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Space-time scaling laws for self-triggered control

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***Abstract*— Feedback control laws have been traditionally treated as periodic tasks when implemented on digital platforms. However, the growing complexity of systems calls for efficient implementations of control tasks that reduce resource utilization while keeping desired levels of performance. In this paper we drop the periodicity assumption in favour of self-triggered strategies for the execution of control laws. Such strategies determine the next execution time based on the current state of the plant. Under the self-triggered policy, the inter-execution times scale in a predictable manner: a scaling of the state of the plant entails a scaling in the inter-execution times. This property allows us to derive a simple formula for the next execution time guaranteeing performance. We illustrate the proposed techniques on the control of a jet engine compressor.**

I. Introduction

Digital implementations of feedback controllers offer many advantages (accuracy, flexibility,..) with respect to analog implementations. Under a digital implementation, a physical system is controlled by measuring its state at discrete instants in time, by using the measured state to compute a feedback control law, and by updating the actuator with the computed law also at discrete instants in time. A natural design problem is to determine the requirements to be imposed on these instants of time to achieve desired performance. Traditionally, engineers and researchers have opted for conservative strategies, such as periodic sampling of signals and periodic execution of control laws. However, due to the growing complexity of systems, more efficient implementations of controllers are required, since resources are usually shared between several subsystems. For instance, in the case of embedded systems, a single computation unit is in charge of many different tasks, ranging from image processing to communication decoding, in addition to one or several control tasks. In distributed systems, where communication is required between sensors, computational units and actuators, a communication link is shared between different subsystems. Periodic implementations execute the controller every T units of time, regardless the state of the control system. Since this period T is chosen from worst-case conditions (to guarantee performance for all operation points), the periodic policy leads to an overly conservative implementation.

To overcome these drawbacks, several researchers ([Årz99], [ÅB02], [Tab07], [HSvdB08]) suggested the idea of event-triggered control. Under this paradigm, the controller execution is triggered according to the state of the plant. The event-triggered technique reduces resource usage and provides a high degree of robustness (since the system is measured continuously). Unfortunately, in many cases it requires dedicated hardware to monitor the plant permanently, not available in most general purpose devices.

In this paper we propose to take advantage of the event-triggered technique without resorting to extra hardware. In most feedback laws, the state of the plant has to be measured (or estimated) to compute the next value of the controller; hence, this information could be used to decide when the control task needs to be applied again. This technique is known as self-triggered control, since the controller decides its next execution time. We could regard this technique as a way to introduce feedback in the triggering process, in contrast with the open-loop triggering process of periodic implementations. A first attempt to explore self-triggered models for linear control systems was developed in [VFM03], by discretizing the plant, and in [WL08] for linear $\mathcal{H}_\infty$ controllers. In the context of nonlinear systems, to the best of our knowledge, the first results appeared in [AT08b] where, under a homogeneity assumption, scaling laws for the inter-execution times were derived as a function of the state norm.

Under self-triggered implementations, the execution times for a control law should be defined by a simple formula (since it has to be computed online) that depends on the dynamical model of the system, the desired performance, and the current measurement of the state. In this paper the inter-execution times are given by a scaling law that generalizes the one appearing in [AT08b] for a wider class of nonlinear systems. The key technical contribution consists in introducing a state-dependent notion of homogeneity. To derive the scaling law we first review an event-triggered condition that guarantees stability under sample-and-hold implementations, previously studied in [Tab07]. Then, we define state-dependent homogeneous systems, and analyse the properties of their trajectories. These systems possess infinitesimal symmetries that can be exploited to find spatio-temporal relations for the inter-execution times. The aforementioned scaling law motivates us to derive self-triggered conditions that determine the next execution time in order to achieve desired performance. We conclude the paper by illustrating our results on the control of a jet engine compressor. Due to space limitations, proofs are included in the journal version of this paper [AT08c].

