\mathcal{R}^{[1]}(x_1,u)Q_{22}\mathcal{R}_h(x_1,u) + 2u^TS_2\mathcal{R}_h(x_1,u) \\ &+ \mathcal{R}_h(x_1,u)^TQ_{22}\mathcal{R}_h(x_1,u)\end{aligned} \tag{22c}$$

B. Existence and Computation of the Solution

Now when the series expansions of $\mathcal{L}$ and $\mathcal{R}$ are computed, the method in Section III can be used. That is, Theorem 1 can be modified to the nonlinear DAE case as follows.

Theorem 2: Consider the optimal control problem (1) and (2). Assume that it satisfies Assumptions 1 and 2. Furthermore, assume that the quadratic part of the cost function satisfies $\begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R}\end{pmatrix} \succ 0$. Then an optimal feedback law $u_*(x)$ satisfying Assumption 3 exists if the ARE

$$0 = (\hat{A}+\tfrac{\lambda}{2}I)^TP + P(\hat{A}+\tfrac{\lambda}{2}I) - (P\hat{B}+\hat{S})\hat{R}^{-1}(P\hat{B}+\hat{S})^T + \hat{Q} \tag{23a}$$

has a unique positive-semidefinite solution such that the matrix $\hat{A} + \hat{B}D$ with D given by

$$D = -\hat{R}^{-1}(\hat{S}^T + \hat{B}^TP) \tag{23b}$$

satisfies

$$\mathrm{Re}\,\mathrm{eig}(\hat{A} + \hat{B}D) < \min(0, -\tfrac{\lambda}{2})$$

The higher higher order terms are given by (12) with the system and cost function replaced by $\mathcal{L}(x_1, u)$ and $\hat{L}(x_1, u)$, respectively.

Proof: Follows from Theorem 1 after reformulation of the DAE model as state-space system. For this model, the power series can be computed to any order. ■

Some cases for which the ARE (23a) has solutions satisfying the conditions is provided by Lemma 1. The solution is computed in the same manner as described in Section III That is, the optimal solution can be found recursively and in this case, the sequence becomes

$$\bar{V}^{[2]}(x_1),\ u_*^{[1]}(x_1),\ \mathcal{R}^{[1]}(x_1, u_*^{[1]}),\ \mathcal{L}^{[1]}(x_1, u_*^{[1]})\ \ldots \tag{24}$$

In the Theorem 2, one possible approach to compute the optimal solution was described, namely to first find the power series of $\mathcal{R}(x_1, u)$ and $\mathcal{L}(x_1, u)$ and then use (12) to compute the solution. The main motivation for choosing this approach, is that it makes it easier to prove existence. However, it is also possible, as shown in [7], to use the following equivalent set equations

$$\begin{aligned}0 &= L_u - V_{x_1}\hat{F}_{1;\dot{x}_1}^{-1}\hat{F}_{1;u} \\ &\quad - \big(L_{x_3} - V_{x_1}\hat{F}_{1;\dot{x}_1}^{-1}\hat{F}_{1;x_3}\big)\hat{F}_{2;x_3}^{-1}\hat{F}_{2;u}\end{aligned} \tag{25a}$$

$$0 = L + \lambda V + V_{x_1}\dot{x}_1 \tag{25b}$$

$$0 = \hat{F}_1 \tag{25c}$$

$$0 = \hat{F}_2 \tag{25d}$$

where L is evaluated in (x_1, x_3, u), V in (x_1), $\hat{F}_1$ in $(\dot{x}_1, x_1, x_3, u)$ and $\hat{F}_2$ in (x_1, x_3, u). The advantage with this is that one solves for $V(x_1)$, $u_*(x_1)$, $\mathcal{L}(x_1)$ and $\mathcal{R}(x_1)$ simultaneously which for certain systems can simplify the computations. For more details see [7].

