Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Performance Limitation of Tracking Control Problem for a Class of References

Hiroshi Okajima and Toru Asai

Abstract— **This paper deals with analysis of fundamental limitations in tracking control problems. In the existing results, the fundamental limitations are analyzed based on detailed assumptions of reference signals such as the step, the trigonometrical signals and so on. On the other hand, we define a class of reference signals in a general manner. For the general class of reference inputs, we give the analysis of the tracking performance limitations. The obtained results extend the existing ones and give a uniform description. Moreover, the analysis results clearly separate the contributions of the plant and the reference characteristics. The results are illustrated by examples.**

I. INTRODUCTION

In the history of control theory, much attention has been paid to analysis of fundamental limitations in control system design. Recently, many results have been report [1], [2], [3], [4], [5], [6], [7], [8]. These papers derive explicit expressions of the achievable optimal H_2 performance described in terms of plant parameters such as unstable zeros/poles and so on. This kind of analysis results not only enable us to understand the relationship between the parameters and the achievable performance, but also they can be used to design a 'good' plant when we can change some parts of the plant. Of cource, the optimal performance level itself can be calculated shortly by using numerical techniques such as the LMIs, the Ricatti equations and so on. However, those a posteriori results hardly provide information about what kind of changes in plants will improve the performance.

Other than the optimal H_2 performance, the analysis has been done for the tracking control problems [2], [3], [4], [5]. In [2], the performance limitations for the step reference input have been analyzed for control systems with one and two degree of freedom (DOF). In addition to the tracking error, magnitude of control inputs is also treated in [3]. The results for trigonometrical functions are given in [4], [5].

The above results provide us useful information about the relationship between the control performance and the plant characteristics. However, since these results are derived based on the specific definitions of the reference signals, the contributions of the plant and the reference to the performance limitations are not separated clearly. This prevents us to obtain a clear insight about the essential sources of the performance limitations.

H. Okajima is with Graduate School of Science and Technokogy, Kumamoto University, Kumamoto, JAPAN, E-mail: okajima@cs.kumamoto-u.ac.jp

T. Asai is with Graduate School of Engineering, Osaka University, Osaka, JAPAN, E-mail: tasai@mech.eng.osaka-u.ac.jp

This paper aims to find the tracking performance limitations for a class of reference inputs. Specifically, we define the class of the reference inputs in a general manner; Even unstable poles of reference inputs are not specified. Based on the definitions of the reference inputs, we will give the performance limitations on the tracking control problems. Both the one and the two DOF control systems are dealt with. The obtained optimal performance indices describe the contribution of the plant and the reference inputs separately.

This paper is organized as follow: The problem formulation is defined in Section II. The analysis results for the 2DOF system are given in Section III, while those for the 1DOF system are in Section IV. The given results are illustrated by using examples in Section V.

Notation is standard. We denote the Laplace transform of $u(t)$ as $\hat{u}(s)$. L_∞ is the space of signals defined by

$$L_\infty = \left\{ e(t) : \sup_t |e(t)| < \infty \right\}.$$

For signal $e(t)$, $\|e\|_2$ represents the L_2 norm

$$\|e\|_2 = \left(\int_0^\infty |e(t)|^2 dt \right)^{\frac{1}{2}}.$$

On the other hand, for Laplace transform $\hat{u}(s)$, $\|\hat{u}\|_2$ is the L_2 norm of $\hat{u}(j\omega)$, i.e,

$$\|\hat{u}\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^\infty \hat{u}(-j\omega)\hat{u}(j\omega) d\omega \right)^{\frac{1}{2}}.$$

If $\hat{u}(s)$ is a rational function that is analytic in the closed right half plane, $\|\hat{u}\|_2$ can be called the H_2 norm. $\mathcal{S}$ is set of functions that are proper real rational and analytic in the closed right half plane.

