Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Feedback Linearizability of Strict Feedforward Systems

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***Abstract*— For any strict feedforward system that is feedback linearizable we provide (following our earlier results) an algorithm, along with explicit transformations, that linearizes the system by change of coordinates and feedback in two steps: first, we bring the system to a newly introduced Nonlinear Brunovský canonical form (NBr) and then we go from (NBr) to a linear system. The whole linearization procedure includes diffeo-quadratures (differentiating, integrating, and composing functions) but not solving PDE's. Application to feedback stabilization of strict feedforward systems is given.**

I. Introduction

Consider a smooth nonlinear single-input control system

$$\Xi: \quad \dot{z} = F(z,u), \quad z \in Z \subseteq \mathbb{R}^n, \; u \in \mathbb{R},$$

where $z \in Z$, an open subset of $\mathbb{R}^n$. Assume that it is, via a smooth change of coordinates $x = \varphi_1(z)$, equivalent to the *strict feedforward form*, shortly (SFF)-form,

$$(SFF) \qquad \begin{cases} \dot{x}_1 &= G_1(x_2,\ldots,x_n,u) \\ & \cdots \\ \dot{x}_{n-1} &= G_{n-1}(x_n,u) \\ \dot{x}_n &= G_n(u). \end{cases}$$

If the system Ξ is feedback linearizable, then (as it is well known, see, e.g. [2], [4], [12]) it takes, in some coordinate system $y = \varphi_2(x)$, the *feedback form*, shortly (FB)-form:

$$(FB) \qquad \begin{cases} \dot{y}_1 &= \bar{G}_1(y_1,y_2) \\ & \cdots \\ \dot{y}_{n-1} &= \bar{G}_{n-1}(y_1,\ldots,y_n) \\ \dot{y}_n &= \bar{G}_n(y_1,\ldots,y_n,u). \end{cases}$$

If Ξ takes in some x-coordinates the (SFF)-form and in some y-coordinates the (FB)-form, then a natural question arises whether there exist coordinates $w = \varphi_3(z)$ in which Ξ would take simultaneously both the (SSF)-form and the (FB)-form. Comparing (SFF) and (FB), that are dual with respect to each other, we see that in w-coordinates (if they exist), Ξ would take the following nonlinear generalization of the Brunovský canonical form:

$$(NBr) \qquad \begin{cases} \dot{w}_1 &= \hat{G}_1(w_2) \\ & \cdots \\ \dot{w}_{n-1} &= \hat{G}_{n-1}(w_n) \\ \dot{w}_n &= \hat{G}_n(u). \end{cases}$$

Recall that the *Brunovský canonical form* is the following linear control system on $\mathbb{R}^n$ (consisting of a chain of n integrators):

$$(Br) \qquad \begin{cases} \dot{\tilde{w}}_1 &= \tilde{w}_2 \\ & \cdots \\ \dot{\tilde{w}}_{n-1} &= \tilde{w}_n \\ \dot{\tilde{w}}_n &= \tilde{u}. \end{cases}$$

One of the main results of this paper asserts that the answer to the above question is indeed positive: if a system is via (different, in general) changes of coordinates equivalent to the (SFF)-form and to the (FB)-form, then it is also equivalent to the above *Nonlinear Brunovský canonical form* (NBr), which is simultaneously (SFF) and (FB).

A similar question arises if we assume that Ξ is equivalent to the (SFF)-form and, instead of supposing feedback linearizability, we assume that it is linearizable via a change of coordinates only. Then we would like to know whether the system can be put into a linear form that would simultaneously be (SFF). This question was answered positively in our previous paper [19], inspired by that of Krstic [7], where we provided an algorithm, along with necessary and sufficient conditions, to linearize (via a change of state coordinates) a strict feedforward system.

Let us recall that strict feedforward systems and their stabilization were first investigated by Teel in his pioneering papers [20], [21]. Since then, it has been followed by a growing literature [1], [5], [6], [7], [8], [9], [10], [11], [13], [14], [15], [17], [18]. Recently, Krstic [7] addressed the problem of linearizability of nonlinear systems in strict feedforward form, and provided two classes (type I and type II) that are linearizable by change of coordinates. By providing linearizing changes of coordinates in some examples, Krstic mentioned the lack of a systematic way of finding those changes of coordinates. We addressed this problem in [19] and provided an efficient algorithm to find linearizing transformations for strict feedforward systems that are linearizable by change of state coordinates. The aim of this paper is to study the class of feedback linearizable strict feedforward systems in its full generality.

The problem of transforming a control system into a linear controllable system via change of coordinates and feedback was solved in the early eighties in [2] and [4], where necessary and sufficient geometric conditions, expressed in terms of involutivity of certain distributions (see Theorem IV.4 below), were obtained (see also [3], [12]). Those conditions are easy to check but if they are satisfied, then finding linearizing coordinates and feedback transformations requires, in general, solving a system of partial differential equations (whose solvability is guaranteed by the involutivity). For strict feedforward systems, however,

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finding linearizing coordinates and feedback turns out to be much easier: our algorithm can be performed using at most $\frac{n(n-1)}{2}$ steps, quadratures, each involving composition and integration of functions only (but not solving PDEs) followed by a sequence of n derivations. That "simplest possible" way of calculating linearizing feedback transformations (using diffeo-quadratures only which is crucial for applications) for any strict feedforward system shows importance of the presented algorithm. Moreover, if the system is not feedback linearizable, the algorithm fails after a finite number of steps, which thus provides a simple way of testing the feedback linearizability of strict feedforward systems.