C. Conditions on the Original Cost Function

The conditions in Theorems 2 are expressed in terms of the reduced optimal control problem, *e.g.*, $\hat{A}$, $\hat{B}$ and $\hat{Q}$. However, in some cases these conditions can be translated to conditions on the original data. First consider the condition

$$\begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{pmatrix} \succ 0 \tag{26}$$

Since the variable transformation matrix Π in (22b) has full column rank, it follows that

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \succ 0 \quad \Rightarrow \quad \begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{pmatrix} \succ 0$$

However, note that the arrow only goes in one direction and the cost matrix (26) may be positive definite also for indefinite matrices in the original problem.

In some cases, it is not desired to penalize the variables x_3. In these cases the cost matrix is given by

$$\begin{pmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{pmatrix} = \Pi^T \begin{pmatrix} Q_{11} & 0 & S_1 \\ 0 & 0 & 0 \\ S_1^T & 0 & R \end{pmatrix} \Pi = \begin{pmatrix} Q_{11} & S_1 \\ S_1^T & R \end{pmatrix}$$

which means that if the cost matrix for x_1 and u is positive definite, the cost matrix for the reduced system is too.

V. Example

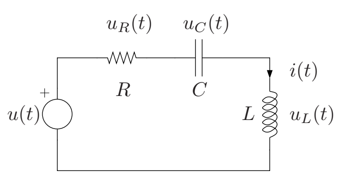

Fig. 1. Electrical ciruit

In order to illustrate the method a small example, namely an electrical circuit. The circuit, which can be seen in Figure 1, consists of an ideal voltage source, an inductor with a ferromagnetic core, a capacitor and a resistor. Because of the ferromagnetic core, the flux of the inductor saturates for large currents. The complete model can then be written as

$$\dot{u}_C = \frac{i}{1+10^{-2}u_C} \tag{27a}$$

$$\dot{\Phi} = u_L \tag{27b}$$

$$0 = \Phi - \arctan(i) \tag{27c}$$

$$0 = u_R - i - i^3 \tag{27d}$$

$$0 = u - u_R - u_C - u_L \tag{27e}$$

where u_C is the voltage over the capacitance, Φ is the flux, u_L is the voltage over the inductor, i is the current, u_R is the voltage over the resistor and u is the voltage over the voltage source. The dynamic variables are in this case chosen as $x_1 = (u_C, \Phi)$ and the algebraic variables are $x_2 = (i, u_L, u_R)$. The control input is the voltage over the voltage source u. This model satisfies Assumptions 1 and 2. The cost function is chosen as

$$L(u_C, \Phi, i, u_L, u_R) = i^2 + i^4 + \tfrac{1}{2}u^2$$

Figure 2 shows the corresponding solution for $u_c(t)$ when the third order approximation of the optimal feedback law is used. The initial conditions for the dynamic variables are chosen as $u_c(0) = 0.5$ and $\Phi(0) = -0.5$, while the other variables are chosen consistently. The different curves are: $\lambda = 3$ (dashed), $\lambda = 0$ (dotted) and $\lambda = -3$ (solid). As can be seen, the convergence time increase when λ decrease, *i.e.*, a more negative λ yields a longer settling time.

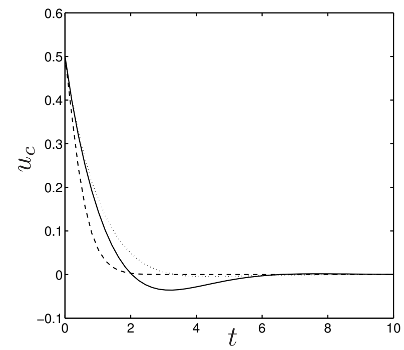

Fig. 2. The solution $u_c(t)$ for the mathematical example with the initial condition $u_c(0) = 0.5$, $\Phi(0) = -0.5$ and the other variables chosen consistently. The cases are $\lambda = 3$ (dashed), $\lambda = 0$ (dotted) and $\lambda = -3$ (solid).

VI. Conclusions

In this paper, it has been shown that for models described by convergent power series, it is possible to include a discount factor in the cost function and still a time-invariant and real analytic optimal solution is ensured to exist under certain regularity conditions. Furthermore, a recursive method to find the optimal solution is presented. Finally, the method is extended to also handle nonlinear DAE models.

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