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II. NOTATION AND INPUT-TO-STATE STABILITY

A. Notation

We shall use the notation $|x|$ to denote the Euclidean norm of an element $x \in \mathbb{R}^n$. A continuous function $\alpha : [0, a[\to \mathbb{R}_0^+$, $a > 0$, is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to be of class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

A function is said to be of class C^∞ or smooth if it can be differentiated infinitely many times. All the objects in this document are considered to be smooth unless otherwise stated.

Given vector fields X, Y in an n-dimensional manifold M, we let [X,Y] denote their Lie product which, in local coordinates $x = (x_1, x_2, ..., x_n)$, we take as $\frac{\partial X}{\partial x}(x)Y(x) - \frac{\partial Y}{\partial x}(x)X(x)$. We denote the tangent bundle of M by TM. We shall use the notations $\psi(t, \cdot)$ and $\psi_t(\cdot)$ interchangeably to denote a map $\psi : \mathbb{R} \times M \to M$.

B. Input-to-state stability (ISS)

We consider a control system:

$$\dot{x} = f(x, u), \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \tag{II.1}$$

where x denotes the state and u the input. We will not use the standard definition of ISS in this paper but rather the following characterization.

Definition 2.1: A smooth function $V : \mathbb{R}^n \to \mathbb{R}_0^+$ is said to be an ISS Lyapunov function for the system (II.1) if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}, \alpha$ and γ satisfying:

$$\underline{\alpha}(|x|) \le V(x) \le \overline{\alpha}(|x|) \tag{II.2}$$

$$\frac{\partial V}{\partial x} f(x, u) \le -\alpha(|x|) + \gamma(|u|) \tag{II.3}$$

The system (II.1) is said to be ISS with respect to the input u if and only if there exists an ISS Lyapunov function for (II.1).

III. EVENT-TRIGGERED STABILIZATION OF NONLINEAR SYSTEMS

We start exploring the behaviour of the plant under the event-triggered implementation introduced in [Tab07] and reviewed in this section. We consider the control system described in (II.1), for which a feedback controller $u = k(x)$ has been designed. The implementation of such feedback law on an embedded processor is typically done by sampling the state at time instants t_i, computing $u(t_i) = k(x(t_i))$ and updating the actuator values at time instants $t_i + \Delta_i$. For ease of exposition, it is assumed that $\Delta_i = 0$ for all $i \in \mathbb{N}$, that is, every time the state is sampled the controller is computed without delay[1]. Furthermore, the sequence of times $\{t_i\}$ is typically periodic meaning that $t_{i+1} - t_i = T$ where $T > 0$ is the period. In this paper we drop the periodicity assumption in favour of self-triggered implementations. To derive a stabilizing execution rule, we study the inter-sample behaviour of the control system. We define the measurement error e as the difference between the last measured state and the current value of the state:

$$t \in [t_i, t_{i+1}[\quad \Longrightarrow \quad e(t) = x(t_i) - x(t) \tag{III.1}$$

With this definition, the closed loop $\dot{x} = f(x, k(x(t_i)))$ becomes:

$$\dot{x} = f(x, k(x + e)) \tag{III.2}$$

Let the control law $u = k(x)$ render the system ISS with respect to the measurement error e. Under that assumption, there exists a Lyapunov function V for the system that satisfies the following inequality:

$$\dot{V} \le -\alpha(|x|) + \gamma(|e|) \tag{III.3}$$

where α and γ are $\mathcal{K}_\infty$ functions. Stability of the closed loop system (III.2) can be guaranteed if we restrict the error to satisfy:

$$\gamma(|e|) \le \sigma\alpha(|x|), \qquad \sigma > 0 \tag{III.4}$$

since the dynamics of V will be bounded by:

$$\dot{V} \le (\sigma - 1)\alpha(|x|) \tag{III.5}$$

guaranteeing that V decreases provided that $\sigma < 1$. If α^{-1} and γ are Lipschitz continuous on compacts, inequality (III.4) is implied by this simpler inequality:

