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A Necessary and Sufficient Condition for Stabilization of Decentralized Time-Delay Systems with Commensurate Delays

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Abstract— **This paper investigates the stabilization problem for interconnected linear time-invariant (LTI) time-delay systems by means of linear time-invariant output feedback decentralized controllers. The delays are assumed to be commensurate and can appear in the states, inputs, and outputs of the system. First, the canonical forms for this type of time-delay systems are introduced and centralized fixed modes (CFM) for this type of systems are defined. It is then shown that a time-delay system which is both controllable and observable does not have any CFMs. Furthermore, an efficient technique for characterizing CFMs of any LTI time-delay system with commensurate delays is obtained. Decentralized fixed modes (DFM) are then defined accordingly, and a necessary and sufficient condition for decentralized stabilizability of the interconnected time-delay systems is proposed. Finally, a numerical example is given to illustrate the importance of results.**

I. INTRODUCTION

Design of a high-performance controller for interconnected systems is an important challenge in control theory [1], [2], [3]. Networked unmanned aerial vehicles (UAV), automated highway systems and automated manufacturing processes all involve multiple entities which are highly dynamic [4], [5]. These interacting subsystems are distributed in space and need to be coordinated with each other using sensing and communication networks. The representation of an interconnected system often involves high-order dynamics with several input and output channels. For such systems, it is typically not feasible to carry out all the control computations in one single point, and is desirable to have a distributed control scheme. By means of a distributed implementation, a more reliable control system is obtained which is less sensitive to failures and has less computational requirements [2]. In addition, distributed implementation of a high-performance centralized controller requires a high degree of connectivity among the subsystems. Therefore, since it is not realistic to assume that all output measurements can be transmitted to every local controller, there are some constraints on information exchange between different subsystems; i.e., full output observation is rarely possible.

A special case of a constrained control structure is the one with diagonal (or block-diagonal) information flow matrix, which is often referred to as a decentralized control structure. Each control station in this type of structure has only access to the measurements of its corresponding subsystem for generating the local control input [1]. All control stations are involved, however, in the overall control operation. An essential question in the study of decentralized control systems is that under what conditions a set of local feedback control laws exist to achieve stability and arbitrary pole-placement. The notion of a decentralized fixed mode (DFM) was introduced in [6] to address this question for finite-dimensional linear time-invariant (LTI) systems.

On the other hand, actuators, sensors, and communication networks in feedback control systems often introduce delays in dynamics. There are numerous physical applications in biology, chemistry, economics, high-speed communication networks, and robotics where the effect of delay cannot be neglected in control design and analysis [7], [8]. Time-delay systems have been studied extensively in the past few decades and several results have been reported in the literature (for example, see [8]). The dynamics of this type of systems are represented by a class of functional differential equations (FDE) which are infinite dimensional, as opposed to ordinary differential equations (ODE). Since neglecting the effect of delay in the model of the system can result in the degradation of the system performance, it is essential to have a sufficiently accurate model for delay in control design. For instance, the stability margin of the overall system can be highly sensitive to delay and a small variation in delay may lead to instability [9].

The stability analysis of time-delay systems has been a topic of longstanding interest, and LTI systems with commensurate delays, in particular, have been investigated intensively [8]. To study the stability of this class of time-delay systems, a two-variable criterion was introduced [10]. Further development of this technique led to a variety of stability tests for systems with commensurate delays, such as polynomial elimination and pseudo-delay methods [8]. Other important methodologies to analyze the stability of systems with commensurate delays include frequency sweeping tests [8]. It is worth mentioning that most of the existing results in this area have been developed for centralized control structure. This motivates the investigation of decentralized control systems with commensurate delays.

This paper deals with the problem of stabilizability of interconnected time-delay systems via decentralized controllers. A LTI interconnected system with commensurate delays in states, inputs and outputs is considered. It is shown that if all system delay operators are considered as elements

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of a properly defined ring of polynomials, the original delay-differential system representation can be converted into a ring model description. In addition, it is supposed that each local controller is desired to have a LTI output feedback form. Stabilizability conditions are obtained for the underlying system. To this end, the concept of controllability and observability are used to obtain a canonical state-space representation (namely the Kalman canonical form) for this class of time-delay systems. Next, the notion of centralized fixed modes (CFM) is introduced for this class of time-delay systems, and it is shown that a time-delay system which is both controllable and observable does not have any CFMs. This important result is then utilized to characterize the CFMs for any arbitrary LTI time-delay system with commensurate delays. Consequently, DFMs for this type of systems are defined in a similar manner. This notion is used to provide a necessary and sufficient condition for asymptotic stabilizability of the system with respect to the decentralized LTI controllers.

The remainder of the paper is organized as follows. In Section II, a convenient notation is given and the problem statement is introduced. The main results of the paper which are the stabilizability conditions for decentralized LTI time-delay systems are then presented in Section III. A numerical examples is provided in Section IV to illustrate the importance of the results. Finally, some concluding remarks are drawn in Section V.

II. Problem Formulation

A. Notation

- The set of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively.
- h is the delay, and λ is the delay operator; i.e. $\lambda f(t) = f(t-h)$ for a function of time t.
- $R[\lambda]$ denotes the ring of polynomials in λ with real coefficients, where λ is the delay operator.
- $A(\lambda) \in R^{m\times n}[\lambda]$ denotes the set of $m \times n$ matrices over $R[\lambda]$.
- For $A(\lambda) \in R^{n\times n}[\lambda]$ with degree k in λ, let $A(\lambda)x(t)$ be defined as follows

$$A(\lambda)x(t) = \sum_{j=0}^{k} A^j x(t-jh)$$

where $A^j \in \mathbb{R}^{n\times n}$ is a constant matrix for any $j \in \{0, 1, \ldots, k\}$.

B. Problem Statement

Consider the following interconnected LTI time-delay system with ν subsystems subject to commensurate delays [8]

$$\begin{aligned} \dot{x}(t) &= \sum_{j=1}^{k_1} A^j x(t-jh) + \sum_{i=1}^{\nu}\sum_{j=1}^{k_2} B_i^j u_i(t-jh) \\ y_i(t) &= \sum_{j=1}^{k_3} C_i^j x(t-jh), \qquad i = 1, 2, \ldots, \nu \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u_i(t) \in \mathbb{R}^{m_i}$ and $y_i(t) \in \mathbb{R}^{p_i}$ are the input and output of the ith local subsystem, respectively. The matrices $A^j \in \mathbb{R}^{n\times n}$, $B_i^j \in \mathbb{R}^{n\times m_i}$ and $C_i^j \in \mathbb{R}^{p_i\times n}$ are assumed to be real and constant. It is to be noted that in (1), commensurate delays can exist in input, state and output.

