Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

On the Jordan structure of the spectral-zero dynamics in multivariable analytic interpolation

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Abstract— **The parametrization of solutions to scalar interpolation problems with a degree constraint relies on the concept of spectral-zeros –these are the poles of the inverse of a corresponding spectral factor. In fact, under a certain degree constraint, the spectral-zeros are free (modulo a stability requirement) and parameterize all solutions. The subject of this paper is the multivariable analog of a Nehari-like analytic interpolation with a degree constraint. Our main result is based on Rosenbrock's pole assignability theorem and addresses the freedom in assigning the Jordan structure of the spectral-zero dynamics.**

I. Introduction

A complete parametrization of bounded degree solutions for scalar analytic interpolation problems has been obtained in terms of the zeros of certain corresponding spectral factors [1]–[5]. These are referred to as the spectral-zeros. For scalar problems the dynamics of the inverse spectral factors are completely determined by the spectral-zeros. However, in the case of multivariable interpolation problems nontrivial (i.e., noncyclic) Jordan structures are possible. The purpose of this work is to study the freedom in specifying the invariant factors for the inverse spectrum. We utilize Rosenbrock's theorem on assignability of dynamics via linear state feedback to characterize interpolants and their zero dynamics.

In this paper, we consider analytic interpolation with $m \times m$ *matrix-valued, positive-real functions*. Very much as in H_∞-control the dimension of the interpolants relates to the complexity of a model, a filter, or a controller. The complexity of the interpolation is characterized by the number of interpolation conditions, n. In the multivariable case where the interpolation conditions are constraints along different directions, *n is the rank of a corresponding Pick matrix* (see [6]), or equivalently of a Hankel operator. While the standard approach [7], based on linear fractional transformations, describes the complete solution set as a function-ball around a central interpolant, it gives no insight as to possible minimal-degree solutions.

Historically, over the last few decades there has been effort to describe minimal-degree solutions for analytic interpolation problems (see e.g., [1], [2], [8], [9]), with significant developments over the past 10 years [3], [4], [10]. These studies led to a complete parametrization of bounded-degree interpolants for the most general Sarason-type analytic interpolation in the scalar case [5]. Although attention to analogous results in matrix-valued interpolation with bounded McMillan degree has already been drawn more than 20 years ago in [1], progress has been slow. We mention some recent nice work by Blomqvist et al. [11].

The main contribution of the current paper is to characterize a family of solutions to multivariable analytic interpolation with degree constraints. This generalizes scalar results where a parametrization of solutions is given in terms of admissible spectral-zeros. This is no longer the case in multivariable problems where the dynamics relate to noncyclic Jordan structures, in general. Thus, a variety of solutions may have zero-dynamics that relate to the same Jordan form. The generically minimal degree of solutions for multivariable problems is $n-m$ (n being the rank of a Pick matrix and $m \times m$ the size of interpolants). We utilize a formalism based on a multivariable moment problem in [12] that describes a class of solutions of degree $n-m$ in a convenient factored form; the real part of interpolants gives rise to a matrix-valued spectral density function that solves the moment problem. We then utilize Rosenbrock's theorem on pole assignability by state feedback to specify all Jordan structures that are possible for the inverse of corresponding spectral factors. We will discuss a number of questions that pertain to the complete parametrization of solutions that remain open.

II. Notation and preliminaries

We denote by $\mathcal{C}$ the set of square matrix-valued functions which are analytic in $\mathbb{D}$, the open unit disk of the complex plane, and have positive Hermitian-part. There is a natural correspondence between elements in $\mathcal{C}$ and the class of finite Hermitian positive matrix-valued measures on $(-\pi, \pi]$ denoted by $\mathcal{M}$. In fact, analytic interpolation constraints on functions in $\mathcal{C}$ can be cast as moment constraints on corresponding measures in $\mathcal{M}$ [13], [14]. We now make the connection between the two as our results in this paper draw heavily on recent developments in the multivariable moment problem [12].