The paper is organized as follows. In Section II we give some basic notations. In Section III we formulate our main result on F-linearizable strict feedforward systems. Then we show that the constructed transformations are, indeed, diffeo-quadratures in Section IV and discuss applications to the stabilization problem in Section V. Finally, the proof of the main result, together with the Algorithm, form Section VI.

II. Definitions and Notations

Throughout the paper, the word smooth will always mean C^∞-smooth. We assume, except otherwise stated, that Ξ is affine in control, i.e., $\frac{\partial^2 F}{\partial u^2} = 0$, and we denote it by Σ.

Consider two control systems

$$\Sigma \;:\; \dot{z} = f(z) + g(z)u, \quad z \in Z \subseteq \mathbb{R}^n,\ u \in \mathbb{R},$$

where f and g are smooth vector fields on Z, an open subset of $\mathbb{R}^n$, and

$$\tilde{\Sigma} \;:\; \dot{\tilde{z}} = \tilde{f}(\tilde{z}) + \tilde{g}(\tilde{z})\tilde{u}, \quad \tilde{z} \in \tilde{Z} \subseteq \mathbb{R}^n,\ \tilde{u} \in \mathbb{R},$$

where $\tilde{f}$ and $\tilde{g}$ are smooth vector fields on $\tilde{Z}$, an open subset of $\mathbb{R}^n$. They are called *state equivalent*, shortly *S-equivalent*, if there exists a smooth diffeomorphism $\varphi : Z \to \tilde{Z}$, such that

$$\varphi_* f = \tilde{f} \quad \text{and} \quad \varphi_* g = \tilde{g};$$

(we take $u = \tilde{u}$). Recall that for any smooth vector field f on Z and any smooth diffeomorphism $\tilde{z} = \varphi(z)$ we denote

$$(\varphi_* f)(\tilde{z}) = D\varphi(z) \cdot f(z), \quad \text{with } z = \varphi^{-1}(\tilde{z}).$$

Two control systems Σ and $\tilde{\Sigma}$ are called *feedback equivalent*, shortly *F-equivalent*, if there exist a smooth diffeomorphism $\varphi : Z \to \tilde{Z}$ and smooth $\mathbb{R}$-valued functions α, β, satisfying $\beta(\cdot) \neq 0$, such that

$$\varphi_*(f + g\alpha) = \tilde{f} \quad \text{and} \quad \varphi_*(g\beta) = \tilde{g}.$$

A control-affine system that is strict feedforward takes the following *affine strict feedforward form*

$$(ASFF) \quad \begin{cases} \dot{z}_1 &= f_1(z_2, \ldots, z_n) + g_1(z_2, \ldots, z_n)u \\ & \cdots \\ \dot{z}_{n-1} &= f_{n-1}(z_n) + g_{n-1}(z_n)u \\ \dot{z}_n &= f_n + g_n u, \end{cases}$$

where f_n, $g_n \in \mathbb{R}$ and $z \in Z$, an open subset of $\mathbb{R}^n$.

Throughout the paper we assume that the drift f has an equilibrium which, without loss of generality, is taken to be $0 \in \mathbb{R}^n$, that is $f(0) = 0$. In particular, $f_n = 0$ in $(ASFF)$.

III. Main Result: F-Linearizable (SFF)-Sytems

In this section we will give our main result on F-linearizable strict feedforward systems.

Theorem III.1 *Assume that a control-affine system Σ is S-equivalent to the $(ASFF)$-form. Then the following conditions are equivalent:*

(i) *Σ is F-equivalent to a linear controllable system;*
(ii) *Σ is S-equivalent to the following Affine Nonlinear Brunovský form:*

$$(ANBr) \quad \begin{cases} \dot{w}_1 &= \hat{G}_1(w_2) \\ & \cdots \\ \dot{w}_{n-1} &= \hat{G}_{n-1}(w_n) \\ \dot{w}_n &= u. \end{cases}$$

Remark III.2 The above theorem holds both locally and globally. More precisely, assume that Σ is, locally at $0 \in Z \subset \mathbb{R}^n$, S-equivalent to the $(ASFF)$-form. Then (i) and (ii) are equivalent locally around z_0. Now assume that Σ is globally S-equivalent to the $(ASFF)$-form on $\mathbb{R}^n$, then (i), satisfied locally around any point $z_0 \in Z$, is equivalent to the global S-equivalence to $(ANBr)$ on $\mathbb{R}^n$.

Proof of this Theorem, together with the Algorithm on which it is based, is given in Section VI. Of course, if Σ is F-linearizable (that is, if (i) of the above theorem holds), then it is F-equivalent to the Brunovský (Br)-form. Our result states, that F-linearizable systems that are S-equivalent to the affine strict feedforward $(ASFF)$-form exhibit also a nice form under a change of coordinates only. Namely, they are S-equivalent to the affine nonlinear Brunovský $(ANBr)$-form.