Using the λ-operator, the system (1) can be written as

$$\begin{aligned} \dot{x}(t) &= A(\lambda)x(t) + \sum_{i=1}^{\nu} B_i(\lambda)u_i(t) \\ y_i(t) &= C_i(\lambda)x(t), \qquad i = 1, 2, \ldots, \nu \end{aligned} \tag{2}$$

where $A(\lambda) \in R^{n\times n}[\lambda]$, $B_i(\lambda) \in R^{n\times m_i}[\lambda]$, and $C_i(\lambda) \in R^{p_i\times n}[\lambda]$. In the problem of decentralized control system design, the primary goal is to find ν local output controllers to stabilize the system. In this work, it is desired to design ν local stabilizing controllers of the following form

$$\begin{aligned} \dot{z}_i(t) &= \Gamma_i z_i(t) + R_i y_i(t) \\ u_i(t) &= Q_i z_i(t) + K_i y_i(t), \qquad i = 1, 2, \ldots, \nu \end{aligned} \tag{3}$$

where $z_i(t) \in \mathbb{R}^{\eta_i}$ is the state of the ith local controller. Γ_i, R_i, Q_i and K_i are the real constant matrices of appropriate size. The objective is to find a necessary and sufficient condition for the stabilizability of the interconnected system (1) under the decentralized output feedback of the form (3).

III. Main Results

A. Preliminaries

Definition 1: Consider the LTI time-delay interconnected system (1). Corresponding to $A(\lambda) \in R^{n\times n}[\lambda]$, $A(e^{-sh})$ is defined as

$$A(e^{-sh}) := A(\lambda)|_{\lambda=e^{-sh}} \tag{4}$$

It is straightforward to verify that

$$\mathcal{L}\{A(\lambda)x(t)\} = A(e^{-sh})X(s)$$

where $\mathcal{L}\{.\}$ denotes the Laplace transform operator, and $X(s)$ is the Laplace transform of $x(t)$.

Definition 2: Similar to $A(e^{-sh})$ and corresponding to the system (1), matrices $B_i(e^{-sh})$ and $C_i(e^{-sh})$ can also be defined, $i = 1, 2, \ldots, \nu$

$$B_i(e^{-sh}) := B_i(\lambda)|_{\lambda=e^{-sh}}, \quad C_i(e^{-sh}) := C_i(\lambda)|_{\lambda=e^{-sh}}$$

Furthermore, let $B(\lambda)$ and $C(\lambda)$ be constructed as follows

$$B(\lambda) = \left[\begin{array}{cccc} B_1(\lambda) & B_2(\lambda) & \ldots & B_\nu(\lambda) \end{array}\right] \tag{5}$$

$$C^T(\lambda) = \left[\begin{array}{cccc} {C_1}^T(\lambda) & {C_2}^T(\lambda) & \ldots & {C_\nu}^T(\lambda) \end{array}\right] \tag{6}$$

and define

$$B(e^{-sh}) := B(\lambda)|_{\lambda=e^{-sh}}, \quad C(e^{-sh}) := C(\lambda)|_{\lambda=e^{-sh}} \tag{7}$$

Remark 1: It is important to recognize that $A(e^{-sh})$, $B(e^{-sh})$ and $C(e^{-sh})$ are matrix quasi-polynomials of s. This property is very important in developing the main results of the paper.

Definition 3: Consider ν local controllers given in (3). Define the following matrices

$$\begin{aligned} \Gamma &:= \text{block diagonal}\left[\Gamma_1, \Gamma_2, \ldots, \Gamma_\nu\right], \\ R &:= \text{block diagonal}\left[R_1, R_2, \ldots, R_\nu\right], \\ Q &:= \text{block diagonal}\left[Q_1, Q_2, \ldots, Q_\nu\right], \\ K &:= \text{block diagonal}\left[K_1, K_2, \ldots, K_\nu\right] \end{aligned}$$

Define also

$$K^e = \begin{bmatrix} K & Q \\ R & \Gamma \end{bmatrix} \tag{8}$$

Lemma 1: The system (1) under the decentralized controller with the local dynamic compensators of (3) is asymptotically stable if and only if all roots of the quasi-polynomial

$$\det(sI - A^e(e^{-sh}) - B^e(e^{-sh})K^eC^e(e^{-sh}))$$

are located in the open left-half complex plane, where

$$\begin{aligned} A^e(e^{-sh}) &= \begin{bmatrix} A(e^{-sh}) & 0 \\ 0 & 0 \end{bmatrix}, \\ B^e(e^{-sh}) &= \begin{bmatrix} B(e^{-sh}) & 0 \\ 0 & I \end{bmatrix}, \\ C^e(e^{-sh}) &= \begin{bmatrix} C(e^{-sh}) & 0 \\ 0 & I \end{bmatrix} \end{aligned} \tag{9}$$

and $A(e^{-sh})$, $B(e^{-sh})$, $C(e^{-sh})$ are all given in Definitions 1 and 2.

Proof: Due to its similarity to [6], the proof is omitted here. For detail, see [11]. ■

B. Canonical Forms for LTI Time-Delay Systems with Commensurate Delays

In this subsection, the following LTI time-delay system with commensurate delays is considered

$$\begin{aligned} \dot{x}(t) &= A(\lambda)x(t) + B(\lambda)u(t) \\ y(t) &= C(\lambda)x(t) \end{aligned} \tag{10}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$. $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are matrices over $R[\lambda]$ with appropriate size. The transfer function for the system (10) is given by

$$G(s) = C(e^{-sh}) \left(sI - A(e^{-sh})\right)^{-1} B(e^{-sh})$$

where $A(e^{-sh})$, $B(e^{-sh})$ and $C(e^{-sh})$ are given by (4)-(7). Controllability and observability of the time-delay systems will be defined next.

Definition 4: In this work, the system (10) is called controllable if the matrix

$$\begin{bmatrix} B(\lambda) & A(\lambda)B(\lambda) & \cdots & (A(\lambda))^{n-1} B(\lambda) \end{bmatrix} \tag{11}$$

is full-rank for all $\lambda \in \mathbb{C}$ [12].

Definition 5: In this work, the system (10) is called observable if the matrix

$$\left[C^T(\lambda) \; A^T(\lambda)C^T(\lambda) \; \cdots \; \left(A^T(\lambda)\right)^{n-1} C^T(\lambda)\right] \tag{12}$$

is full-rank for all $\lambda \in \mathbb{C}$ [12].