A. Analytic Interpolation

We consider the problem of parameterizing all functions $F \in \mathcal{C}$ of size $m \times m$ which satisfy

$$F(z) = F_{\mathrm{o}}(z) + Q(z)V(z), \quad \text{for } z \in \mathbb{D}, \qquad \text{(1a)}$$

with known $F_{\mathrm{o}}(z), V(z)$, and the parameter $Q(z)$ all being square matrix-functions. Equation (1a) represents an interpolation condition. Indeed, along directions where $V(z)$ vanishes $F(z) - F_{\mathrm{o}}(z)$ vanishes as well, and thus, $F(z)$

This work was supported in part by the National Science Foundation and the Vincentine Hermes-Luh endowment.

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978-1-4244-3124-3/08/$25.00 ©2008 IEEE

interpolates $F_{\mathrm{o}}(z)$. The matrix-function $Q(z)$ is required to be analytic in $\mathbb{D}$, and thereby, without loss of generality, we can always assume $V(z)$ is inner (all-pass), i.e.,

$$V(z)^*V(z) = V(z)V(z)^* = I, \quad \text{for all } |z| = 1, \tag{1b}$$

where "$*$" denotes "conjugate-transpose". In cases where $V(z)$ is a scalar multiple of the identity, $F(z)$ possesses the same value as $F_{\mathrm{o}}(z)$ at the roots of $V(z)$, whereas in general with an arbitrary inner $V(z)$, constraint (1a) is in the category of tangential interpolation (see [13]).

We are interested in finite dimensional interpolation problems where $V(z)$ is a rational function of McMillan degree n. Therefore, we write

$$F_{\mathrm{o}}(z) = H(I - zA)^{-1}B, \quad \text{and} \tag{1c}$$

$$V(z) = D + Cz(I - zA)^{-1}B, \tag{1d}$$

where $A \in \mathbb{R}^{n\times n}$, B full column rank in $\mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{m\times n}$, (A, B) is a reachable pair, and the eigenvalues of A lie in $\mathbb{D}$. Throughout the paper, "I" denotes the identity matrix of compatible size. Moreover, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$ are suitably chosen so that (C, A) is an observable pair and $V(z)$ satisfies (1b). The completion of (A, B) into an inner matrix-valued function $V(z)$ is well known and is part of the bounded real lemma, cf. [15].

Equations (1a-d), which are standing assumptions throughout, form the standard Nehari-type problem with the triplet of matrices (A, B, H) as the interpolation data. It turns out that a positive Hermitian-part solution of (1a-d) exists if and only if the Lyapunov equation

$$R - ARA^* = BH + H^*B^* \tag{2}$$

has a positive semi-definite solution for R, which is thought of as the corresponding Pick matrix (see e.g., [6], [16]). In cases where R is strictly positive definite there exist infinitely many solutions. Our interest in this paper is to identify the family of interpolants, $F(z)$'s, which satisfy (1a-d) and have low McMillan degree. We now outline the correspondence between interpolation problem (1a-d) and a related moment problem.

B. Connection with Moment Problem

Consider the linear discrete-time dynamical system

$$x_k = Ax_{k-1} + Bu_k, \quad \text{for } k \in \mathbb{Z}, \tag{3}$$

where $x_k \in \mathbb{R}^n$, and $u_k \in \mathbb{R}^m$ are the state and the input vectors, respectively. The input-to-state transfer function of this system is

$$G(z) := (I - zA)^{-1}B,$$

where "z" stands for the transform of the delay operator, and thus, "stability" of $G(z)$ corresponds to "analyticity in $\mathbb{D}$". Let the input u_k be a stationary zero-mean random process with the matrix-valued density function $\Phi_{uu}(\theta)$, $\theta \in (-\pi, \pi]$, where $\Phi_{uu}(\theta)d\theta \in \mathcal{M}$. Then, the state covariance matrix of (3) is in the form of the following integral

$$E\{x_k x_k^*\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})\Phi_{uu}(\theta)G(e^{j\theta})^* \, d\theta.$$