If we consider a general nonlinear system Ξ and assume that it is S-equivalent (locally or globally) to the (SFF)-form, then the above theorem remains valid (locally or globally) with the form $(ANBr)$ replaced by (NBr), that is, the equation $\dot{w}_n = u$ in (ii) replaced by $\dot{w}_n = \hat{G}_n(u)$.

IV. Calculating Normalizing and Linearizing Transformations

In this section we will examine the class of state and feedback transformations that bring $(ASFF)$-systems to the $(ANBr)$-form and to the linear Brunovský form (Br). Let us start with the following result of the authors [19], where we studied strict feedforward systems that are S-linearizable.

Theorem IV.1 *Assume that Σ is S-equivalent to the $(ASFF)$-form. Then the following conditions are equivalent:*

(i) *Σ is S-equivalent to a linear controllable system;*
(ii) *Σ is S-equivalent to the Brunovský form (Br).*

This result was proved (in a slightly different context) in [19] in a constructive way that gives the linearizing diffeomorphism in an explicit form calculated via integrations and compositions of functions only. We will formalize this important property as follows.

We will say that a transformation is calculated by *quadratures* if it is defined by a finite sequence consisting of elementary operations, composing functions and calculating integrals. We say that a transformation is calculated by *diffeo-quadratures* if it is defined by a finite sequence consisting of elementary operations, composing functions, calculating integrals, *and* differentiating.

We say that two systems are S-equivalent by bi-quadratures if there exists a diffeomorphism φ conjugating them so that φ and φ^{-1} are calculated by quadratures.

Proposition IV.2 *Consider a control-affine system Σ in the affine strict feedforward $(ASFF)$-form. If the system is S-linearizable, then it is S-equivalent to the Brunovský canonical form (Br) by bi-quadratures.*

For systems in affine strict feedforward $(ASFF)$-system that are F-linearizable, described by Theorem III.1, the picture is slightly different.

Proposition IV.3 *Consider a control system Σ in the affine strict feedforward $(ASFF)$-form. If the system is F-linearizable, then it is S-equivalent to the affine nonlinear Brunovský form $(ANBr)$ by bi-quadratures. Moreover, it is F-transformable to the Brunovský canonical form (Br) (and thus F-linearizable) by a transformation that is calculated by diffeo-quadratures.*

Proof. The proof of the first statement follows directly from the Algorithm given in Section VI which provides explicit formulas for the components of the diffeomorphism $w = \varphi(z)$ transforming $(ASFF)$ into $(ANBr)$. Those formulas involve, indeed, compositions, elementary operations, and integrations only. Notice that at the first glance we can suspect that in order to find $\varphi_l(z)$ (see the general substep of the general step of Algorithm) we have to integrate b_l and in order to know b_l we have to calculate *derivatives* since $b_l = \frac{\partial f_l}{\partial z_i}(\frac{\partial f_{i-1}}{\partial_{z_i}})^{-1}$. It is crucial to observe that there exists another way to calculate b_l as $b_l = (f_{i-1}(z_i))^{-1}(f_l(z_{l+1}, \ldots, z_i) - f(z_{l+1}, \ldots, z_{i-1}, 0))$ which involves composition and elementary operations only. Now, consider the $(ANBr)$-form and denote

$$f(w) = \hat{G}_1(w_2)\frac{\partial}{\partial w_1} + \cdots + \hat{G}_{n-1}(w_n)\frac{\partial}{\partial w_{n-1}},\ g(w) = \frac{\partial}{\partial w_n}.$$

It is well known (see, e.g., [3] and [12]) that in order to transform (via feedback) $(ANBr)$ into (Br) we use the following finite sequence of derivations

$$(LF) \qquad \begin{cases} \tilde{w}_1 &= h(w), \\ \tilde{w}_i &= L_f^{i-1}h(w),\ i = 2, \ldots, n, \\ \tilde{u} &= L_f^n h(w) + (L_g L_f^{n-1}h(w))u, \end{cases}$$

where $h(w) = w_1$. □

Notice that we do not claim that Σ is F-equivalent to a linear system (to the Brunovský form (Br) , for instance) by bi-diffeo-quadratures. Indeed, calculating the inverse feedback transformation (actually both, the inverse of the linearizing diffeomorphism and the inverse of control transformation) involves compositions, integrations, and *inverting* nonlinear functions.

A few comments are to be given. Consider a (control-affine, for simplicity) single-input nonlinear system

$$\dot{z} = f(z) + g(z)u,$$

which is not necessarily in the $(ASSF)$-form. We attach to Σ the sequence of nested distributions $\mathcal{D}^1 \subset \mathcal{D}^2 \subset \cdots \subset \mathcal{D}^n$:

$$\mathcal{D}^k = \operatorname{span}\left\{g, ad_f g, \ldots, ad_f^{k-1}g\right\},\ k = 1, 2, \ldots, n,$$

with $ad_f^0 g = g$, and inductively, $ad_f^{k-1}g = [f, ad_f^{k-2}g]$. Necessary and sufficient conditions for feedback linearization obtained in [2], [4] (see also [3] and [12]) are as follows:

Theorem IV.4 *A control-affine system $\Sigma : \dot{z} = f(z) + g(z)u$ is locally equivalent, via a change of coordinates $\tilde{w} = \varphi(z)$ and feedback $\tilde{u} = \tilde{\alpha}(z) + \tilde{\beta}(z)u$, to a linear controllable Brunovský canonical form (Br) if and only if*

(F1) $\dim \mathcal{D}^n(z) = n$

(F2) $\mathcal{D}^{n-1}$ *is involutive.*

As it is well known (see, e.g., [2], [3], [4], [12]), in order to F-linearize Σ we have to find a linearizing output, that is a function h whose differential $\mathrm{d}h$ does not vanish and annihilates the involutive distribution $\mathcal{D}^{n-1}$. Then h defines the linearizing coordinates and linearizing feedback by the formula (LF) given in the proof of Proposition IV.3. In order to find h we have to solve the system of first order PDE's

$$L_g L_f^{i-1}h = 0, \quad 1 \le i \le n-1, \qquad L_g L_f^{n-1}h \neq 0.$$

This system admits a solution (assured by involutivity of $\mathcal{D}^{n-1}$) but its solvability is, in general, a highly nontrivial task. A partial corollary of Proposition IV.3 is that for F-linearizable systems Σ that are in the $(ASFF)$-form, the problem of finding a linearizing output h can by solved by quadratures. Indeed, the first statement of Proposition IV.3 asserts that we can find, by quadratures (of the components $f_i(z)$ and $g_i(z)$), the diffeomorphism $w = \varphi(z)$ that transforms Σ into the Affine Nonlinear Brunovsky canonical form $(ANBr)$. A linearizing output is now the first component $w_1 = \varphi_1(z)$ of that diffeomorphism (compare the second part of the proof of Proposition IV.3 to see this).

V. Stabilization of F-Linearizable (ASFF)-Systems

It is well known that any F-linearizable system is (locally) asymptotically stabilizable by a state feedback that is linear with respect to the linearizing coordinates, see, e.g., [3]. The difficulty of implementing this result resides on the fact that the linearizing coordinates and feedback law are not always easy to find. For F-linearizable system that are in the $(ASFF)$-form, our algorithm provides, however, an easy way of finding the linearizing transformations and, as a consequence, a stabilizing controller via diffeo-quadratures. Namely, Proposition IV.3 implies the following result:

Proposition V.1 *Consider a system $\Sigma : \dot{z} = f(z)+g(z)u$ in $(ASFF)$-form, locally around $0 \in \mathbb{R}^n$ (resp. globally on $\mathbb{R}^n$) that is F-linearizable locally at $0 \in \mathbb{R}^n$ (resp. locally at any $z \in \mathbb{R}^n$). Let $w = \varphi(z)$ be the coordinates change (given by the Algorithm) that takes Σ into the $(ANBr)$-form, and $\tilde{w} = \psi(w), \tilde{u} = \tilde{\alpha}(w)+\tilde{\beta}(w)u$, given by (LF) in the proof of Proposition IV.3, be the feedback transformation that maps the $(ANBr)$-form into the (Br)-form. Then the feedback law*

$$u = -\frac{\tilde{\alpha}(\varphi(z)) + \sum_{i=1}^{n} k_i \psi_i(\varphi(z))}{\tilde{\beta}(\varphi(z))}, \tag{1}$$

where the polynomial $p(\lambda) = \lambda^n + \sum_{i=0}^{n-1} \lambda^i k_{i+1}$ is Hurwitz, locally (resp. globally on $\mathbb{R}^n$) asymptotically stabilizes the origin $0 \in \mathbb{R}^n$. Moreover, this stabilizing control law can be calculated by diffeo-quadratures (in terms of the components $f_i(z)$ and $g_i(z)$ of the original system Σ).

Proof. The proof is a direct consequence of Proposition IV.3 and stabilizability of feedback linearizable systems. Indeed, applying the controller (1) yields the closed loop linear system $\dot{\tilde{w}} = A\tilde{w}$, where $\tilde{w} = \Phi(z) = \psi \circ \varphi(z)$, and A is Hurwitz. Thus, by Proposition IV.3, the components Φ_i of Φ are calculated by diffeo-quadratures in terms of the components $f_i(z)$ and $g_i(z)$ of the original system Σ. □

It is interesting to observe that the Lyapunov function V can also be calculated by diffeo-quadratures (in terms of the components of the original system). Indeed, let P be the positive definite symmetric matrix solution of the Riccati equation $A^\top P + PA = -I$. Then $V(z) = \Phi^\top(z)P\Phi(z)$.

To illustrate this result we consider the following example.

Example V.2 Let us consider the system $(ASFF)$

$$\begin{cases} \dot{z}_1 &=& \sin(z_2+z_{n-1}) - 2(z_2+z_{n-1})\sin z_3, \\ \dot{z}_2 &=& \sin z_3 - \sin z_n, \\ \dot{z}_i &=& \sin z_{i+1}, \quad 3 \le i \le n-1 \\ \dot{z}_n &=& u, \end{cases}$$

which is control-normalized, i.e., $g(z) = \Big(0, \ldots, 0, 1\Big)^\top$.