Definition 6: $T(\lambda) \in R^{n\times n}(\lambda)$ is defined to be unimodular if $(T(\lambda))^{-1} \in R^{n\times n}(\lambda)$. For the sake of simplicity, the inverse of $T(\lambda)$ is denoted by $T^{-1}(\lambda)$ in the sequel.

Lemma 2: Suppose that the rank of the controllability matrix (11) for the system (10) is n_1, where $n_1 < n$. Also, let $n_2 := n - n_1$. Then,

1) There exists a unimodular matrix $T(\lambda)$ such that the triple $\left(\bar{C}(\lambda), \bar{A}(\lambda), \bar{B}(\lambda)\right)$ defined as

$$\left(C(\lambda)T(\lambda), T^{-1}(\lambda)A(\lambda)T(\lambda), T^{-1}(\lambda)B(\lambda)\right)$$

has the following form

$$\begin{aligned} &\bar{A}(\lambda) = \begin{bmatrix} \bar{A}_1(\lambda) & \bar{A}_{12}(\lambda) \\ 0 & \bar{A}_2(\lambda) \end{bmatrix}, \quad \bar{B}(\lambda) = \begin{bmatrix} \bar{B}_1(\lambda) \\ 0 \end{bmatrix}, \\ &\bar{C}(\lambda) = \begin{bmatrix} \bar{C}_1(\lambda) & \bar{C}_2(\lambda) \end{bmatrix} \end{aligned} \tag{13}$$

where $\bar{A}_1(\lambda) \in R^{n_1\times n_1}[\lambda]$, $\bar{A}_2(\lambda) \in R^{n_2\times n_2}[\lambda]$, $\bar{B}_1(\lambda) \in R^{n_1\times m}[\lambda]$, $\bar{C}_1(\lambda) \in R^{p\times n_1}[\lambda]$, $\bar{C}_2(\lambda) \in R^{p\times n_2}[\lambda]$, and the pair $\left(\bar{A}_1(\lambda), \bar{B}_1(\lambda)\right)$ is controllable.

2) The transfer function matrix is given by

$$G(s) = \bar{C}_1(e^{-sh}) \left(sI - \bar{A}_1(e^{-sh})\right)^{-1} \bar{B}_1(e^{-sh})$$

where $\bar{C}_1(e^{-sh})$, $\bar{A}_1(e^{-sh})$, and $\bar{B}_1(e^{-sh})$ are obtained from $\bar{C}_1(\lambda)$, $\bar{A}_1(\lambda)$, and $\bar{B}_1(\lambda)$, respectively, by a simple substitution similar to Definitions 1 and 2.

Proof: The first part of Lemma is proven in [12], and the proof of the second part is straightforward [13]. ■

Remark 2: The triple $\left(\bar{C}_1(\lambda), \bar{A}_1(\lambda), \bar{B}_1(\lambda)\right)$ can be viewed as the controllable component of the system (10).

Lemma 3: Suppose that the rank of the observability matrix (12) for the system (10) is n_1, where $n_1 < n$. Also, let $n_2 := n - n_1$. Then,

1) There exists a unimodular matrix $T(\lambda)$ such that the triple $\left(\bar{C}(\lambda), \bar{A}(\lambda), \bar{B}(\lambda)\right)$, defined as

$$\left(C(\lambda)T(\lambda), T^{-1}(\lambda)A(\lambda)T(\lambda), T^{-1}(\lambda)B(\lambda)\right)$$

has the following form

$$\begin{aligned} &\bar{A}(\lambda) = \begin{bmatrix} \bar{A}_1(\lambda) & 0 \\ \bar{A}_{21}(\lambda) & \bar{A}_2(\lambda) \end{bmatrix}, \quad \bar{B}(\lambda) = \begin{bmatrix} \bar{B}_1(\lambda) \\ \bar{B}_2(\lambda) \end{bmatrix}, \\ &\bar{C}(\lambda) = \begin{bmatrix} \bar{C}_1(\lambda) & 0 \end{bmatrix} \end{aligned}$$

where $\bar{A}_1(\lambda) \in R^{n_1\times n_1}[\lambda]$, $\bar{A}_2(\lambda) \in R^{n_2\times n_2}[\lambda]$, $\bar{B}_1(\lambda) \in R^{n_1\times m}[\lambda]$, $\bar{B}_2(\lambda) \in R^{n_2\times m}[\lambda]$, $\bar{C}_1(\lambda) \in R^{p\times n_1}[\lambda]$, and the pair $\left(\bar{C}_1(\lambda), \bar{A}_1(\lambda)\right)$ is observable.

2) The transfer function matrix is given by

$$G(s) = \bar{C}_1(e^{-sh}) \left(sI - \bar{A}_1(e^{-sh})\right)^{-1} \bar{B}_1(e^{-sh})$$

where $\bar{C}_1(e^{-sh})$, $\bar{A}_1(e^{-sh})$, and $\bar{B}_1(e^{-sh})$ are obtained from $\bar{C}_1(\lambda)$, $\bar{A}_1(\lambda)$, and $\bar{B}_1(\lambda)$, respectively, by a simple substitution similar to Definitions 1 and 2.

Proof: The first part of Lemma is proven in [12], and the proof of second part is straightforward [13]. ■

Theorem 1: Consider the system (10). Assume that

- The rank of the controllability matrix corresponding to the pair $(A(\lambda), B(\lambda))$ is $n_1 < n$. Let $n_2 := n - n_1$.
- The rank of the observability matrix corresponding to the pair $\left(\bar{C}_1(\lambda), \bar{A}_1(\lambda)\right)$ is $n_{11} < n_1$, where $\left(\bar{C}_1(\lambda), \bar{A}_1(\lambda), \bar{B}_1(\lambda)\right)$ is the controllable component of the system (10). Also, let $n_{12} := n_1 - n_{11}$.