II. PROBLEM FORMULATION

We consider the tracking control problem for a SISO plant given by a real rational transfer function $P(s)$. We assume that $P(s)$ has ℓ_p and m_p number of unstable poles $p_1, \dots, p_{\ell_p}$ and unstable zeros $z_1, \dots, z_{m_p}$, respectively. The relative degree of $P(s)$ is h_p. To keep derivations simple, we further assume that all the unstable poles and zeros lie in the open right half plane and distinct, i.e.

$$\begin{aligned} z_i \neq z_j \quad \forall i \neq j, \quad \mathrm{Re}(z_i) > 0 \quad \forall i, \\ p_k \neq p_l \quad \forall k \neq l, \quad \mathrm{Re}(p_k) > 0 \quad \forall k. \end{aligned} \tag{1}$$

We assume that the reference signal is given by its Laplace transform and that it belongs to the following set:

$$\mathcal{R} = \{ \hat{r} \in \mathcal{N} : \mathcal{L}^{-1}[\hat{r}] \in L_\infty \} \tag{2}$$

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where $\mathcal{N}$ is the set of strictly proper real rational functions. $\mathcal{R}$ is the set of real rational functions whose inverse transforms are bounded. The following lemma is given in [9]:

Lemma 1: Let $\hat{r} \in \mathcal{N}$ be given. Then, $\hat{r} \in \mathcal{R}$ holds, iff all the poles of $\hat{r}(s)$ lie in the closed left half plane and the multiplicity of the pure imaginary poles is at most one.

$\hat{r} \in \mathcal{R}$ can be any linear combination of the Laplace transforms of the trigonometric functions, the step function and the decaying exponential functions $e^{-\alpha t}$ where $\alpha > 0$.

We assume that $\hat{r} \in \mathcal{R}$ has m_r number of unstable zeros $z_{m_p+1}, \dots, z_{m_p+m_r}$ and that the relative degree of $\hat{r}$ is h_r. Moreover, all the unstable zeros of $P(s)$ and $\hat{r}(s)$ are distinct, i.e. (1) holds, even if the unstable zeros of $\hat{r}(s)$ are taken into account. In the sequel, we denote the total number of the unstable zeros and the relative degree as m_a and h_a respectively, i.e.

$$m_a = m_p + m_r, \quad h_a = h_p + h_r.$$

Moreover, we assume that there is no unstable pole/zero cancellation between $\hat{r}(s)$ and $P(s)$.

For the tracking control, we deal with control systems with one and two degree of freedom (DOF) depicted in Figs. 1 and 2, respectively. In Fig. 1, r and y are the reference input and the control output, respectively. $C_1(s)$ is the feedback controller, while $G_1(s)$ is the closed-loop system from r to y. In Fig. 2, $C_2(s)$ is the feedback controller, while $G_2(s)$ and $P(s)^{-1}G_2(s)$ are the feedforward controllers. Note that the structure in Fig. 2 does not lose any generality [11].

The aim of this paper is to analyze the performance limitations in the tracking control problem. The precise problem formulation is as follows:

Problem 1: Let $P(s)$ and $\hat{r}(s)$ be given. Then, for each of the 1DOF and the 2DOF control systems, find $J = \inf \|e\|_2^2$, where $e(t) = y(t) - r(t)$ is the error signal.

Note that J depends on both $P(s)$ and $\hat{r}(s)$. In the sequel, the infima in Problem 1 will be denote by J_1 and J_2 for the one and the two DOF control systems, respectively.

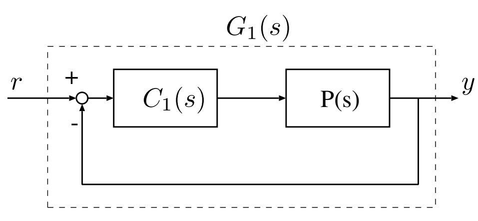

Fig. 1. 1DOF control system

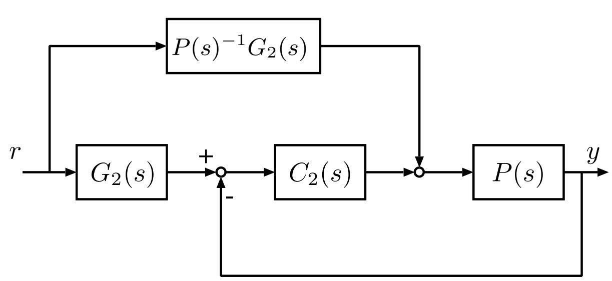

Fig. 2. 2DOF Control system

The performance limitations have been analyzed for the several cases of reference inputs and control structures. Most of those results are given by assuming fine details of reference inputs. On the other hand, we do not assume details of the reference input and allow the fairly general class $\mathcal{R}$ of the reference inputs. In particular, we do not assume any details of the unstable poles of $\hat{r}(s)$ other than $r(t)$ is bounded.