The change of coordinates (see Example V.2-bis below)

$$w = \varphi(z) \triangleq \begin{cases} w_1 &=& z_1 + (z_2+z_{n-1})^2, \\ w_2 &=& z_2 + z_{n-1}, \\ w_i &=& z_i, \qquad 3 \le i \le n \end{cases}$$

transforms the system into the $(ANBr)$-form

$$\begin{cases} \dot{w}_i &=& \sin w_{i+1}, \quad 1 \le i \le n-1 \\ \dot{w}_n &=& u, \end{cases}$$

where $(w_1, \ldots, w_n) \in (-\pi,\pi) \times \cdots \times (-\pi,\pi)$. It is thus feedback linearizable by $\tilde{w} = \psi(w)$, $\tilde{u} = \tilde{\alpha}(w) + \tilde{\beta}(w)u$:

$$\psi_1 = w_1, \psi_2 = \sin w_2 \frac{\partial \psi_1}{\partial w_1}, \ldots, \psi_n = \sum_{i=1}^{n-1} \sin w_{i+1} \frac{\partial \psi_{n-1}}{\partial w_i},$$

$$\tilde{u} = \sum_{i=1}^{n-1} \sin w_{i+1} \frac{\partial \psi_n}{\partial w_i} + \frac{\partial \psi_n}{\partial w_n} u.$$

Since $\dfrac{\partial \psi_i}{\partial w_i} = \cos w_i \dfrac{\partial \psi_{i-1}}{\partial w_{i-1}},\ i = 2, \ldots, n$, then

$$\frac{\partial \psi_n}{\partial w_n} = \cos w_n \frac{\partial \psi_{n-1}}{\partial w_{n-1}} = \cdots = \cos w_n \cos w_{n-1} \cdots \cos w_2,$$

and thus for any $0 < \epsilon < \pi/4$, the feedback law

$$u = -\frac{\sum_{i=1}^{n-1} \sin z_{i+1} \frac{\partial \psi_n}{\partial z_i}(\varphi(z)) + \sum_{i=1}^{n} k_i \psi_i(\varphi(z))}{\cos(z_2+z_{n-1}) \cos z_3 \cdots \cos z_n}$$

locally asymptotically stabilizes the system on $(-\epsilon, \epsilon)^n$.

VI. Proof of Theorem III.1

(i)⇒(ii). It is clear that a system in $(ANBr)$-form is F-linearizable by the change of coordinates and feedback (LF).
(ii)⇒(i). We show that an F-linearizable $(ASFF)$-form, can be taken, via a (local) diffeomorphism, to the $(ANBr)$-form.
Algorithm. Assume that Σ is control-normalized (see [19]). It is well known that the involutivity of $\mathcal{D}^{n-1}$ (see $(F2)$ of Theorem IV.4) implies that of all distributions $\mathcal{D}^k$, $1 \le k \le n$.
Step 1. The involutivity of $\mathcal{D}^2$ implies

$$[g, ad_f g] = \gamma_1 ad_f g + \gamma_0 g,$$

where γ_1 and γ_0 are smooth functions. Because

$$f(z) = \sum_{j=1}^{n-1} f_j(z_{j+1}, \ldots, z_n)\partial_{z_j} \quad \text{and} \quad g = \partial_{z_n},$$

it follows that $\gamma_0 = 0$ and $\gamma_1 = \gamma_1(z_n)$. Above, $\partial_{z_j} = \frac{\partial}{\partial z_j}$. Thus the involutivity of $\mathcal{D}^2$ reduces to the condition

$$(\mathcal{F}_n) \qquad \frac{\partial^2 f_j}{\partial z_n^2} = \gamma_1(z_n) \frac{\partial f_j}{\partial z_n} \quad \text{for all } 1 \le j \le n-1.$$

Condition $(\mathcal{F}_n)$ is necessary for F-linearization, i.e., if it fails (γ_1 depends on other variables than z_n or γ_1 is not the same for all components f_j) then the algorithm stops. If $(\mathcal{F}_n)$ holds, we can simplify the system using $n-2$ substeps.

Let $j = n-1$ in $(\mathcal{F}_n)$. Since $f_{n-1} = f_{n-1}(z_n)$, we get $f''_{n-1}(z_n) = \gamma_1(z_n) f'_{n-1}(z_n)$, which gives γ_1 uniquely as $\gamma_1 = f''_{n-1}(z_n)/f'_{n-1}(z_n)$. Two successive integrations yield

$$f_{n-1}(z) = \int_0^{z_n} a_{n-1} \exp\left(\int_0^t \gamma_1(s) \mathrm{d}s\right) \mathrm{d}t,$$

with $a_{n-1} \in \mathbb{R}^* = \mathbb{R} \setminus 0$.
Substep 1. Take $j = n-2$ in $(\mathcal{F}_n)$ and denote $h_{n-2} = \frac{\partial f_{n-2}}{\partial z_n}$. We obtain after integration

$$h_{n-2}(z_{n-1}, z_n) = a_{n-2}(z_{n-1}) \exp\left(\int_0^{z_n} \gamma_1(s) \mathrm{d}s\right)$$

which implies, after a second integration, that

$$f_{n-2}(z_{n-1}, z_n) = c_{n-2}(z_{n-1}) + f_{n-1}(z_n) b_{n-2}(z_{n-1}),$$

for some smooth functions c_{n-2} and $b_{n-2} = a_{n-2}/a_{n-1}$.