Then,

1) There exists a unimodular matrix $\tilde{T}(\lambda)$ such that the triple $\left(\tilde{C}(\lambda), \tilde{A}(\lambda), \tilde{B}(\lambda)\right)$ defined as

$$\left(C(\lambda)\tilde{T}(\lambda), \tilde{T}^{-1}(\lambda)A(\lambda)\tilde{T}(\lambda), \tilde{T}^{-1}(\lambda)B(\lambda)\right)$$

has the following form

$$\tilde{A}(\lambda) = \begin{bmatrix} \tilde{A}_{11}(\lambda) & 0 & \tilde{A}_{13}(\lambda) \\ \tilde{A}_{21}(\lambda) & \tilde{A}_{22}(\lambda) & \tilde{A}_{23}(\lambda) \\ 0 & 0 & \tilde{A}_{33}(\lambda) \end{bmatrix}$$
$$\tilde{B}(\lambda) = \begin{bmatrix} \tilde{B}_1(\lambda) \\ \tilde{B}_2(\lambda) \\ 0 \end{bmatrix}, \quad \tilde{C}(\lambda) = \begin{bmatrix} \tilde{C}_1(\lambda) & 0 & \tilde{C}_2(\lambda) \end{bmatrix} \tag{14}$$

where $\tilde{A}_{11}(\lambda) \in R^{n_{11}\times n_{11}}[\lambda]$, $\tilde{A}_{22}(\lambda) \in R^{n_{12}\times n_{12}}[\lambda]$, $\tilde{A}_{33}(\lambda) \in R^{n_2\times n_2}[\lambda]$, $\tilde{B}_1(\lambda) \in R^{n_{11}\times m}[\lambda]$, $\tilde{B}_2(\lambda) \in R^{n_{12}\times m}[\lambda]$, $\tilde{C}_1(\lambda) \in R^{p\times n_{11}}[\lambda]$, $\tilde{C}_2(\lambda) \in R^{p\times n_2}[\lambda]$, and the triple $\left(\tilde{C}_1(\lambda), \tilde{A}_{11}(\lambda), \tilde{B}_1(\lambda)\right)$ is both controllable and observable.

2) The transfer function matrix is given by

$$G(s) = \tilde{C}_1(e^{-sh}) \left(sI - \tilde{A}_{11}(e^{-sh})\right)^{-1} \tilde{B}_1(e^{-sh})$$

where $\tilde{C}_1(e^{-sh})$, $\tilde{A}_{11}(e^{-sh})$, and $\tilde{B}_1(e^{-sh})$ are obtained from $\tilde{C}_1(\lambda)$, $\tilde{A}_{11}(\lambda)$, and $\tilde{B}_1(\lambda)$, respectively, by a simple substitution similar to Definitions 1 and 2.

Proof: Using Lemmas 2 and 3, the Theorem can be proven similar to the non-delay case [13]. For detail, see [11]. ■

Remark 3: The triple $\left(\tilde{C}(\lambda), \tilde{A}(\lambda), \tilde{B}(\lambda)\right)$ will be referred to as the Kalman canonical equivalence of the original system $(C(\lambda), A(\lambda), B(\lambda))$ (analogously to the non-delay case).

C. Centralized Fixed Modes for LTI Time-Delay Systems with Commensurate Delays

Let K_c denote the set of all $m \times p$ matrices with real entries. The following definition is essential in the presentation of the main results of the paper.

Definition 7: Consider the system (10), and let $K \in \mathbb{R}^{m\times p}$ be a constant matrix. For a constant $\mu \in \mathbb{R}$, the set of μ-centralized fixed modes of the system (10), denoted by $\Lambda_\mu (C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c)$, is defined as follows

$$\Lambda_\mu (C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c) = \{s | s \in \mathbb{C}, \mathrm{Re}\{s\} \geq \mu, \phi(s) = 0, \ \forall K \in \mathrm{K}_c\}$$

where

$$\phi(s) = \det\left(sI - A(e^{-sh}) - B(e^{-sh})KC(e^{-sh})\right)$$

Lemma 4: Assume that the triple $(C(\lambda), A(\lambda), B(\lambda))$ in (10) is both controllable and observable. Then,

$$\Lambda_\mu (C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c) = \emptyset$$

for any finite $\mu \in \mathbb{R}$.

Sketch of Proof: From Definition 7, it is implied that

$$\Lambda_\mu (C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c) \subseteq \Lambda_\mu (A(\lambda))$$

where $\Lambda_\mu (A(\lambda))$ denotes the set of all $s \in \mathbb{C}$ which are the roots of $\det\left(sI - A(e^{-sh})\right) = 0$ and $\mathrm{Re}\{s\} \geq \mu$. It is well-known that $\Lambda_\mu (A(\lambda))$ is a finite set [14]; denote this finite set with $\Lambda_\mu (A(\lambda)) = \{s_1, s_2, \ldots, s_q\}$.

Consider an arbitrary $s_i \in \Lambda_\mu (A(\lambda))$, $i = 1, 2, \ldots, q$, and define

$$\rho_i(K) = \det\left(s_iI - A(e^{-s_ih}) - B(e^{-s_ih})KC(e^{-s_ih})\right)$$

as a $(m \times p)$-variable polynomial in entries of K. In the sequel, it is shown that $\rho_i(K)$ cannot be identically zero. Suppose that

$$\rho_i(K) \equiv 0$$

Since $A(e^{-s_ih})$, $B(e^{-s_ih})$, $C(e^{-s_ih})$ can be treated as constant matrices, it can be shown in a manner similar to the techniques used in [15] that one of the following statements must hold

- rank $\begin{bmatrix} s_iI - A(e^{-s_ih}) & B(e^{-s_ih}) \end{bmatrix} < n$;
- rank $\begin{bmatrix} C^T(e^{-s_ih}) & s_iI - A^T(e^{-s_ih}) \end{bmatrix} < n$.

This means that the triple $(C(\lambda), A(\lambda), B(\lambda))$ is either uncontrollable or unobservable [16], which contradicts the assumption of the theorem. Therefore, there exists a (nonzero) $K_i^0 \in \mathbb{R}^{m\times p}$ for which $\rho_i(K_i^0) \neq 0$. This means that

$$s_i \notin \Lambda_\mu (C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c)$$

This completes the proof. ■

Theorem 2: Suppose that $\left(\tilde{C}(\lambda), \tilde{A}(\lambda), \tilde{B}(\lambda)\right)$ is the corresponding Kalman canonical form for the triple $(C(\lambda), A(\lambda), B(\lambda))$. Then,

$$\Lambda_\mu (C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c) = \{s | s \in \mathbb{C}, \mathrm{Re}\{s\} \geq \mu, \phi(s) = 0, \ \forall K \in \mathrm{K}_c\}$$

where

$$\phi(s) = \prod_{i=2}^{3} \det\left(sI - \tilde{A}_{ii}(e^{-sh})\right)$$

and $\tilde{A}_{22}(e^{-sh})$, $\tilde{A}_{33}(e^{-sh})$ are obtained from $\tilde{A}_{22}(\lambda)$, $\tilde{A}_{33}(\lambda)$, respectively, similar to Definitions 1 and 2.