$$b|e| \le \sigma a|x| \tag{III.6}$$

for a and b appropriately chosen according to the Lipschitz constants of α^{-1} and γ. Inequality (III.6) can be enforced by executing the control task whenever:

$$|e| = \sigma\frac{a}{b}|x| \tag{III.7}$$

Upon the execution of the control task, the state is measured and the error becomes 0, since $x(t_i) = x(t)$ implies $e(t) = x(t_i) - x(t) = 0$. An event-triggered implementation based on this equality would require testing (III.7) frequently. Unless this testing process is implemented in hardware, one might run the risk of consuming the processor time freed-up by using an event-triggered implementation to test (III.7). To overcome this drawback, we opt for self-triggered strategies, where the current state measurement $x(t_i)$ is used to determine its next execution time t_{i+1}.

Hence the inter-execution times that preserve the stability of the system $\dot{x} = f(x, k(x+e))$ are determined by the ratio $|e|/|x|$. The inter-execution time implicitly defined by (III.7) is the time it takes for $\frac{|e|}{|x|}$ to evolve from[2] 0 to $\sigma\frac{a}{b}$. The dynamics of such ratio are difficult to analyse, specially for general nonlinear systems. We will then study the evolution not for the ratio $|e|/|x|$ but for the inter-execution times described by inequality (III.6). To describe this evolution, we study the properties of the trajectories of state-dependent homogeneous systems in the next section.

[1]The results can be generalized for nonzero Δ_i by following the procedure described in [Tab07].

[2]Recall that at the execution instant $t = t_i$ we have $e(t) = x(t_i) - x(t) = 0$ and thus $\frac{|e|}{|x|} = 0$.

IV. State dependent homogeneity and space-time dilations

Homogeneous vector fields are vector fields possessing a symmetry with respect to a family of dilations. They appear as local approximations for general nonlinear systems [Her91] since we can always decompose an analytic function in an infinite sum of homogeneous functions. Moreover, many physical systems such as the rigid body can be described as homogeneous systems (see [Bai80] for more examples).

Definition 4.1: A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called homogeneous of order ζ if for all $\lambda > 0$, $x \in \mathbb{R}^n$, there exist $r_i > 0$, $i = 1..n$ such that:

$$f_i(\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n) = \lambda^{\zeta}\lambda^{r_i}f_i(x_1, \ldots, x_n) \quad \text{(IV.1)}$$

where $\zeta > -\min_i r_i$.
We will now recall the more general, coordinate-free geometric notion of homogeneity introduced in [Kaw95].

Definition 4.2: Let $D : M \rightarrow TM$ be a vector field such that $\dot{x} = -D(x)$ is globally asymptotically stable. The vector field $X : M \rightarrow TM$ is called homogeneous of degree ζ with respect to D if it satisfies the following relation:

$$[D, X] = \zeta X \quad \text{(IV.2)}$$

The trajectories $\psi : \mathbb{R} \times M \rightarrow M$ of the vector field D are called the homogeneous rays of the system.
We are mainly interested in the trajectories of the vector field X.

Theorem 4.3: Let X and D be vector fields on a manifold M, giving rise to flows $\phi : \mathbb{R} \times M \rightarrow M$ and $\psi : \mathbb{R} \times M \rightarrow M$, respectively. The vector field X is homogeneous of degree ζ with respect to D if and only if:

$$\phi_t \circ \psi_s = \psi_s \circ \phi_{\mathbf{e}^{\zeta s}t}, \qquad s, t \in \mathbb{R} \quad \text{(IV.3)}$$

This theorem implies that the flows of homogeneous vector fields commute in a particular way: applying the flow ψ before the flow ϕ to a point $x \in M$ is equivalent to applying the flow ϕ before ψ but with a scaling in time, given by $\mathbf{e}^{\zeta s}$, where ζ is the degree of homogeneity. Similar results were obtained in [Tun05] for hybrid homogeneous systems. In this document we generalize Definition 4.2 to the following state-dependent notion of homogeneity.