The classical moment problem amounts to the (inverse) problem of finding an input spectral density function $\Phi_{uu}(\theta)$ which is consistent with a given state covariance matrix $E\{x_k x_k^*\}$. It turns out that a non-negative definite matrix R admits such a representation, i.e.,

$$R = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})\Phi_{uu}(\theta)G(e^{j\theta})^* \, d\theta, \tag{4}$$

and hence qualifies as a state covariance of (3) if and only if the following equivalent conditions hold (see [6], [16]):

$$\operatorname{rank}\begin{bmatrix} R - ARA^* & B \\ B^* & 0 \end{bmatrix} = 2m, \tag{5a}$$

$$R \text{ satisfies (2) for a choice of } H. \tag{5b}$$

Further, consistent spectral densities, Φ_{uu}'s, for the moment problem (4) are in a bijective correspondence with solutions to the interpolation problem (1a-d). More precisely, corresponding to any positive-real matrix-function $F(z)$ which satisfies (1a-d) for a given triplet of (A, B, H), there exists a solution to the moment problem (4) with the triplet of data (A, B, R) such that

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1 + ze^{j\theta}}{1 - ze^{j\theta}}\right)\Phi_{uu}(\theta)d\theta + jc, \tag{6}$$

where jc is an arbitrary skew-Hermitian constant. This is the content of Riesz-Herglotz's theorem [17]. Conversely, any density function of more general measure that satisfies (4) originates as the boundary limit of the real part of a positive-real solution to (1a-d), i.e.,

$$\Phi_{uu}(\theta) = \lim_{r\nearrow 1} \Re e\, F(re^{j\theta}). \tag{7}$$

We wish to emphasize that H (interpolation data) and R (moment data) relate to each other via (2).

C. Solutions to the general moment problem

Reference [12], building on earlier works, e.g., [3]–[5], characterized all positive solutions to the moment problem (4) as minimizers of suitably weighted relative entropy functionals. More specifically, let $\Psi(\theta)d\theta$ and $\Phi(\theta)d\theta$ belong to $\mathcal{M}$ and define the relative entropy functional

$$\mathbf{S}(\Psi\|\Phi) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{trace}(\Psi\log\Psi - \Psi\log\Phi)d\theta.$$

Then, for a given admissible (A, B, R), i.e., such that (5a-b) hold, all positive solutions to the moment problem (4) can be obtained as minimizers of $\mathbf{S}(\Psi\|\Phi_{uu})$ for a choice of Ψ, i.e., they are of the form

$$\operatorname*{argmin}_{\Phi_{uu}}\left\{\mathbf{S}(\Psi\|\Phi_{uu}) : R = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})\Phi_{uu}(\theta)G(e^{j\theta})^* \, d\theta\right\}, \tag{8}$$

with Ψ a free parameter. Minimizers of this optimization problem are shown to be in the form of

$$\Phi_{uu}(\theta) = \sigma(e^{j\theta})\left(G(e^{j\theta})^*\lambda G(e^{j\theta})\right)^{-1}\sigma(e^{j\theta})^*,$$

with λ, a Hermitian matrix of Lagrange multipliers, and σ, a matrix-valued spectral factor of Ψ, i.e., $\sigma\sigma^* := \Psi$. Given σ, the value of λ can be obtained numerically via a continuation method (see [12]).

For the special choice of $\sigma(e^{j\theta}) = I + Ke^{j\theta}G(e^{j\theta})$, selected so that $A - BK$ is Hurwitz, we obtain a solution of the form

$$\Phi_{uu}(\theta) = \left(G_{\rm o}(e^{j\theta})^* \lambda G_{\rm o}(e^{j\theta})\right)^{-1}, \quad \text{with} \tag{9a}$$

$$G_{\rm o}(e^{j\theta}) = \left(I - e^{j\theta}(A - BK)\right)^{-1} B, \tag{9b}$$

where $\Phi_{uu}(\theta)$ in (9a-b) is a rational spectral density function of degree at most $2n$, and thereby, corresponds (via (6)) to a positive-real $F(e^{j\theta})$ of McMillan degree at most n. This F is, indeed, the solution of the corresponding interpolation problem (1a-d) for the triplet of data (A, B, H). In this paper, we identify and characterize a subclass of an even lower McMillan degree. In fact, the minimal McMillan degree which is generically feasible is $n - m$, and we show that for a choice of K we can generate a family of solutions of degree $n - m$.