Proof: Consider $\phi(s)$ in Definition 7. It can be shown that

$$\phi(s) = \det\left(sI - \tilde{A}_{11}(e^{-sh}) - \tilde{B}_1(e^{-sh})K\tilde{C}_1(e^{-sh})\right) \times \prod_{i=2}^{i=3} \det\left(sI - \tilde{A}_{ii}(e^{-sh})\right)$$

From Theorem 1, it is known that $\left(\tilde{C}_1(\lambda), \tilde{A}_{11}(\lambda), \tilde{B}_1(\lambda)\right)$ is both controllable and observable. On the other hand, according to Lemma 4, there is no finite $s \in \mathbb{C}$ such that

$$\det\left(sI - \tilde{A}_{11}(e^{-sh}) - \tilde{B}_1(e^{-sh})K\tilde{C}_1(e^{-sh})\right) = 0$$

for any $K \in \mathrm{K}_c$. This results that s belongs to $\Lambda_\mu (C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_c)$ if and only if it is a root of $\prod_{i=2}^{i=3} \det\left(sI - \tilde{A}_{ii}(e^{-sh})\right)$. ■

D. Decentralized Fixed Modes for LTI Time-Delay Systems with Commensurate Delays

Definition 8: Consider the system (1) and let K_d denote the set of all block diagonal matrices given below

$$\mathrm{K}_d = \{K | K = \text{block diagonal} \left[K_1, K_2, \ldots, K_\nu\right], K_i \in \mathbb{R}^{m_i\times p_i}, i = 1, 2, \ldots, \nu\} \tag{15}$$

For a constant $\mu \in \mathbb{R}$, the set of μ-decentralized fixed modes of the system (1), denoted by $\Lambda_\mu(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d)$, is defined as follows

$$\Lambda_\mu(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d) = \{s | s \in \mathbb{C}, \mathrm{Re}\{s\} \geq \mu, \phi(s) = 0,\ \forall K \in \mathrm{K}_d\}$$

where

$$\phi(s) = \det\left(sI - A(e^{-sh}) - B(e^{-sh})KC(e^{-sh})\right)$$

Lemma 5: Consider the system (1) and define

$$A^e(\lambda) = \begin{bmatrix} A(\lambda) & 0 \\ 0 & 0 \end{bmatrix}, \quad B^e(\lambda) = \begin{bmatrix} B(\lambda) & 0 \\ 0 & I \end{bmatrix}, \quad C^e(\lambda) = \begin{bmatrix} C(\lambda) & 0 \\ 0 & I \end{bmatrix}$$

Denote with K_d^e the set of all $(m+p)\times(m+p)$ real constant matrices shown in (8). Then, for any given set of integers $\eta_1 \geq 0, \ldots, \eta_\nu \geq 0$ and any finite $\mu \in \mathbb{R}$

$$\Lambda_\mu(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d) \subseteq \Lambda_\mu(C^e(\lambda), A^e(\lambda), B^e(\lambda), \mathrm{K}_d^e) \tag{16}$$

Proof: The proof is carried out for the special case of $\eta_1 = 1$ and $\eta_i = 0$, $i = 2, \ldots, \nu$; the general case can be easily followed from induction. The matrix K_d^e has the same form as the matrix given in (8), i.e.

$$K_d^e = \begin{bmatrix} K_1 & & \bigcirc & & q_1 \\ & K_2 & & & 0 \\ & & \ddots & & \vdots \\ & \bigcirc & & K_\nu & 0 \\ r_1 & 0 & \cdots & 0 & \gamma_1 \end{bmatrix}$$

In addition, let K be defined as

$$K = \text{block diagonal}\,[K_1, K_2, \ldots, K_\nu]$$

It is easy to verify that for any $K \in \mathrm{K}_d$

$$\Lambda_\mu(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d) = \Lambda_\mu(C(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B(\lambda), \mathrm{K}_d)$$

Similar to non-delay case [6], it can be shown that

$$\Lambda_\mu(C(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B(\lambda), \mathrm{K}_d) \subseteq \Lambda_\mu(C_1(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B_1(\lambda), \mathrm{K}_{c_1})$$

where $\mathrm{K}_{c_1} = \mathbb{R}^{m_1 \times p_1}$. Thus, one can conclude that

$$\Lambda_\mu(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d) \subseteq \Lambda_\mu(C_1(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B_1(\lambda), \mathrm{K}_{c_1}) \tag{17}$$

For an arbitrary $K \in \mathrm{K}_d$, consider the triple

$$(C_1(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B_1(\lambda))$$

Using Theorem 1, there exists a unimodular matrix $T(\lambda) \in R^{n\times n}[\lambda]$ that transforms the state-space model to the Kalman canonical form given below

$$T^{-1}(\lambda)\left(A(\lambda) + B(\lambda)KC(\lambda)\right)T(\lambda) = \begin{bmatrix} \tilde{A}_{11}(\lambda) & 0 & \tilde{A}_{13}(\lambda) \\ \tilde{A}_{21}(\lambda) & \tilde{A}_{22}(\lambda) & \tilde{A}_{23}(\lambda) \\ 0 & 0 & \tilde{A}_{33}(\lambda) \end{bmatrix}$$

and

$$T^{-1}(\lambda)B(\lambda) = \begin{bmatrix} \tilde{B}_1(\lambda) \\ \tilde{B}_2(\lambda) \\ 0 \end{bmatrix},$$

$$\tilde{C}(\lambda)T(\lambda) = \begin{bmatrix} \tilde{C}_1(\lambda) & 0 & \tilde{C}_2(\lambda) \end{bmatrix}$$

It results from Theorem 2 that

$$\Lambda_\mu(C_1(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B_1(\lambda), \mathrm{K}_{c_1}) = \{s | s \in \mathbb{C}, \mathrm{Re}\{s\} \geq \mu, \phi(s) = 0,\ \forall K^* \in \mathrm{K}_{c_1}\} \tag{18}$$

where

$$\phi(s) = \det\left(sI - A(e^{-sh}) - B(e^{-sh})KC(e^{-sh}) - B_1(e^{-sh})K^*C_1(e^{-sh})\right) = \prod_{i=2}^{i=3} \det\left(sI - \tilde{A}_{ii}(e^{-sh})\right)$$

On the other hand,

$$\Lambda_\mu(C^e(\lambda), A^e(\lambda), B^e(\lambda), \mathrm{K}_d^e) = \{s | s \in \mathbb{C}, \mathrm{Re}\{s\} \geq \mu, \psi(s) = 0,\ \forall K_d^e \in \mathrm{K}_d^e\} \tag{19}$$

where

$$\psi(s) = \det(sI - A^e(e^{-sh}) - B^e(e^{-sh})K_d^eC^e(e^{-sh}))$$

One can show that

$$\psi(s) = \left[(s-\gamma_1) - r_1C_1(e^{-sh}) \times \left(sI - A(e^{-sh}) - B(e^{-sh})KC(e^{-sh})\right)^{-1} B_1(e^{-sh})q_1\right] \times \det\left(sI - A(e^{-sh}) - B(e^{-sh})KC(e^{-sh})\right)$$