Definition 4.4: Let $D : M \rightarrow TM$ be a vector field such that $\dot{x} = -D(x)$ is globally asymptotically stable. The vector field $X : M \rightarrow TM$ is called homogeneous with degree function $m : M \rightarrow \mathbb{R}$ with respect to D if it satisfies the following relation:

$$[D, X] = mX \quad \text{(IV.4)}$$

Example 4.5: Let the vector fields D and X be:

$$\begin{aligned} D &= \alpha x_1 \frac{\partial}{\partial x_1} + \alpha x_2 \frac{\partial}{\partial x_2}, \qquad \alpha > 0 \\ X &= (-x_1 - x_1^3)\frac{\partial}{\partial x_1} + (-x_2 - x_1^2 x_2)\frac{\partial}{\partial x_2} \end{aligned} \quad \text{(IV.5)}$$

where $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$ is the canonical basis for the tangent bundle TM. It is straightforward to verify that the vector fields X and D satisfy (IV.4) for the following m:

$$m(x) = \frac{3x_1^2 + 1}{x_1^2 + 1}$$

As for the standard homogeneity case, a relation between trajectories can also be derived.

Theorem 4.6: Let X and D be vector fields on a manifold M, giving rise to flows $\phi : \mathbb{R} \times M \rightarrow M$ and $\psi : \mathbb{R} \times M \rightarrow M$, respectively. The vector field X is homogeneous with degree function $m : M \rightarrow \mathbb{R}$ with respect to D if and only if:

$$\begin{aligned} \phi_t \circ \psi_s &= \psi_s \circ \phi_{\mathbf{e}^{\rho(s)}t} \qquad s, t \in \mathbb{R} \quad \text{(IV.6)} \\ \text{with} \qquad \rho(s) &= \int_0^s m \circ \psi_\tau \, d\tau \quad \text{(IV.7)} \end{aligned}$$

The vector field D generates a flow ψ that can be considered as a spatial dilation operator acting on points in M. This operator determines how times are scaled in the flow of ϕ. That is, given a point x in a manifold M, applying this spatial operator ψ to x entails a scaling in time $\mathbf{e}^{\rho(s)}$ in the flow of a vector field X.

In the case of the standard homogeneity, ρ is linear in time, $\rho(s) = ms$ (m being a constant function), and equation (IV.6) becomes (IV.3) for $m = \zeta$.

Example 4.7: To illustrate the previous theorem, we recover the previous example (4.5). The flow for the dilation vector field D is:

$$\psi_s(x) = \mathbf{e}^{\alpha s}x \quad \text{(IV.8)}$$

where $x = (x_1, x_2)^T$. Hence the flow of the vector field X satisfies equation (IV.6):

$$\begin{aligned} \phi_t(\mathbf{e}^{\alpha s}x) &= \mathbf{e}^{\alpha s}\phi_{\mathbf{e}^{\rho(s)}t}(x) \\ \text{with} \qquad \rho(s) &= \frac{1}{\alpha}\log\left(\frac{\mathbf{e}^{3\alpha s}x_1^2 + \mathbf{e}^{\alpha s}}{x_1^2 + 1}\right) \end{aligned}$$

We see that the spatial dilation $\mathbf{e}^{\alpha s}$ induces a temporal dilation $\mathbf{e}^{\rho(s)}$ in the trajectory ϕ.

Remark 4.8: For linear systems, the degree of homogeneity m is 0, therefore $\rho(s) = 0$ and the vector fields X and D commute, that is:

$$\phi_t \circ \psi_s = \psi_s \circ \phi_t \quad \text{(IV.9)}$$

The commutative properties herein described lead us to develop a self-triggered strategy in the next section.