The diffeomorphism $x = \varphi(z)$ whose components are

$$\tilde{z}_j = \varphi_j(z) = z_j, \quad j \ne n-2$$

$$\tilde{z}_{n-2} = \varphi_{n-2}(z) = z_{n-2} - \int_0^{z_{n-1}} b_{n-2}(s) \mathrm{d}s$$

transforms the system, by *quadratures*, into the form

$$\tilde{\Sigma}\ :\ \dot{\tilde{z}} = \tilde{f}(\tilde{z}) + \tilde{g}(\tilde{z})u, \quad \tilde{z} \in \mathbb{R}^n,$$

with $\tilde{g}(\tilde{z}) = (0, \ldots, 0, 1)^\top$ and

$$\begin{aligned} \tilde{f}(\tilde{z}) &= \sum_{j=1}^{n-3} \tilde{f}_j(\tilde{z}_{j+1}, \ldots, \tilde{z}_n)\partial_{\tilde{z}_j} \\ &+ \tilde{f}_{n-2}(\tilde{z}_{n-1})\partial_{\tilde{z}_{n-2}} + \tilde{f}_{n-1}(\tilde{z}_n)\partial_{\tilde{z}_{n-1}}. \end{aligned}$$

General Substep. Assume that for some $1 \le i \le n-2$, a sequence of quadratures exists whose composition has brought the original system into (we keep the z-notation)

$$\Sigma\ :\ \dot{z} = f(z) + g(z)u, \quad z \in \mathbb{R}^n,$$

with $g(z) = (0, \ldots, 0, 1)^\top$ and

$$\begin{aligned} f(z) &= \sum_{j=1}^{i} f_j(z_{j+1}, \ldots, z_n)\partial_{z_j} \\ &+ \sum_{j=i+1}^{n-2} f_j(z_{j+1}, \ldots, z_{n-1})\partial_{z_j} + f_{n-1}(z_n)\partial_{z_{n-1}}. \end{aligned}$$

Taking $j = i$ in the condition $(\mathcal{F}_n)$ we have

$$\frac{\partial^2 f_i}{\partial z_n^2} = \gamma_1(z_n)\frac{\partial f_i}{\partial z_n}.$$

Denoting $h_i(z_{i+1}, \ldots, z_n) = \frac{\partial f_i}{\partial_{z_n}}$, we obtain

$$h_i = a_i(z_{i+1}, \ldots, z_{n-1}) \exp\left(\int_0^{z_n} \gamma_1(s)\mathrm{d}s\right)$$

and after a second integration

$$f_i = c_i(z_{i+1}, \ldots, z_{n-1}) + f_{n-1}(z_n)b_i(z_{i+1}, \ldots, z_{n-1}),$$

for some smooth functions c_i and $b_i = a_i/a_{n-1}$.

The diffeomorphism $x = \varphi(z)$ whose components are

$$\begin{aligned} \tilde{z}_j &= \varphi_j(z) = z_j, \quad j \neq i \\ \tilde{z}_i &= \varphi_i(z) = z_i - \int_0^{z_{n-1}} b_i(z_{i+1}, \ldots, z_{n-2}, s)\mathrm{d}s \end{aligned}$$

transforms the system, by *quadratures*, into the form

$$\tilde{\Sigma}\ :\ \dot{\tilde{z}} = \tilde{f}(\tilde{z}) + \tilde{g}(\tilde{z})u, \quad \tilde{z} \in \mathbb{R}^n,$$

with $\tilde{g}(\tilde{z}) = (0, \ldots, 0, 1)^\top$ and

$$\begin{aligned} \tilde{f}(\tilde{z}) &= \sum_{j=1}^{i-1} \tilde{f}_j(\tilde{z}_{j+1}, \ldots, \tilde{z}_n)\partial_{\tilde{z}_j} \\ &+ \sum_{j=i}^{n-2} \tilde{f}_j(\tilde{z}_{j+1}, \ldots, \tilde{z}_{n-1})\partial_{\tilde{z}_j} + \tilde{f}_{n-1}(\tilde{z}_n)\partial_{\tilde{z}_{n-1}}. \end{aligned}$$

Notice that, at each substep, the inverse ψ of the diffeomorphism $x = \varphi(z)$ is easily computable as

$$\begin{aligned} z_j &= \psi_j(\tilde{z}) = \tilde{z}_j, \quad j \neq i \\ z_i &= \psi_i(\tilde{z}) = \tilde{z}_i + \int_0^{\tilde{z}_{n-1}} b_i(\tilde{z}_{i+1}, \ldots, \tilde{z}_{n-2}, s)\mathrm{d}s. \end{aligned}$$

Moreover, for any $1 \le j \le n-2$, we have

$$\tilde{f}_j(\tilde{z}_{j+1}, ..., \tilde{z}_{n-1}) = f_j(\psi_{j+1}(\tilde{z}_{j+1}, ..., \tilde{z}_{n-1}), ..., \psi_{n-1}(\tilde{z}_{n-1})).$$

The original system is thus brought, via $n-2$ substeps, to

$$\Sigma\ :\ \dot{z} = f(z) + g(z)u, \quad z \in \mathbb{R}^n,$$

with $g = (0, \ldots, 0, 1)^\top$ and

$$\begin{aligned} f(z) &= \sum_{j=1}^{i} f_j(z_{j+1}, \ldots, z_n)\partial_{z_j} \\ &+ \sum_{j=i+1}^{n-2} f_j(z_{j+1}, \ldots, z_{n-1})\partial_{z_j} + f_{n-1}(z_n)\partial_{z_{n-1}}. \end{aligned}$$

This ends the first step of the algorithm. We will denote by φ^1 the composition of the diffeomorphisms of step 1.