Using Theorem 1, $\psi(s)$ can be rewritten as

$$\psi(s) = \Big\{(s-\gamma_1) \times \det\left(sI - \tilde{A}_{11}(e^{-sh})\right) - r_1C_1(e^{-sh}) \left[\mathrm{adj}\left(sI - \tilde{A}_{11}(e^{-sh})\right)\right] B_1(e^{-sh})q_1\Big\} \times \prod_{i=2}^{3} \det\left(sI - \tilde{A}_{ii}(e^{-sh})\right)$$

Therefore, for any $K \in \mathrm{K}_d$, and for all $q_1 \in \mathbb{R}^{m_1\times 1}$, $r_1 \in \mathbb{R}^{1\times p_1}$, $\gamma_1 \in \mathbb{R}$, any root of $\phi(s)$ will be a root of $\psi(s)$ as well. Thus, along with (17), (18), and (19), (16) is obtained. ■

Theorem 3: A necessary condition for the existence of an asymptotically stabilizing decentralized controller for the system (1) with the local dynamic compensator given by (3) is that

$$\Lambda_0(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d) = \emptyset$$

Proof: Assume that

$$\Lambda_0(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d) \neq \emptyset$$

Then, there exists a $s_0 \in \mathbb{C}$ such that $\mathrm{Re}\{s_0\} \geq 0$ and

$$s_0 \in \Lambda_0(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d)$$

From Lemma 5, one can conclude that

$$s_0 \in \Lambda_\mu \left(C^e(\lambda), A^e(\lambda), B^e(\lambda), \mathrm{K}_d^e\right)$$

According to Definition 8, for any $K_d^e \in \mathrm{K}_d^e$

$$\det(s_0 I - A^e(e^{-s_0 h}) - B^e(e^{-s_0 h}) K_d^e C^e(e^{-s_0 h})) = 0$$

Using Lemma 1, one can infer that there is no asymptotically stabilizing decentralized controller for the system (1) with the local dynamic compensator given by (3). This completes the proof. ■

In the remainder of this section, it is shown that the condition given by Theorem 3 is not only necessary but also sufficient. To prove that, Lemmas 6, 7 and 8 will be needed.

Lemma 6: Given $A(\lambda) \in R^{n\times n}[\lambda]$ with degree k in λ, let

$$\phi(s) := \det\left(sI - A(e^{-sh})\right)$$

Furthermore, let $s = a + \mathbf{i}b$ be an arbitrary root of $\phi(s)$, where a and b are real numbers and $\mathbf{i}^2 = -1$. Then,

- a is not an arbitrarily large positive number ($a \neq +\infty$);
- if a is finite (i.e., if $a \neq -\infty$), then b is finite as well.

Proof: The characteristic equation $\phi(s)$ has the following form

$$\phi(s) = \zeta_0(s) + \sum_{l=1}^{l_f} \zeta_l(s) e^{-lhs}$$

where $l_f := k^n$, $\zeta_0(s)$ is a monotone polynomial of degree n, and the functions $\zeta_l(s)$, $l = 1, 2, \ldots, l_f$, are polynomials of degree at most $n-1$ [8]. Since $\phi(s)$ has a principal term, a cannot be arbitrarily large positive number [8], and hence $a \neq +\infty$. On the other hand, if s in (20) is replaced by $a + \mathbf{i}b$, two equations (in terms of a and b) will be obtained, which correspond to the real and imaginary parts of (20). Both of these equation can be expressed as a combination of polynomials, exponentials, sinusoidals and their products. More specifically, one of these two equations (depending on whether n is even or odd), can be written as

$$P_0(a,b) + \sum_{l=1}^{l_f} P_l^1(a,b) e^{-lha} sin(lhb) + \sum_{l=1}^{l_f} P_l^2(a,b) e^{-lha} cos(lhb) = 0 \quad (20)$$

where $P_0(a,b)$ is a polynomial of degree n with respect to b, and the functions $P_l^1(a,b)$ and $P_l^2(a,b)$, $l = 1, 2, \ldots, l_f$, are polynomials of degree at most $n-1$ with respect to b. Now, let a be a fixed finite number and assume that b goes to infinity. In this case, one can verify that the left side of (20) will go to infinity as well. Therefore, $a \pm \mathbf{i}\infty$ cannot be a root of $\phi(s)$. This completes the proof. ■

Lemma 7: Let the arbitrary positive real scalar σ_0 and complex scalar s_0 be given. Define the disk $\mathcal{D}(s_0, \sigma_0)$ as

$$\mathcal{D}(s_0, \sigma_0) = \{s | s \in \mathbb{C}, |s - s_0| < \sigma_0\}$$

Consider the system (1) and the set K_d of block diagonal matrices defined in (15). For any $K \in \mathrm{K}_d$, define

$$\phi(s, K) := \det\left(sI - A(e^{-sh}) - B(e^{-sh}) K C(e^{-sh})\right) \quad (21)$$

Define also $\bar{\mathcal{D}}(s_0, \sigma_0)$ as the boundary of the disk $\mathcal{D}(s_0, \sigma_0)$; i.e.

$$\bar{\mathcal{D}}(s_0, \sigma_0) := \{s | s \in \mathbb{C}, |s - s_0| = \sigma_0\}$$

If $\phi(s, 0)$ is nonzero on $\bar{\mathcal{D}}(s_0, \sigma_0)$, there exists a positive γ such that for all $K \in \mathrm{K}_d$ with $\|K\| < \gamma$, the number of roots of $\phi(s, K)$ and $\phi(s, 0)$ inside $\mathcal{D}(s_0, \sigma_0)$ are the same, where $\|.\|$ denotes any induced norm.