V. Inter-execution time scaling laws for homogeneous systems

For simplicity, we use the following dilation vector field:

$$D = \sum_{i=1}^{n} r_i x_i \frac{\partial}{\partial x_i} \quad \text{(V.1)}$$

The flow of this vector field (i.e., the homogeneous ray) is:

$$\psi_i(s, x) = \mathbf{e}^{r_i s}x_i \qquad i = 1, \ldots, n \quad \text{(V.2)}$$

since it satisfies:

$$\frac{d}{ds}\psi_s(x) = D(\psi_s(x)), \qquad \psi_0(x) = x \quad \text{(V.3)}$$

Moreover, we assume $r_1 = r_2 = \ldots = r_n$ (called standard homogeneity), since the general case of (V.1) can be reduced to this one by means of a change of coordinates, as explained in [Grü00]. Using the commutative properties of homogeneous flows, we can derive the following scaling law for the inter-execution times under the event-triggered policy described in section (III).

Theorem 5.1: Let $\dot{x} = X(x, u)$ be a control system for which a feedback control law $u = k(x)$ has been designed, rendering the closed loop homogeneous with degree function m with respect to the standard dilation vector field (V.1) with $r_1 = r_2 = \ldots = r_n$. The inter-execution times τ implicitly defined by the execution rule $|e| = c|x|$, with $c > 0$, scale according to:

$$\tau(\mathbf{e}^{rs}p) \;=\; \mathbf{e}^{-\rho(s)}\tau(p) \quad \forall s \in \mathbb{R}, r > 0 \tag{V.4}$$

$$\text{with} \qquad \rho(s) \;=\; \int_0^s m(\mathbf{e}^{r\tau}p)d\tau \tag{V.5}$$

where $p \in \mathbb{R}^n$ represents any point in the state space.
This theorem relates the inter-execution time at a point with the inter-execution time at any other point lying along the same homogeneous ray. That is, once the time is known for just one initial condition p, we can infer the times for all initial conditions of the type λp, for any $\lambda \geq 0$.

Remark 5.2: For linear systems, the degree function m is 0, therefore $\rho(s) = 0$ and $\mathbf{e}^{-\rho(s)} = 1$. That is, for linear systems the inter-execution times remain constant as we move along homogeneous rays.

Remark 5.3: When the degree function m is a constant (Definition 4.2), the scaling in time depends entirely on the value of s. That is, to see how times scale it is enough to determine the position of the state p on a homogeneous ray, but not on which particular ray p lies, since all rays scale in the same way. In other words, the scaling in time is just a function of the current norm of the state. On the other hand, for state-dependent homogeneity the scaling in time is determined by the ray where p lies and the position of p in that ray, that is, ρ is a function of s and p, since each homogeneous ray scales in a different manner.

Theorem 5.1 allows us to use the estimate of the inter-execution times at some x in order to determine the inter-execution times for the whole ray through x. Therefore, it is enough to find estimates of these times on any $n-1$ sphere, and then extend the results along homogeneous rays. Moreover, since a lower bound τ can be easily computed for linear systems[3], we can always choose a $n-1$ sphere where a linear over-approximation for the control system can be obtained. To do so, we rewrite equation (IV.4) in local coordinates. A homogeneous function g of order $m-1$ satisfies:

$$(m + r_i)g(x) = \sum_{i=1}^{n} r_i x_i \frac{\partial g}{\partial x_i}$$

Hence, for the closed loop system $\dot{x} = \tilde{f}(x, e) = f(x, k(x + e))$ we can find a bound for $|\tilde{f}(x, e)|$ linear in $|x|$ and $|e|$:

$$\begin{aligned} |\tilde{f}(x, e)| &= |H(x, e)x + G(x, e)e| \\ &\leq |H(x, e)||x| + |G(x, e)||e| \\ &\leq |H(x_a^*, e_a^*)||x| + |G(x_b^*, e_b^*)||e| \end{aligned}$$

where:

$$H(x, e) := \begin{bmatrix} \frac{r_1}{m+r_1}\frac{\partial \tilde{f}_1}{\partial x_1} & \cdots & \frac{r_1}{m+r_1}\frac{\partial \tilde{f}_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{r_n}{m+r_n}\frac{\partial \tilde{f}_n}{\partial x_1} & \cdots & \frac{r_n}{m+r_n}\frac{\partial \tilde{f}_n}{\partial x_n} \end{bmatrix}$$

$$G(x, e) := \begin{bmatrix} \frac{r_1}{m+r_1}\frac{\partial \tilde{f}_1}{\partial e_1} & \cdots & \frac{r_1}{m+r_1}\frac{\partial \tilde{f}_1}{\partial e_n} \\ \vdots & \ddots & \vdots \\ \frac{r_n}{m+r_n}\frac{\partial \tilde{f}_n}{\partial e_1} & \cdots & \frac{r_n}{m+r_n}\frac{\partial \tilde{f}_n}{\partial e_n} \end{bmatrix} \tag{V.6}$$

and (x_a^*, e_a^*) and (x_b^*, e_b^*) are such that:

$$|H(x, e)| \leq |H(x_a^*, e_a^*)|, \qquad |G(x, e)| \leq |G(x_b^*, e_b^*)|$$

for all (x, e) in a neighbourhood Ω around the origin. So given this set Ω we can find where the norm of these matrices H and G attain its maximum values and then work with the linear model:

$$\dot{x} = H(x_a^*, e_a^*)x + G(x_b^*, e_b^*)e \tag{V.7}$$

It is important to emphasize that we are not trying to find a linearized model, as it would not guarantee stability for the original nonlinear system. To summarize, the computation of the self-triggered execution strategy is made in 4 steps:

1) Define an invariant set Ω around the equilibrium point, for instance a level set of the Lyapunov function.
2) Find H and G and compute the point(s) $\{x^*, e^*\} \in \Omega$ where $|H|$ and $|G|$ are maximized.
3) Find a stabilizing inter-execution time τ^* for the linear model (V.7). Among others, one possible technique to compute τ^* was described in our previous work [AT08b]. This time τ^* is a stabilizing sampling period of our original system for any initial condition lying in Ω.
4) Let Γ be the largest ball inside Ω, and let d be its radius. Relate the current state $x(t_i)$ with some point in the boundary[4] of Γ via homogeneous rays, that is, find $\mathbf{e}^{rs}$ (the dilation in space) such that $\mathbf{e}^{rs}(y) = x(t_i)$ for some y in the boundary of Γ. Since we are working with the standard dilation and since we have an estimate τ^* valid for any point in the boundary of Γ we can compute the next inter-execution time $\tau(x(t_i))$ of the control task by using (V.4):

$$\tau(x(t_i)) = \mathbf{e}^{-\rho(s)}\tau^* \tag{V.8}$$

As τ^* can be precomputed offline, the evaluation of (V.8) can be performed online in a very short time. It is important to notice that the self-triggered technique is trying to emulate the event-triggered condition defined in (III.6). In this context, the conservativeness of this approach (due

[3] See [AT08b] for one possible method to find such τ^*.

[4] Note that the boundary of Γ is an $n-1$ sphere.

to the mismatch between the self-triggered and the event-triggered policy) relies entirely on the accuracy of the period guaranteeing stability for the linear system (V.7). That is, no conservativeness is added when the scaling law is applied.