General Step. For simplicity, we skip the tildes. Assume that Σ has been brought, via quadratures, to the form

$$\Sigma\ :\ \dot{z} = f(z) + g(z)u, \quad z \in \mathbb{R}^n,$$

where $g = (0, \ldots, 0, 1)^\top$ and for some $3 \le i \le n-2$

$$f(z) = \sum_{j=1}^{i-2} f_j(z_{j+1}, \ldots, z_i)\partial_{z_j} + \sum_{j=i-1}^{n-1} f_j(z_{j+1})\partial_{z_j}.$$

We will show that Σ can be brought, via quadratures, to $\tilde{\Sigma} : \tilde{f}(\tilde{z}) + \tilde{g}(\tilde{z})u$, where $\tilde{g} = (0, \ldots, 0, 1)^\top$ and

$$\tilde{f}(\tilde{z}) = \sum_{j=1}^{i-3} \tilde{f}_j(\tilde{z}_{j+1}, \ldots, \tilde{z}_{i-1})\partial_{\tilde{z}_j} + \sum_{j=i-2}^{n-1} \tilde{f}_j(\tilde{z}_{j+1})\partial_{\tilde{z}_j}.$$

We deduce from above that for any $1 \le i \le k \le n-1$

$$ad_f^{n-k} g = \mu_k(z_{k+1}, \ldots, z_n)\partial_{z_k} + \vartheta_k(z),$$

where the vector field $\vartheta_k \in \mathcal{D}^{n-k} = \mathrm{span}\,\{\partial_{z_{k+1}}, \ldots, \partial_{z_n}\}$ and μ_k is a smooth function. In particular for $k = i$ we have

$$ad_f^{n-i} g = \mu_i(z_{i+1}, \ldots, z_n)\partial_{z_i} + \vartheta_i(z),$$

from which, and the expression of f, we deduce that

$$ad_f^{n-i+1} g = \sum_{j=1}^{i-1} \mu_j(z_{j+1}, \ldots, z_n)\partial_{z_j} + \vartheta_{i-1}(z),$$

where $\vartheta_{i-1} \in \Delta^{n-i+1}$ and for any $1 \le j \le i-1$

$$\mu_j(z_{j+1}, \ldots, z_n) = -\mu_i(z_{i+1}, \ldots, z_n)\frac{\partial f_j}{\partial z_i}.$$

A simple calculation shows that

$$\left[ad_f^{n-i+1} g, ad_f^{n-i} g\right] = \mu_i^2 \cdot \sum_{j=1}^{i-1} \frac{\partial^2 f_j}{\partial z_i^2}\partial_{z_j} + \tilde{\vartheta}_{i-1}(z),$$

where $\tilde{\vartheta}_{i-1} \in \mathcal{D}^{n-i+1} = \mathrm{span}\,\{\partial_{z_i}, \ldots, \partial_{z_n}\}$.

The involutivity of $\mathcal{D}^{n-i+2}$ implies that

$$\begin{aligned} \left[ad_f^{n-i+1} g, ad_f^{n-i} g\right] &= \sum_{k=i-1}^{n} \gamma_{n-k} ad_f^{n-k} g \\ &= \gamma_{n-i+1} ad_f^{n-i+1} g + \hat{\vartheta}_{i-1} \end{aligned}$$

for some smooth functions $\gamma_0, \gamma_1, \ldots, \gamma_{n-i+1}$.

Comparing the two Lie brackets it follows that

$$(\mu_i)^2 \cdot \sum_{j=1}^{i-1} \frac{\partial^2 f_j}{\partial z_i^2}\partial_{z_j} = -(\mu_i)\gamma_{n-i+1} \cdot \sum_{j=1}^{i-1} \frac{\partial f_j}{\partial z_i}\partial_{z_j},$$

that is, the condition

$$(\mathcal{F}_i) \qquad \frac{\partial^2 f_j}{\partial z_i^2} = \tilde{\gamma}_{n-i+1}\frac{\partial f_j}{\partial z_i},\ 1 \le j \le i-1.$$

For $j = i-1$ we get $f''_{i-1}(z_i) = \tilde{\gamma}_{n-i+1}(z) f'_{i-1}(z_i)$ which determines $\tilde{\gamma}_{n-i+1} = f''_{i-1}(z_i)/f'_{i-1}(z_i)$ uniquely in terms of z_i. The components f_{i-1} and $\tilde{\gamma}_{n-i+1}$ are related by

$$f_{i-1}(z_i) = \int_0^{z_i} a_{i-1}\exp\left(\int_0^t \tilde{\gamma}_{n-i+1}(s)\mathrm{d}s\right)\mathrm{d}t,\ a_{i-1} \in \mathbb{R}^*.$$