Proof: Since $\phi(s, 0)$ is nonzero on $\bar{\mathcal{D}}(s_0, \sigma_0)$, one can find $\eta > 0$ such that $|\phi(s, 0)| \geq \eta$ for all $s \in \bar{\mathcal{D}}(s_0, \sigma_0)$. On the other hand, $\phi(s, K)$ can be written in the following form

$$\phi(s, K) = \xi_0(s, K) + \sum_{l=1}^{l_f} \xi_l(s, K) e^{-lhs} \quad (22)$$

where

$$\xi_0(s, K) = \sum_{\tau=0}^{n} a_\tau(K) s^\tau$$
$$\xi_l(s, K) = \sum_{\tau=0}^{n_l} b_{\tau,l}(K) s^\tau \quad (23)$$

In the above equations, $a_\tau(K)$ and $b_{\tau,l}(K)$, are polynomials in $k_i(\alpha, \beta)$; i.e., the (α, β) element of the matrix K_i, $i = 1, 2, \ldots, \nu$, $\alpha = 1, 2, \ldots, m_i$ and $\beta = 1, 2, \ldots, p_i$. One can conclude from (22) and (23) that

$$|\phi(s, K) - \phi(s, 0)| \leq \sum_{\tau=0}^{n} |a_\tau(K) - a_\tau(0)||s|^\tau + \sum_{l=1}^{lf} |e^{-lhs}| \sum_{\tau=0}^{n_l} |b_{\tau,l}(K) - b_{\tau,l}(0)||s|^\tau$$

Furthermore, if $|s - s_0| = \sigma_0$, then $|s| \leq |s_0| + \sigma_0$. Therefore, for $s \in \bar{\mathcal{D}}(s_0, \sigma_0)$,

$$|\phi(s, K) - \phi(s, 0)| \leq \sum_{\tau=0}^{n} |a_\tau(K) - a_\tau(0)|(|s_0| + \sigma_0)^\tau + \sum_{l=1}^{lf} e^{lh(|s_0|+\sigma_0)} \sum_{\tau=0}^{n_l} |b_{\tau,l}(K) - b_{\tau,l}(0)|(|s_0| + \sigma_0)^\tau \quad (24)$$

Since $a_\tau(K)$ and $b_{\tau,l}(K)$ are continuous functions of $k_i(\alpha, \beta)$, thus there exists a $\gamma > 0$ such that if $\|K\| < \gamma$, then

$$|a_\tau(K) - a_\tau(0)| < \frac{\eta}{(n+1)(l_f+1)(|s_0| + \sigma_0)^\tau}$$
$$|b_{\tau,l}(K) - b_{\tau,l}(0)| < \frac{\eta e^{-lh(|s_0|+\sigma_0)}}{(l_f+1)(n_l+1)(|s_0| + \sigma_0)^\tau} \quad (25)$$

Consequently, from (24) and (25) it can be deduced that for any K with $\|K\| < \gamma$

$$|\phi(s, K) - \phi(s, 0)| < \eta \leq |\phi(s, 0)|, \quad \forall s \in \bar{\mathcal{D}}(s_0, \sigma_0)$$

The proof follows directly from Rouché Theorem [17]. ■

Lemma 8: Consider the system (1) and the set K_d of block diagonal matrices defined in (15), and the characteristics equation $\phi(s, K)$, $K \in \mathrm{K}_d$, defined in (21). Let s_j, $j \in \mathbb{N}$, denote roots of $\phi(s, 0)$, and assume that the set of closed right-half plane roots of $\phi(s,0)$ (referred to as unstable roots, hereafter) is represented by $\{s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_\beta}\}$. If

$$\Lambda_0\left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d\right) = \emptyset,$$

then

1) There exists a positive γ such that for all $K \in \mathrm{K}_d$ with $\|K\| < \gamma$, the number of unstable roots of $\phi(s,K)$ is not greater than the number of unstable roots of $\phi(s,0)$.
2) For any $\xi > 0$, there exists a $\hat{K} \in \mathrm{K}_d$ with $\|K\| < \xi$ such that $\phi(s_j, \hat{K}) \neq 0$ for all $j \in \{\alpha_1, \alpha_2, \dots, \alpha_\beta\}$.

Proof: Given an arbitrary $\epsilon > 0$, define Θ_ϵ as

$$\Theta_\epsilon = \{s | s \in \mathbb{C}, \mathrm{Re}\{s\} > -\epsilon\}$$

Since the roots of $\phi(s,0)$ are separated in the right side of any line parallel to imaginary axis [14], one can find ϵ^* such that Θ_{ϵ^*} does not include any stable poles of $\phi(s,0)$. Furthermore, consider the disk $\mathcal{D}(\rho-\epsilon^*, \rho)$, which is centered at $\rho - \epsilon^*$ in the complex plane and has radius ρ. It is easy to show that the point $-\epsilon^*$ lies on the boundary of $\mathcal{D}(\rho-\epsilon^*, \rho)$. In addition,

$$\lim_{\rho\to\infty} \mathcal{D}(\rho-\epsilon^*, \rho) = \Theta_{\epsilon^*}$$

From Lemma 6, it can be deduced that there exists a ρ^* such that for any $\rho > \rho^*$, all the unstable roots of $\phi(s,0)$ are placed in $\mathcal{D}(\rho-\epsilon^*, \rho)$. In this case, $\phi(s,0)$ is nonzero over the boundary of $\mathcal{D}(\rho-\epsilon^*, \rho)$ for any $\rho > \rho^*$. In addition, according to Lemma 7, there exists a γ such that the number of roots of $\phi(s,K)$ in the disk $\mathcal{D}(\rho-\epsilon^*, \rho)$, for all $K \in \mathrm{K}_d$ with $\|K\| < \gamma$, is equal to the number of roots of $\phi(s,0)$ in the same disk if $\rho > \rho^*$. This implies the first statement of the lemma.

In order to prove the second part, define the following set for all $j \in \{\alpha_1, \alpha_2, \dots, \alpha_\beta\}$

$$\Pi_j := \{K | K \in \mathrm{K}_d \text{ and } \phi(s_j, K) = 0\}$$

Since s_j, $j \in \{\alpha_1, \alpha_2, \dots, \alpha_\beta\}$ is not a DFM, $\phi(s_j, K)$ is a non-constant polynomial in K. Thus, Π_j is a hyper-surface in the parameter space of K (for the definition of hyper-surface, see [18]). Moreover, in any non-empty open set of the parameter space of K, there exists a $\hat{K}$ such that $\hat{K} \notin \bigcup_j \Pi_j$. This completes the proof of the second part of the lemma. ■

Theorem 4: Consider the system (1) with its equivalent model given by (2). Let K_d be the set of block diagonal matrices defined in (15). If

$$\Lambda_0\left(C(\lambda), A(\lambda), B(\lambda), \mathrm{K}_d\right) = \emptyset,$$

then there exists a decentralized controller of the form (3) to asymptotically stabilize the system.

Proof: Using Lemmas 4, 8, and the fact that a dynamic output feedback to place an observable and controllable mode in the left half-hand complex plane can always be found for time-delay systems [19], the Theorem can be proven similar to the non-delay case [6]. For detail, see [11]. ■

Remark 4: Although in the development of the main results it is assumed that the delay of the system is known and fixed, one can apply the results to nominal model of a system with uncertain delay. If the corresponding model has an unstable DFM for nominal model, one can conclude that there is no robust decentralized LTI controller to stabilize the system.