VI. EXAMPLE: JET ENGINE COMPRESSOR

To illustrate the previous results, we consider the control of a jet engine compressor. We borrow the following model from [KK95]:

$$\begin{aligned}\dot{\phi} &= -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \\ \dot{\psi} &= \frac{1}{\beta^2}(\phi - \phi_T)\end{aligned} \quad \text{(VI.1)}$$

where ϕ is the mass flow, β is a constant positive parameter, ψ is the pressure rise and ϕ_T corresponds to the throttle mass flow. In the model (VI.1) we already have translated the origin to the desired equilibrium point, hence the objective of the control system consists in steering both state variables (ϕ, ψ) to zero. A control law $\phi_T = g(\phi, \psi)$ is designed to render the closed loop globally asymptotically stable. The closed loop equations are:

$$\begin{aligned}\dot{\phi} &= -\frac{1}{2}(\phi^2+1)(\phi+y) \\ \dot{y} &= -(\phi^2+1)y\end{aligned} \quad \text{(VI.2)}$$

where we have applied the nonlinear change of coordinates $y = \frac{3\phi^2+2\psi-\phi}{\phi^2+1}$. An ISS Lyapunov function for this system can be found using SOStools [PPSP04]:

$$V = 1.46\phi^2 - 0.35\phi y + 1.16y^2$$

We also use SOStools to find bounds for the Lie derivative of V along the trajectories:

$$\dot{V} \leq -0.74 \cdot 10^8|x|^4 + 0.90 \cdot 10^8|x|^2|e|^2$$

where $x = (\phi, y)^T$. Hence we can guarantee stability if we satisfy the following inequality:

$$0.90|e|^2 \leq 0.74\sigma^2|x|^2$$

The closed loop system (VI.2) is homogeneous with respect to (V.1) for $m(x) = m(\phi, y) = \frac{2\phi^2}{\phi^2+1}$. For simplicity, we pick $r = 1$ in the dilation vector field (V.1). We select a value of $\sigma < 1$ guaranteeing stability under rule (III.6), for instance $\sigma = 0.33$. The operation region is a ball of radius 5 centered at the origin. Applying equation (V.8), we obtain the following formula describing the inter-execution times for the control task:

$$\tau\big(\phi(t_i), y(t_i)\big) = \frac{29\phi(t_i) + d^2}{5.36d\phi(t_i)^2 + d^2} \cdot \tau^* \quad \text{(VI.3)}$$

where d is the norm of the previously measured state, $d = \sqrt{\phi(t_i)^2 + y(t_i)^2}$, and $\tau^* = 7.63$ms for the selected value of σ. From this formula, we can see that times tend to enlarge as the system approaches the equilibrium point, since a linear term in d only appears in the denominator of (VI.3). In order to show the effectiveness of the approach, we consider 50 different initial conditions equally distributed along the boundary of the operation region. In Figures 1 and 2, we compare the behaviour of both strategies, periodic and self-triggered. To choose a stabilizing period for our system, we select the worst case inter-execution time obtained from (VI.3). Other possible way to compute a stabilizing period for a nonlinear system appeared in [Lai03]; both techniques lead to similar values for the period. The systems exhibit a similar behaviour for the state variables for any initial condition (see Figure 1 for one particular initial condition). Figure 2 shows the evolution of the input for the control system. At the beginning, both the periodic and self-triggered use the same inter-execution time, but as the system tends to the equilibrium point the self-triggered policy increases the time between executions, whereas the periodic policy keeps updating the controller at the same rate. The right side of Figure 2 zooms the last part of the simulation, where the inter-execution times for the self-triggered strategy is already 24 times larger than the periodic. Hence the self-triggered implementation leads to a much smaller number of executions, while achieving a similar performance. The number of executions required for both implementations are shown in Table I, for different values of σ (and averaged over all initial conditions considered): the self-triggered policy reduces the number of executions by a factor of 8, for a simulation time of 3s.

σ	periodic	self-triggered
0.11	890	123
0.22	506	68
0.33	397	53

TABLE I
NUMBER OF EXECUTIONS OF ONE CONTROL TASK FOR SIMULATION TIME OF 3S.