General Substep. Let us assume that the original system has been brought, via *quadratures*, to the form

$$\Sigma\ :\ \dot{z} = f(z) + g(z)u, \quad z \in \mathbb{R}^n,$$

where $g = (0, \ldots, 0, 1)^\top$ and for some $2 \le l < i \le n-2$

$$f(z) = \sum_{j=1}^{l} f_j(z_{j+1}, \ldots, z_i)\partial_{z_j} + \sum_{j=l+1}^{n-1} f_j(z_{j+1})\partial_{z_j}.$$

Taking $j = l$ in $(\mathcal{F}_i)$ and denoting $h_l = \frac{\partial f_l}{\partial_{z_i}}$ we have

$$h_l = a_l(z_{l+1}, \ldots, z_{i-1})\exp\left(\int_0^{z_i} \tilde{\gamma}_{n-i+1}(s)\mathrm{d}s\right)$$

and after integration

$$f_l(z_{l+1}, ..., z_i) = c_l(z_{l+1}, ..., z_{i-1}) + f_{i-1}(z_i) b_l(z_{l+1}, ..., z_{i-1}),$$

for some smooth functions c_l and $b_l = a_l/a_{i-1}$.

The new coordinates $x = \varphi(z)$ whose components are

$$\begin{aligned} \tilde{z}_j &= \varphi_j(z) = z_j, \quad j \neq l \\ \tilde{z}_l &= \varphi_l(z) = z_l - \int_0^{z_{i-1}} b_l(z_{l+1}, \ldots, z_{i-2}, s)\mathrm{d}s \end{aligned}$$

transforms the system, by *quadratures*, into the form

$$\tilde{\Sigma}\ :\ \dot{\tilde{z}} = \tilde{f}(\tilde{z}) + \tilde{g}(\tilde{z})u, \quad \tilde{z} \in \mathbb{R}^n,$$

where $\tilde{g} = (0, \ldots, 0, 1)^\top$ and

$$\tilde{f}(\tilde{z}) = \sum_{j=1}^{l-1} \tilde{f}_j(\tilde{z}_{j+1}, \ldots, \tilde{z}_i)\partial_{\tilde{z}_j} + \sum_{j=l}^{n-1} \tilde{f}_j(\tilde{z}_{j+1})\partial_{\tilde{z}_j}.$$

This ends the general step. Denote by φ^i the composition of the coordinates changes for the i-th step. Thus the composition $\varphi^{n-2} \circ \cdots \circ \varphi^1$ defines the coordinates change taking Σ into the $(ANBr)$-form, which completes the proof of Theorem III.1.

Example V.2-bis. Reconsider Example V.2. Then $(\mathcal{F}_n)$ holds with $\gamma_1 = -\tan z_n$. For $3 \le i \le n-1$, the decomposition

$$f_i(z) = c_i(z_{i+1}, \ldots, z_{n-1}) + f_{n-1}(z_n) b_i(z_{i+1}, \ldots, z_{n-1})$$

yields $b_i = 0$ and $c_i = \sin z_{i+1}$ because $f_i(z) = \sin z_{i+1}$. Moreover, $b_2 = -1$ and $c_2 = \sin z_3$ since $f_{n-1}(z) = \sin z_n$ and $f_2(z) = \sin z_3 - \sin z_n$. The transformation

$$\begin{aligned} \tilde{z}_j &= z_j, j \neq 2 \\ \tilde{z}_2 &= z_2 - \int_0^{z_{n-1}} (-1)\mathrm{d}s = z_2 + z_{n-1} \end{aligned}$$

brings the system into the form

$$\begin{cases} \dot{\tilde{z}}_1 &= \sin\tilde{z}_2 - 2\tilde{z}_2\sin\tilde{z}_3, \\ \dot{\tilde{z}}_i &= \sin\tilde{z}_{i+1}, \qquad 2 \le i \le n-1 \\ \dot{\tilde{z}}_n &= u. \end{cases}$$

Next, we apply the last step (Step $n-3$) with

$$(\mathcal{F}_3) \qquad \frac{\partial^2 \tilde{f}_j}{\partial \tilde{z}_3^2} = \tilde{\gamma}_{n-2}(\tilde{z})\frac{\partial \tilde{f}_j}{\partial \tilde{z}_3}, \quad 1 \le j \le 2,$$

which holds for $\tilde{\gamma}_{n-2} = -\tan\tilde{z}_3$. The decomposition of

$$\tilde{f}_1(\tilde{z}) = \tilde{c}_1(\tilde{z}_2) + \tilde{f}_2(\tilde{z}_3)\tilde{b}_1(\tilde{z}_2)$$

yields $\tilde{c}_1(\tilde{z}_2) = \sin\tilde{z}_2$ and $\tilde{b}_1(\tilde{z}_2) = -2\tilde{z}_2$. Hence

$$\begin{aligned} w_j &= \tilde{z}_j, j \neq 1 \\ w_1 &= \tilde{z}_1 - \int_0^{\tilde{z}_2} (-2s)\mathrm{d}s = \tilde{z}_2 + \tilde{z}_2^2 \end{aligned}$$

The composition gives the linearizing coordinates system.

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