IV. An Illustrative Numerical Example

Example: Consider an interconnected system $\mathcal{S}$ consisting of two subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$. Let the respective state-space representation of subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ be given by

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} -4 & 7 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} z_1(t) \\
&+ \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-2h) \\ x_2(t-h) \end{bmatrix} \\
&+ \begin{bmatrix} 6 \\ 1 \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1(t-h) \\
y_1(t) &= 8x_1(t) - 6x_2(t) - 2x_2(t-h) + w_1(t)
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} z_2(t) \\
&+ \begin{bmatrix} 0 \\ 2 \end{bmatrix} x_4(t-2h) \\
&+ \begin{bmatrix} 0 \\ -1 \end{bmatrix} u_2(t) + \begin{bmatrix} 0 \\ -4 \end{bmatrix} u_2(t-2h) \\
y_2(t) &= -2x_3(t) + x_4(t) - 2x_4(t-2h) + w_2(t)
\end{aligned}$$

where $u_i(t) \in \mathbb{R}$ and $y_i(t) \in \mathbb{R}$ are the local input and output corresponding to $\mathcal{S}_i$, for $i = 1, 2$. In addition, $[x_1^T \ x_2^T]^T$ and $[x_3^T \ x_4^T]^T$ are the state vectors of the subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. The signals $z_1(t)$ and $z_2(t)$ are the incoming interconnection signals of the subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, and are assumed to be as follows

$$\begin{aligned}
z_1(t) &= \frac{1}{3}x_4(t) - x_4(t-h) \\
z_2(t) &= -4x_1(t) + 3x_2(t) + x_2(t-h)
\end{aligned}$$

The signals $w_1(t)$ and $w_2(t)$ represent the direct effect of the state of one subsystem on the output of the other subsystem, and are considered to be

$$\begin{aligned}
w_1(t) &= x_3(t) - e^2 x_3(t-2h) \\
w_2(t) &= x_2(t) - e x_2(t-h)
\end{aligned}$$

Using λ-operator, the state space model for the interconnected system $\mathcal{S}$ can be written as

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -4-\lambda^2 & 7+\lambda & 0 & -1-3\lambda \\ -1-\lambda^2 & 5 & 0 & -1-3\lambda \\ -4 & 3+\lambda & 1 & 0 \\ 0 & 0 & 0 & -2-2\lambda \end{bmatrix} x(t) \\
&+ \begin{bmatrix} 6 & 0 \\ 1+\lambda & 0 \\ 0 & 0 \\ 0 & -1-4\lambda^2 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 8 & -6-2\lambda & 1-e^2\lambda^2 & 0 \\ 0 & 1-e\lambda & -2 & 1-2\lambda^2 \end{bmatrix} x(t)
\end{aligned}$$

where

$$x = \begin{bmatrix} x_1{}^T & x_2{}^T & x_3{}^T & x_4{}^T \end{bmatrix}^T$$
$$u = \begin{bmatrix} u_1{}^T & u_2{}^T \end{bmatrix}^T$$
$$y = \begin{bmatrix} y_1{}^T & y_2{}^T \end{bmatrix}^T$$

One can easily verify that $s = 1$ is a mode of the system $\mathcal{S}$ for all $h \geq 0$. Assume initially that $h = 0$ (finite-dimensional case). In this case, denote the controllability and observability matrices of the system $\mathcal{S}$ with M_{c0} and M_{o0}, respectively. It is easy to show that

$$\text{rank } M_{c0} = 4, \qquad \text{rank } M_{o0} = 4$$

Hence, for $h = 0$, the system $\mathcal{S}$ is both controllable and observable, which implies that it does not have any CFM. Furthermore, it can be verified that in this case the system $\mathcal{S}$ does not have any DFM either [1]. Thus, the modes of the system $\mathcal{S}$, including $s = 1$, can be placed arbitrarily in the complex plane using both centralized and decentralized output feedback controllers.

Now, assume that $h = 1$. It can be verified in this case that $s = 1$ is a controllable and observable mode using the criteria provided by [14]. Therefore, according to Lemma 4 this mode of the system is not a μ-centralized fixed mode, for any finite $\mu \in \mathbb{R}$, and a static output feedback $u(t) = Ky(t)$, $K \in \mathbb{R}^{2\times 2}$ can displace this mode of the system. Next, it is aimed to investigate if there exists a decentralized LTI finite-dimensional output feedback controller to stabilize the system. Consider the following static decentralized output feedback

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

In this case, the matrix $sI - A(e^{-s}) - B(e^{-s})KC(e^{-s})$ is

$$\begin{bmatrix} \phi_{1,1}(s,K) & \phi_{1,2}(s,K) & 0 & 1+3e^{-s} \\ \phi_{2,1}(s,K) & \phi_{2,2}(s,K) & 0 & 1+3e^{-s} \\ 4 & -3-e^{-s} & s-1 & 0 \\ 0 & 0 & \phi_{4,3}(s,K) & \phi_{4,4}(s,K) \end{bmatrix}$$

where

$$\phi_{1,1}(s,K) = s + 2 + e^{-2s} - 48k_1$$
$$\phi_{1,2}(s,K) = -7 - e^{-s} - 6k_1(-6 - 2e^{-s})$$
$$\phi_{2,1}(s,K) = 1 + e^{-2s} - 8k_1e^{-s} - 8k_1$$
$$\phi_{2,2}(s,K) = s - 5 - (1 + e^{-s})k_1(-6 - 2e^{-s})$$
$$\phi_{4,3}(s,K) = 2(1 - 4e^{-2s})k_2$$
$$\phi_{4,4}(s,K) = s + 2 - 2e^{-s} - (-1 - 4e^{-2s})k_2(1 - 2e^{-2s})$$

It can be shown using *Symbolic Math Toolbox* that for $s = 1$, $\det(sI - A(e^{-s}) - B(e^{-s})KC(e^{-s})$ is zero for any 2×2 diagonal matrix K. Thus, it can be concluded that $s = 1$ is an unstable DFM for the underlying system, and as a result (from Theorem 3) there is no LTI finite-dimensional decentralized output feedback controller to stabilize the system.

V. Conclusions

The problem of stabilization of linear time-invariant (LTI) time-delay interconnected systems using decentralized LTI output feedback control is investigated in this work. It is assumed that the system is subject to the input/output and state commensurate delays. The notion of decentralized fixed modes (DFM) introduced in [6] is extended to the underlying class of time-delay systems with known fixed delays, and a necessary and sufficient condition is obtained for the stabilizability of this type of systems, under decentralized LTI controllers. The existing results on decentralized stabilization of LTI time-delay systems provide sufficient conditions only; this substantiates the importance of the results presented in this work. The numerical example elucidates significance of the results.

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