Most of the existing results have been derived based on the parameterization of all the stabilizing controllers. However, this approach would not be easy for the above problem, since there is few assumptions on the unstable poles of $\hat{r}(s)$. Insteadly, for each of the control structures, we parametrize set of all the outputs that are produced by internally stable control systems and that track the reference signal asymptotically. The sets are denote by $\mathcal{Y}_1$ and $\mathcal{Y}_2$ for the one and the two DOF cases, respectively. Based on the parameterizations, we analyze J_1 and J_2.

III. Analysis for the 2DOF system

A. Achievable sets of outputs

We first deal with the 2DOF control system in Fig. 2. The transfer characteristics from the reference input $\hat{r}(s)$ to the output $\hat{y}(s)$ is written by

$$\hat{y}(s) = G_2(s)\,\hat{r}(s). \tag{3}$$

Note that $G_2(s)$ is independent of the feedback controller $C_2(s)$. Hence, under the assumption that $C_2(s)$ internally stabilizes the closed-loop part of the system, the whole system is stable, iff the following conditions hold:

$$G_2 \in \mathcal{S}, \quad P^{-1}G_2 \in \mathcal{S}. \tag{4}$$

Therefore, $\mathcal{Y}_2$ is given by the set of $\hat{y}(s)$ such that (3), (4) and the following condition hold for some $G_2(s)$:

$$\lim_{t\to\infty} r(t) - y(t) = 0. \tag{5}$$

A more concrete characterization of $\mathcal{Y}_2$ can be given as follows:

Lemma 2: Let $P(s)$ and $\hat{r}(s)$ be the given plant and the given reference signal. Then, $\hat{y} \in \mathcal{Y}_2$ holds, iff the following conditions hold:

- $\hat{y} - \hat{r} \in \mathcal{S}$ holds.
- The relative degree of $\hat{y}(s)$ is greater than or equal to h_a.
- The following equations hold:

$$\hat{y}(z_i) = 0 \quad \forall\, i = 1, \dots, m_a. \tag{6}$$

Proof: (Necessity) Suppose that $\hat{y} \in \mathcal{Y}_2$ holds. Then, (5) leads to $\hat{y} - \hat{r} \in \mathcal{S}$. (3) implies that the relative degree of $\hat{y}$ is greater than or equal to h_a, since the relative degree of G_2 must be greater than or equal to h_p due to (4). Similarly, (3) and (4) also imply (6).

(Sufficiency) Suppose that $\hat{y}$ satisfies the conditions. $\hat{y} - \hat{r} \in \mathcal{S}$ leads to (5). We here define $G_2(s)$ as follows:

$$G_2(s) = \frac{\hat{y}(s)}{\hat{r}(s)}.$$

Obviously, (3) holds. The above $G_2(s)$ can be written by

$$G_2(s) = \frac{\hat{y}(s) - \hat{r}(s)}{\hat{r}(s)} + 1.$$

Since $\hat{y} - \hat{r} \in \mathcal{S}$ is assumed and (6) implies that $\hat{y}(z_i) - \hat{r}(z_i) = 0$ holds for $i = m_p+1, \ldots, m_a$. Moreover, owing to the conditions on $\hat{y}$, the relative degree of $G_2(s)$ is greater than or equal to h_p. Then, $G_2 \in \mathcal{S}$ holds. On the other hand, since all the unstable zeros are assumed to be distinct, $G_2(z_i) = 0$ holds for $i = 1, \ldots, m_p$. The relative degree of $P^{-1}G_2$ is greater than or equal to 0. Therefore, $P^{-1}G_2 \in \mathcal{S}$ holds and G_2 satisfies (4). Thus, $\hat{y} \in \mathcal{Y}_2$ is proven. ■

The conditions in Lemma 2 characterize $\mathcal{Y}_2$. By using those conditions, we will characterize $\mathcal{Y}_2$ more explicitly. However, rather than dealing with $\mathcal{Y}_2$, it is technically easier to deal with the set $\mathcal{E}_2$ of the error signals

$$\hat{e}(s) = \hat{y}(s) - \hat{r}(s). \tag{7}$$

Hence, we will find an explicit expression of $\mathcal{E}_2$. If it is given, then $\mathcal{Y}_2$ is given explicitly by

$$\mathcal{Y}_2 = \hat{r} + \mathcal{E}_2.$$

Since $\hat{e}$ is defined by (7) and the set of $\hat{y}(s)$ is characterized by Lemma 2, the set $\mathcal{E}_2$ is characterized by the following conditions:

- $\hat{e} \in \mathcal{S}$
- The relative degree of $\hat{r}(s) + \hat{e}(s)$ is greater than or equal to h_a.
- $\hat{r}(z_i) + \hat{e}(z_i) = 0$ holds for all $i = 1, \ldots, m_a$

Then, the explicit expression of $\mathcal{E}_2$ can be given directly by the results in [9] as follows:

Theorem 3: Let $P(s)$ and $\hat{r}(s)$ be the given plant and the reference input assumed in this paper, respectively. Moreover, let $a > 0$ be an arbitrary given positive real number. Then, the following equivalence holds:

$$\mathcal{E}_2 = U_2 + V_2\,\mathcal{S} \tag{8}$$

where $U_2(s)$ and $V_2(s)$ are given by the following recursion:

$$U_2(s) = K^{(h_a)}(s), \quad V_2(s) = L^{(h_a)}(s) \tag{9}$$

$$K^{(k+1)}(s) = K^{(k)}(s) + \alpha_k L^{(k)}(s) \tag{10}$$

$$L^{(k+1)}(s) = \frac{1}{s+a}\,L^{(k)}(s) \tag{11}$$

$$\alpha_k = -\lim_{s\to\infty}\left(s^k(\hat{r}(s) + K^{(k)}(s))\right) \tag{12}$$

$$K^{(0)}(s) = -\sum_{i=1}^{m_p} H_i(s)\,\hat{r}(z_i) \tag{13}$$

$$H_i(s) = \left(\frac{z_i+a}{s+a}\right)^{m_a-1} \prod_{j=1, j\neq i}^{m_a} \frac{s-z_j}{z_i-z_j} \tag{14}$$

$$L^{(0)}(s) = \frac{\prod_{i=1}^{m_a}(s-z_i)}{(s+a)^{m_a}} \tag{15}$$

Theorem 3 gives the explicit parameterization (8) of $\mathcal{E}_2$. The parameterization is given by the set of the proper stable real rational functions, which is similar to the case of the KYJB parameterization of internally stabilizing controllers [10]. While the KYJB parameterization characterizes the set of controllers, (8) parameterize the set of signals.

The initial functions $K^{(0)}(s)$ and $L^{(0)}(s)$ correspond to the constraints on the unstable zeros. Note that the summation in $K^{(0)}(s)$ is carried out for $i = 1, \ldots, m_p$, since $r(z_i) = 0$ holds for $i = m_p+1, \ldots, m_a$. On the other hand, the recursion (10) and (11) corresponds to the constraint on the relative degree. Notice that α_k in (12) is bounded by the constructions of $K^{(k)}(s)$. Although Theorem 3 assumes that a is independent of k, it can depend on k, provided that $\frac{1}{s+a_k}$ is a stable transfer function.

B. L_2 optimal $\hat{e}(s)$

Based on Theorem 3, we further consider the minimization problem of the L_2 norm of $e(t)$. Owing to the Parseval's equality, the L_2 norm of $e(t)$ is equal to the H_2 norm of $\hat{e}(s)$. Thus, we consider the following minimization problems:

$$\text{minimize } \|\hat{e}\|_2 \text{ subject to } \hat{e} \in \mathcal{E}_2. \tag{16}$$

We denote the minimizer of (16) as follows:

$$\hat{e}^{\rm opt}(s) = \arg \inf_{\hat{e}\in\mathcal{E}_2} \|\hat{e}\|_2. \tag{17}$$

The minimization problem (16) is equivalent to the following problem:

$$\text{minimize } \|U_2 + V_2\,Q\|_2 \text{ subject to } Q \in \mathcal{S}. \tag{18}$$