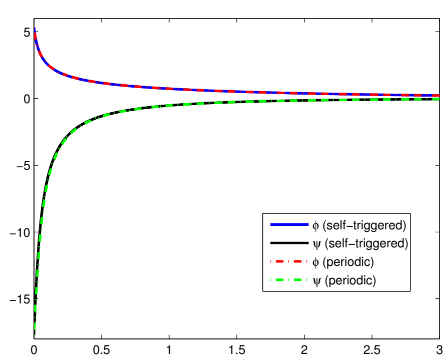

Fig. 1. Evolution of the states for self-triggered and periodic strategies

Finally, the self-triggered technique is robust with respect to disturbances, as expected because of the existence of an ISS Lyapunov function. Figure 3 shows the evolution of the states for both periodic and self-triggered strategies when sensor noise $n(t)$ is considered (noise power being 2% of the signal power). In addition, a disturbance $d(t)$ at the

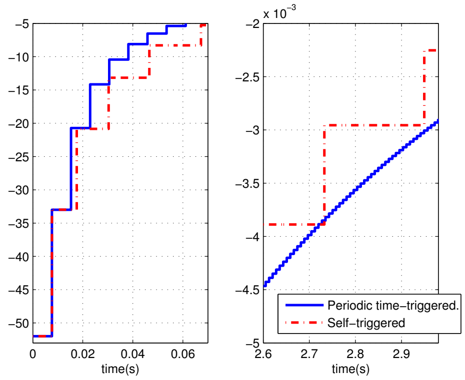

Fig. 2. Control input for periodic and self-triggered implementation

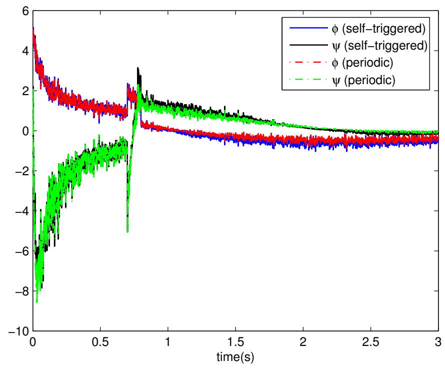

Fig. 3. Evolution of the states for self-triggered and periodic strategies in the presence of disturbances

actuator level is applied at $t = 0.7s$. Again, the self-triggered strategy achieves a similar rate of decay with a much smaller number of executions. Figure 4 depicts the evolution of the inter-execution time under the self-triggered policy with and without the disturbance. When no disturbance $d(t)$ is applied, times enlarge as the states approach the origin, and the inter-execution times at $t = 3$s rises above 210ms. For the second case, the disturbance steers the system far from the origin at $t = 0.7$s, and therefore the self-triggered task reduces the inter-execution times accordingly to guarantee the required performance. As the system reaches the equilibrium point, the inter-execution times start growing again since less executions are required to achieve the desired performance. Other possible real-time implementation effects that could degrade the performance of the system, such as jitter and input-output latency, are discussed in detail in [AT08a].

VII. Discussion

Throughout the paper, we have just considered the standard dilation vector field as defined in (V.1). Nevertheless, Theorem 4.6 holds for any vector field D such that $\dot{x} = -D(x)$ is globally asymptotically stable. Given a vector field X, D and m can be chosen so that X satisfies Definition 4.4. Hence, there exists a family of dilation vector fields D that generates a family of scaling laws in time, analogous to Theorem 5.1. However, since we are looking for self-triggered conditions to be applied online, only pairs

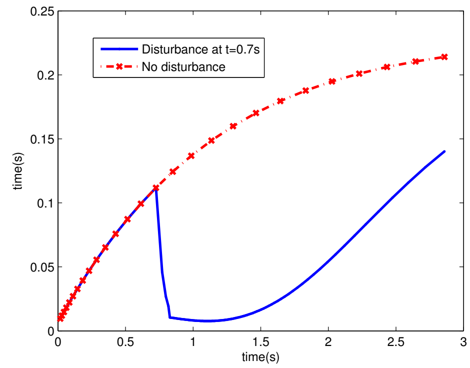

Fig. 4. Evolution of inter-execution times for the self-triggered implementation

(D,m) that lead to simple formulas of the type of (V.8) are currently being investigated by the authors.

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