III. Jordan structure of spectral-zero dynamics

In this section, we use Rosenbrock's theorem [18] to highlight the freedom in assigning the invariant factors of the inverse dynamics of interpolants in the multivariable setting. Earlier results for the scalar case (see e.g., [2]–[4]) suggest that the spectral-zeros can be arbitrarily assigned. However, the associated Jordan structure of these spectral-zeros is always cyclic. In the matricial interpolation this is not the case and the spectral-zero dynamics relate to nontrivial (i.e., noncyclic) Jordan structures as well.

A. Rosenbrock's theorem

We first recall the notion of invariant polynomials. Let $\Pi_t(z)$ for $t = 1, 2, \ldots, r$ denote the greatest common divisor of all the minors of order t of $zI - A$. Then each polynomial in the series

$$\Pi_r(z), \Pi_{r-1}(z), \ldots, \Pi_1(z), \Pi_0(z) \equiv 1$$

is divisible by the succeeding one and the quotients

$$p_1(z) = \frac{\Pi_r(z)}{\Pi_{r-1}(z)}, p_2(z) = \frac{\Pi_{r-1}(z)}{\Pi_{r-2}(z)}, \ldots, p_r(z) = \frac{\Pi_1(z)}{\Pi_0(z)}$$

are the *invariant polynomials* of A (see e.g., [19], [20]).

Next, we need the notion of controllability indices. Let (A, B) be a controllable pair, assume that B has full column rank, and consider the ordered set of vectors

$$b_1, \ldots, b_m, Ab_1, \ldots, Ab_m, A^2b_1, \ldots, A^2b_m, \ldots,$$

where b_i is the ith column of B. Following Popov ([21]), $A^k b_j$ is an "antecedent" of $A^\mu b_\nu$ if it is listed earlier in the above ordered list (i.e., if $km + j < \mu m + \nu$). We denote by κ_i the smallest positive integer for which $A^{\kappa_i} b_i$ is a linear combination of its antecedents. Then, κ_i's for $i \in \{1, 2, \ldots, m\}$ are the *controllability indices* of the pair (A, B). They are also known as *Kronecker invariants* and their sum is equal to n, the dimension of the system. These indices are invariant under state feedback, similarity transformation, and invertible linear transformation on the columns of B.

Rosenbrock's theorem on pole assignment: Consider the controllable pair (A, B) (as before) with controllability indices κ_i's in decreasing order for $i = 1, 2, \ldots, m$. Let $\{p_i(z);\ i = 1, 2, \ldots, m\}$ be any set of polynomials which satisfy

$$p_i(z) \mid p_{i-1}(z),$$

i.e., p_i divides p_{i-1}, and $\sum_{i=1}^m \deg p_i(z) = n$. Then the conditions

$$\sum_{i=1}^{j} \deg\ p_i(z) \geq \sum_{i=1}^{j} \kappa_i$$

for $j = 1, 2, \ldots, m$ are necessary and sufficient for the existence of linear map $K : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that $\{p_i(z)\}$ is the set of invariant polynomials of $A - BK$.

This theorem was proven in [18, page 190] (see also [22] and [23], and an elegant constructive proof based on a geometric argument by Flamm in [24]).