(18) is a standard model-matching problem, since $U_2 \in \mathcal{S}$ and $V_2 \in \mathcal{S}$ hold by their constructions. Moreover, (18) is solvable, since $U_2(s)$ is strictly proper. The solution of the standard model matching problem (18) is known well. In fact, the minimizer and the infimum of (18) are given as follow [10]:

$$Q^{\rm opt}(s) = -W(s)^{-1}\,(\Theta(s)^{-1}U_2(s))_{\rm st}, \tag{19}$$

$$\inf_{Q\in\mathcal{S}} \|U_2 + V_2 Q\|_2 = \|(\Theta^{-1}U_2)_{\rm anst}\|_2. \tag{20}$$

where $\Theta(s)$ and $W(s)$ are the inner and the outer factors of $V_2(s)$, respectively. Moreover, $(\Theta(s)^{-1}U_2(s))_{\rm anst}$ and $(\Theta(s)^{-1}U_2(s))_{\rm st}$ are the anti-stable and the stable parts of $\Theta(s)^{-1}U_2(s)$, respectively. Note that the norm on the right hand side of (20) is the L_2 norm of $(\Theta(s)^{-1}U_2(s))_{\rm anst}|_{s=j\omega}$, while the left hand side is the H_2 norm.

Since $U_2(s)$ and $V_2(s)$ are given explicitly in Theorem 3, the optimal quantities (19) and (20) can be also given explicitly as the following theorem:

Theorem 4: Let $P(s)$ and $\hat{r}(s)$ be the given plant and the reference input. Moreover, $U_2(s)$ and $V_2(s)$ are the real rational functions defined in Theorem 3. Let $\Theta(s)$ be the inner factor of $V_2(s)$. Then, the following equations hold:

$$(\Theta(s)^{-1}U_2(s))_{\rm anst} = -\sum_{i=1}^{m_p} \frac{q_i\,\hat{r}(z_i)}{s-z_i}, \tag{21}$$

$$J_2^2 = \|(\Theta^{-1}U_2)_{\rm anst}\|_2^2 = \rho^* M_2\,\rho, \tag{22}$$

$$\hat{e}_2^{\rm opt}(s) = -\sum_{i=1}^{m_p} H_i^{\rm opt}(s)\,\hat{r}(z_i), \tag{23}$$

where

$$q_i = \frac{\prod_{j=1}^{m_a}(z_i+\overline{z_j})}{\prod_{j=1,j\neq i}^{m_a}(z_i-z_j)}, \tag{24}$$

$$(M_2)_{ij} = \frac{\overline{q_i}\,q_j}{\overline{z_i}+z_j}, \tag{25}$$

$$\rho = \left[\begin{array}{ccc} \hat{r}(z_1) & \cdots & \hat{r}(z_{m_p}) \end{array}\right]^T, \tag{26}$$

$$H_i^{\rm opt}(s) = q_i\,\frac{\prod_{j=1,j\neq i}^{m_a}(s-z_j)}{\prod_{j=1}^{m_a}(s+\overline{z_j})}. \tag{27}$$

Theorem 4 gives the explicit form of the optimal tracking error (23) and the optimal norm (22) for the general class of the reference input. In particular, (22) clarifies the way how J_2 depends on $P(s)$ and $\hat{r}(s)$; J_2 is composed of the matrix M_2 and the vector ρ, where M_2 is a function of the unstable zeros, while ρ is composed of the values of $\hat{r}(s)$ at the unstable zeros z_i. Thus, (22) clearly separates the contributions of $P(s)$ and $\hat{r}(s)$.

If $P(s)$ has an unstable zero z near an unstable pole of $\hat{r}(s)$, $\hat{r}(z)$ has a large magnitude. Then, J_2 can be also large and the resultant tracking performance would be quite poor. This result coincides with our intuition. On the contrary, if $P(s)$ has an unstable zero z near an unstable zero of $\hat{r}(s)$, magnitude of $\hat{r}(z)$ is small. Moreover, if $P(s)$ has no unstable zeros other than z, J_2 can be also small. It follows that high tracking performance may be achieved, even though $P(s)$ has the unstable zero.

The matrix M_2 depends on the unstable zeros of both $P(s)$ and $\hat{r}(s)$. If we are allowed to choose more suitable input signals in Fig. 2 than $r(t)$, the optimal norm is given by replacing m_a in (22) with m_p [9]. The obtained M_2 is independent of $\hat{r}(s)$. This is also true in the case that $\hat{r}(s)$ has no unstable zeros. We may consider that this M_2 describes the essential performance limitation attained by $P(s)$, since it depends only on $P(s)$ and is independent of $\hat{r}(s)$.

If $\hat{r}(s)$ is given by the step or the sinusoidal functions, J_2 in Theorem 4 coincides with the existing results [2], [4], [5]. Even in those cases, Theorem 4 present new information, since it describes the contributions of $\hat{r}(s)$ and $P(s)$ separately.

IV. Analysis for the 1DOF system

In this section, the analysis will be conducted for the 1DOF system, i.e. the unity feedback system depicted in Fig. 1. The output of the system is described by

$$\hat{y}(s) = G_1(s)\,\hat{r}(s) \tag{28}$$

where $G_1(s)$ is the closed-loop system in Fig. 1, i.e.

$$G_1(s) = \frac{P(s)C_1(s)}{1+P(s)C_1(s)}. \tag{29}$$

Conversely, if $G_1(s)$ is given, the controller $C_1(s)$ satisfying (29) is given as follows:

$$C_1(s) = \frac{G_1(s)}{P(s)(1-G_1(s))}. \tag{30}$$

Hence, $C_1(s)$ and $G_1(s)$ have one-to-one correspondence. In the sequel, we consider the problem mainly based on $G_1(s)$ rather than $C_1(s)$.

The analysis is based on the set $\mathcal{Y}_1$ of the outputs, as in the case of the 2DOF system. We first consider the internal stability of the closed-loop system. All the transfer functions defining the internal stability can be written by using $G_1(s)$ as follow:

$$\frac{P(s)}{1+P(s)C_1(s)} = P(s)\,(1-G_1(s)), \tag{31}$$

$$\frac{1}{1+P(s)C_1(s)} = 1-G_1(s), \tag{32}$$

$$\frac{C_1(s)}{1+P(s)C_1(s)} = \frac{G_1(s)}{P(s)}, \tag{33}$$

$$\frac{P(s)C_1(s)}{1+P(s)C_1(s)} = G_1(s). \tag{34}$$

Consequently, $C_1(s)$ in (30) is an internally stabilizing controller, iff the following conditions hold:

$$G_1 \in \mathcal{S}, \quad P^{-1}G_1 \in \mathcal{S}, \quad P\,(1-G_1) \in \mathcal{S}. \tag{35}$$

Then, the output set $\mathcal{Y}_1$ is given by the set of $\hat{y}$ such that (5), (28) and (35) hold for some $G_1(s)$. Compared (35) with (4), we see that the difference is the stability requirement for $P(s)\,(1-G_1(s)) = \frac{P(s)}{1+P(s)C_1(s)}$.

Although we have derived the conditions based on the set $\mathcal{Y}_1$ of the output signals, the obtained results are consistent with the internal model principle. Let $\hat{y} \in \mathcal{Y}_1$ be given and p be an unstable pole of $\hat{r}(s)$. Note that $P(p) \neq 0$ has been assumed. Then, $\hat{y} \in \mathcal{Y}_1$ implies $\hat{r} - \hat{y} = (1-G_1)\,\hat{r} \in \mathcal{S}$. Hence, $1-G_1(s)$ must have the unstable zero at $s = p$, i.e. $1-G_1(p) = 0$ or equivalently $G_1(p) = 1 \neq 0$. As a consequence, $C_1(s)$ in (30) must have the same unstable poles as $\hat{r}(s)$.

Similar to Lemma 2, a more concrete characterization of $\mathcal{Y}_1$ can be given as follows:

Lemma 5: Let $P(s)$ and $\hat{r}(s)$ be the given plant and the given reference signal. Then, $\hat{y} \in \mathcal{Y}_1$ holds, iff $\hat{y} \in \mathcal{Y}_2$ and the following equations hold:

$$\hat{y}(p_i) - \hat{r}(p_i) = 0 \quad \forall\, i = 1, \ldots, \ell_p \tag{36}$$

where $\hat{y} \in \mathcal{Y}_2$ can be examined by Lemma 2.

Proof: (Necessity) Suppose that $\hat{y} \in \mathcal{Y}_1$ holds, i.e. (5), (28) and (35) hold for some $G_1(s)$. By comparing (35) with (4), $\hat{y} \in \mathcal{Y}_2$ obviously holds. Since $G_1(s)$ can be written by $G_1(s) = \frac{\hat{y}(s)}{\hat{r}(s)}$, the following condition must hold:

$$P\,(1-G_1) = \frac{P}{\hat{r}}\,(\hat{r}-\hat{y}) \in \mathcal{S}$$

By assumption, $\hat{r}(s)$ do not share any unstable poles with $P(s)$. Hence, $\hat{r} - \hat{y}$ have to satisfy (36).