B. Spectral-zero dynamics assignability in multivariable interpolation

The (generically) minimal-degree solutions for scalar interpolation have been fully parameterized (see e.g., [2]–[5]) by an arbitrary choice of "spectral-zeros". These spectral-zeros are in fact the zeros of a meromorphic extension of the real part of the interpolating function. Naturally, in the matricial case, there are more degrees of freedom. Herein, we use the notion of "spectral-zero dynamics" which helps draw analogous results in multivariable interpolation. For any $F(z) \in \mathcal{C}$, denote by $W_{\rm o}$ the outer spectral factor of the inverse of the real part of F, i.e.,

$$\left(\Re e\, F(e^{j\theta})\right)^{-1} = W_{\rm o}(e^{j\theta})^* W_{\rm o}(e^{j\theta}), \tag{10}$$

with $W_{\rm o}(z) =: D_{\rm o} + C_{\rm o} z \left(I - zA_{\rm z}\right)^{-1} B_{\rm o}$, a minimal realization of $W_{\rm o}(z)$. We call $A_{\rm z}$ the spectral-zero dynamics of $F(z)$ as it determines the pole-structure of $\left(\Re e\, F(e^{j\theta})\right)^{-1}$. Theorem 1, below, characterizes a family of minimal-degree solutions to multivariable interpolation that correspond to choices of spectral-zero dynamics.

Theorem 1: Consider data (A, B, H) for multivariable interpolation problem defined by (1a-d) and assume that (2) admits a positive definite solution R, i.e., that the problem is solvable. Let κ_i's for $i = 1, 2, \ldots, m$ denote the controllability indices of (A, B) in decreasing order. Then, corresponding to any $A_{\rm z} \in \mathbb{R}^{(n-m)\times(n-m)}$ which is Hurwitz and whose invariant polynomials $\{p_i(z);\ i = 1, 2, \ldots, m\}$ satisfy

$$\sum_{i=1}^{j} \deg\ p_i(z) + j \geq \sum_{i=1}^{j} \kappa_i, \quad \text{for } j = 1, \ldots, m, \tag{11}$$

there exists an interpolant $F(z)$ of McMillan degree at most $n - m$ whose spectral-zero dynamics has the Jordan structure as $A_{\rm z}$.

Corollary 1: Consider the interpolation problem (1a-d) with a corresponding $R \succ 0$ and let A_z be a Hurwitx matrix. If any of the following conditions holds:

1) A_z is cyclic,
2) m divides n and all the controllability indices of (A, B) are equal, i.e., $\kappa_1 = \ldots = \kappa_m = \frac{n}{m}$,
3) m does not divide n and the controllability indices of (A, B) are as follows
$$\kappa_j = \begin{cases} \lfloor \frac{n}{m} \rfloor + 1 & \text{for } j = 1, \ldots, \text{mod}(n, m) \\ \lfloor \frac{n}{m} \rfloor & \text{for } j = \text{mod}(n, m) + 1, \ldots, m \end{cases},$$
where $\lfloor \frac{n}{m} \rfloor$ stands for "the integer part of $\frac{n}{m}$", and "mod(n, m)" gives the remainder of division n/m,

there exists an interpolant $F(z)$ of McMillan degree at most $n - m$ whose spectral-zero dynamics has the Jordan structure as A_z.

The first condition implies that $\deg\ p_1(z) = n - m$, while $p_2 = \ldots = p_m = 1$ and thus Theorem 1 applies. The other two can be argued in a similar way. An example where the second condition holds is that of the trigonometric moment problem in which

$$A = \begin{bmatrix} O & O & \cdots & O & O \\ I & O & \cdots & O & O \\ & \ddots & \ddots & \vdots & \vdots \\ \\ O & O & & I & O \end{bmatrix}, \quad B = \begin{bmatrix} I \\ O \\ \vdots \\ \\ O \end{bmatrix}, \tag{12}$$

with I and O the identity and the zero matrices of size $m \times m$, A an $(l+1) \times (l+1)$ and B an $(l+1) \times 1$ block matrices, respectively. The size of each block is $m \times m$, and hence, the actual size of A, B are $n \times n$ and $n \times m$, with $n = (l+1)m$. These correspond to a block-Topeltiz matrix

$$R = \begin{bmatrix} R_0 & R_1 & \cdots & R_l \\ R_{-1} & R_0 & \cdots & R_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{-l} & R_{-l+1} & \cdots & R_0 \end{bmatrix}, \tag{13}$$

where the size of each R_i is $m \times m$ and

$$F(z) = \frac{1}{2} R_0 + R_1 z + \cdots + R_l z^l + o(z^l).$$

For this case of (A, B) all the controllability indices are equal to $\frac{n}{m} = l + 1$. Section V discusses further the trigonometric moment problem.

It should be noted that in general there may not exist solutions $F(z)$ of degree less than $n - m$. In fact, very much as in the scalar case where $m = 1$, both the data set of admissible triplets (A, B, R) for which a solution of degree less than $n - m$ is possible, as well as the complement where $n - m$ is the minimal degree, have a nonempty interior, i.e., they are both generic conditions (see e.g. [1, pages 50, & 80–83]). This is due to the semi-algebraic nature of the underlying problem.

Note also that the set of conditions (11) are only sufficient for the existence of degree $n - m$ solutions corresponding to the Jordan structure of a given A_z. This will be taken up in the next section. The proof of Theorem 1 requires a couple of preceding steps, given as two separate lemmas.

Lemma 1: Let $F(z)$ be a rational function of McMillan degree $n - m$, which is strictly positive-real with $\Re e\, F(z)$ uniformly bounded in $\mathbb{D}$. Then, there exists $\hat{F}(z)$ which is also strictly positive-real and of McMillan degree $n - m$ such that $\Re e\, \hat{F}(e^{j\theta}) = \left(\Re e\, F(e^{j\theta})\right)^{-1}$.

Proof: The function $\hat{F}$ in the lemma is the analytic part of $(\Re e\, F)^{-1}$. The McMillan degree of $\hat{F}$ is the same as F and this follows by comparing the degrees of the corresponding spectral factors of their real parts. □

The above fact sets a bijection $F \leftrightarrow \hat{F}$ between functions in this class.

Lemma 2: Let $A, B, G_o(z)$ be as in (9b), let $\Delta(K, \lambda) := G_o^* \lambda G_o$ for $\lambda \in \mathbb{R}^{n \times n}$, and let K satisfy:

(i) $A - BK$ is Hurwitz,
(ii) rank$(A - BK) = n - m$.

Then, there exists a rational matrix-valued function $\hat{F}(z)$ of degree $n - m$ such that $\Delta(K, \lambda) = \Re e\, \hat{F}$.

Proof: We write $\Delta(K, \lambda)$ as the two-sided series:

$$\begin{aligned} B^* \left(I - z^{-1}(A - BK)^*\right)^{-1} \lambda \left(I - z(A - BK)\right)^{-1} B = \\ \cdots + z^{-2} B^* A_o^{*2} \Lambda B + z^{-1} B^* A_o^* \Lambda B + B^* \Lambda B + \\ z B^* \Lambda A_o B + z^2 B^* \Lambda A_o^2 B + \cdots, \end{aligned} \tag{14}$$

with $A_o := A - BK$ and Λ the solution of the Lyapunov equation $\Lambda - A_o^* \Lambda A_o = \lambda$. It readily follows from (14) that $\Delta(K, \lambda) = \Re e\, \hat{F}$ with

$$\hat{F}(z) = \frac{1}{2} B^* \Lambda B + B^* \Lambda A_o z (I - z A_o)^{-1} B. \tag{15}$$

Since rank$(A_o) = n - m$, the rank of the observability matrix of $\hat{F}(z)$ cannot exceed $n - m$, hence neither can the McMillan degree of $\hat{F}(z)$. □

Remark 1: The two lemmas establish a mapping $(K, \lambda) \mapsto \hat{F} \mapsto F$. Therefore, the construction of interpolants begins with a choice of K that satisfies the conditions in Lemma 2. We then solve the optimization problem (8) which gives rise to the desired λ, and hence Λ. This Λ via (15) generates the matrix-valued function $\hat{F}$. Finally, the interpolant F is the analytic part of $(\Re e\, \hat{F})^{-1}$.