(Sufficiency) Suppose that $\hat{y} \in \mathcal{Y}_2$ and (36) hold. Define $G_1(s)$ by $\frac{\hat{y}(s)}{\hat{r}(s)}$, and obviously (28) holds. Moreover, due to the definition of $\mathcal{Y}_2$ and Lemma 2, (5), $G_1 \in \mathcal{S}$ and $P^{-1}G_1 \in \mathcal{S}$ hold. Hence, $1 - G_1 = \frac{\hat{r}-\hat{y}}{\hat{r}} \in \mathcal{S}$ holds. Since $\hat{r}(s)$ is assumed to have no zeros at $s = p_i$, (36) yields $1 - G_1(p_i) = 0$. It follows that $P\,(1-G_1) \in \mathcal{S}$ holds. ■

Lemma 5 reveals that $\mathcal{Y}_1$ is obtained by giving the additional constraint (36) to $\mathcal{Y}_2$. Consequently, if $P(s)$ has an unstable pole, $\mathcal{Y}_1$ is a proper subset of $\mathcal{Y}_2$. On the other hand, if $P(s)$ is stable, $\mathcal{Y}_1 = \mathcal{Y}_2$ holds. In other words, if $P(s)$ is stable, the unity feedback control system can attain the same tracking performance as the 2DOF systems.

If we define the set of the error signals as $\mathcal{E}_1$, $\mathcal{Y}_1$ is written by

$$\mathcal{Y}_1 = \hat{r} + \mathcal{E}_1.$$

Moreover, due to Lemma 5, $\mathcal{E}_1$ is given by the set of $\hat{e}(s)$ such that the following conditions hold:

- $\hat{e} \in \mathcal{E}_2$ holds.
- The following equations hold:

$$\hat{e}(p_i) = 0 \quad \forall\, i = 1, \ldots, \ell_p. \tag{37}$$

An explicit characterization of $\mathcal{E}_1$ is given by the following theorem:

Theorem 6: Let $P(s)$ and $\hat{r}(s)$ be the given plant and the reference input, respectively. Let $b > 0$ be an arbitrary given positive real number. Then, the following equivalence holds:

$$\mathcal{E}_1 = U_1 + V_1\, \mathcal{S} \tag{38}$$

where $U_1(s)$ and $V_1(s)$ are real rational functions given by the following recursive equations:

$$U_1(s) = \tilde{K}^{(\ell_p)}(s), \quad V_1(s) = \tilde{L}^{(\ell_p)}(s)$$

$$\tilde{K}^{(k+1)}(s) = \tilde{K}^{(k)}(s) + \beta_k\, \tilde{L}^{(k)}(s) \tag{39}$$

$$\beta_k = -\frac{\tilde{K}^{(k)}(p_{k+1})}{\tilde{L}^{(k)}(p_{k+1})} \tag{40}$$

$$\tilde{L}^{(k+1)}(s) = \frac{s - p_{k+1}}{s + b}\, \tilde{L}^{(k)}(s) \tag{41}$$

$$\tilde{K}^{(0)}(s) = U_2(s) \tag{42}$$

$$\tilde{L}^{(0)}(s) = V_2(s) \tag{43}$$

$U_2(s)$ and $V_2(s)$ are defined in Theorem 3.

By using Theorem 6, J_1 and the optimal error can be given explicitly as follows:

Theorem 7: Let $P(s)$ and $\hat{r}(s)$ be the given plant and the reference input. Moreover, $U_1(s)$ and $V_1(s)$ are the real rational functions defined in Theorem 6. Let $\tilde{\Theta}(s)$ be the inner factor of $V_1(s)$. Then, $(\tilde{\Theta}(s)^{-1}U_1(s))_{\text{anst}}$ and $\hat{e}_1^{\text{opt}}(s)$ can be written as follows:

$$(\tilde{\Theta}(s)^{-1}U_1(s))_{\text{anst}} = -\sum_{i=1}^{m_p} \frac{q_i\, w_i\, \hat{r}(z_i)}{s - z_i}, \tag{44}$$

$$J_1^2 = \|(\tilde{\Theta}^{-1}U_1)_{\text{anst}}\|_2^2 = \rho^* M_1\, \rho \tag{45}$$

$$\hat{e}_1^{\text{opt}}(s) = -\left(\prod_{j=1}^{\ell_p} \frac{s - p_j}{s + \overline{p_j}}\right) \sum_{i=1}^{m_p} H_i^{\text{opt}}(s)\, w_i\, \hat{r}(z_i). \tag{46}$$

where

$$w_i = \prod_{j=1}^{\ell_p} \frac{z_i + \overline{p_j}}{z_i - p_j}, \tag{47}$$

$$(M_1)_{ij} = \frac{\overline{w_i\, q_i}\, q_j\, w_j}{\overline{z_i} + z_j}. \tag{48}$$

Moreover, $H_i^{\text{opt}}(s)$, q_i and ρ are defined in Theorem 4.

Theorem 7 gives the explicit form of the optimal tracking error (46) and the optimal norm (45) for the general class of the reference input. The obtained results are similar to those of Theorem 4. In particular, (45) separately describes the contribution of $\hat{r}(s)$ from that of $P(s)$. Moreover, the result is obtained simply by multiplying w_j to q_j, where w_j is defined by using the unstable poles of $P(s)$. Hence, if there is an unstable zero near the unstable poles p_j, w_j has a quite large magnitude and so is J_1. Again, this result coincides with our intuitions.

V. Numerical Examples

This section illustrates the results in this paper by using numerical examples.

Suppose $m_a = m_p = 1$ and $\ell_p = 1$. Then, Theorems 4 and 7 yield the followings:

$$J_1 = 2\, z \left(\frac{z + p}{z - p}\right)^2 \hat{r}(z)^2,$$

$$J_2 = 2\, z\, \hat{r}(z)^2,$$

where $z > 0$ and $p > 0$ are the unstable zero and pole of $P(s)$, respectively. Note that J_1 depends on p, while J_2 is independent of p. More specifically, J_1 is larger than J_2 by the factor of $\left(\frac{z+p}{z-p}\right)^2$. Note that this factor is irrelevant to choices of $\hat{r}(s)$.

We consider another example. Suppose that $P(s)$ is stable and has an unstable zero z_1, while $\hat{r}(s)$ has an unstable zero z_2. In this case, J_2 is given by

$$J_2^2 = 2\, z_1 \left(\frac{z_1 + z_2}{z_1 - z_2}\right)^2 \hat{r}(z_1)^2.$$

If we can use a suitable input instead of $r(t)$ for the system in Fig. 2, the performance limitation can be improved as

$$J_{2o}^2 = 2\, z_1 \hat{r}(z_1)^2.$$

In fact, if z_1 tends to z_2, J_2 tends to a positive number while J_{2o} tends to zero. This comparison suggests that the configuration in Fig. 2 is not perfect and its performance can be improved by replacing the input $r(t)$ with other signals, when $\hat{r}(s)$ has an unstable zero. For example, when $z_1 = 1$ and

$$\hat{r}(s) = \frac{s - 2}{s^2 + 1}$$

are given, $J_2^2 = 4.5$ and $J_{2o}^2 = 0.5$ can be obtained. The optimal outputs are shown in Fig. 3. The responses in red and blue are corresponding to J_{2o} and J_2, respectively. The response in black represents the reference. We see that the response corresponding to J_{2o} results in better performance.

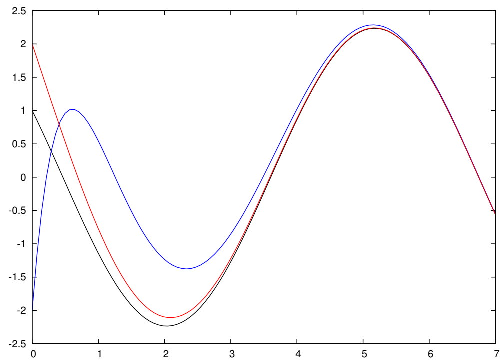

Fig. 3. Optimal Outputs

VI. CONCLUSION

The limitation on the tracking performance has been analyzed for the 1DOF and the 2DOF control systems. We have first characterize the sets of the admissible output signals and parameterize those based on the set of the stable proper real rational functions. Using the parameterization, we have derived the optimal errors and the corresponding norms explicitly. The obtained formula clarifies how the optimal norm depends on the plant and the reference input.

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