Proof of Theorem 1: For any spectral-zero dynamics A_z with invariant polynomials $\{p_i(z);\ i = 1, 2, \ldots, m\}$ that satisfy (11), the sequence of polynomials $z^i p_i(z)$ satisfies the condition of Rosenbrock's theorem for the pair (A, B). Thus there exists a K such that $A - BK$ has $\{z^i p_i(z);\ i = 1, 2, \ldots, m\}$ as its invariant polynomials, and hence

$$A - BK \overset{\text{similar}}{\sim} \begin{bmatrix} A_z & 0 \\ Y & 0 \end{bmatrix}, \tag{16}$$

with Y as a matrix of compatible size. This choice of K gives rise to a matricial power spectral density $\Phi_{uu}(\theta)$ as in (9a-b). Then, from Lemma 2 we conclude that there exists a rational function $\hat{F}(z)$ of McMillan degree $n - m$ such that $\Phi_{uu}(\theta)^{-1} = \Re e\, \hat{F}(e^{j\theta})$. Positivity of its Hermitian part on the circle ensures that $\hat{F}(z)$ is a positive-real function. Finally, application of Lemma 1 implies that there exists a positive-real function $F(z)$ of McMillan degree $n - m$ such that $\Phi_{uu}(\theta) = \Re e\, F(e^{j\theta})$. Indeed, this $F(z)$ is a solution to multivariable interpolation (1a-d) and this proves the claim of the theorem. □

IV. Alternative solutions of degree $n-m$

In this part, we show that for a given triplet of data (A, B, H) in the multivariable interpolation where $m > 1$, there may exist interpolants of McMillan degree $n-m$ with spectral-zero dynamics which do not satisfy inequalities (11). Indeed, the condition rank$(A - BK) = n - m$ that played a central role in the earlier section is only sufficient and not necessary in order to obtain solutions of McMillan degree $n-m$. In other words, such a rank condition, and hence the drop of the degree, originate from the rank deficiency of the observability matrix of $\hat{F}(z)$ (see the proof of Lemma 2). In cases where the rank of this observability matrix equals $n-m$ and rank$(A - BK) > n - m$, there may be more degrees of freedom in assigning spectral-zero dynamics of the interpolants so that instead of (16) we have

$$A - BK \overset{\text{similar}}{\sim} \begin{bmatrix} A_z & 0 \\ Y_1 & Y_2 \end{bmatrix}. \quad (17)$$

Here, Y_1 and Y_2 are free matrices of compatible size, Y_2 is unobservable dynamics, and A_z, whose invariant polynomials do not need to satisfy (11), is the only block which contributes in spectral-zero dynamics of the solution. We illustrate this point with an example (see [25], [26] for details).

Example 1: Consider the interpolation data

$$A = \begin{bmatrix} \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$H = \begin{bmatrix} 0.0012 & -0.3351 & 0.2507 & 0 \\ 0 & 0 & 0 & 0.2507 \end{bmatrix}, \quad (18)$$

where $n = 4, m = 2$. It is shown that for this triplet of (A, B, H), there exists a solution $F(z)$ of McMillan degree $n - m = 2$ corresponding to a zero dynamics which does not fall in the class specified by Theorem 1 (i.e., does not satisfy (11)).

We solve the problem by following the path described in Remark 1. In particular, we take $K = 0$ and this choice results in

$$\hat{F}(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{z}{1-\frac{1}{2}z} & 0 \\ 0 & \frac{z}{1-\frac{1}{2}z} \end{bmatrix}.$$

Although in this example rank$(A - BK) = 4 \neq n - m$, the resulting $\hat{F}(z)$ is of McMillan degree 2 with the minimal realization

$$A_{\hat{F}} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad B_{\hat{F}} = C_{\hat{F}} = D_{\hat{F}} = I_2. \quad (19)$$

Hence, $F(z)$ is of McMillan degree 2 as well.

We now show that the resulting $A_{\hat{F}}$ does not satisfy the set of conditions (11) in Theorem 1. Note that the invariant polynomials of the desired spectral-zero dynamics, $A_{\hat{F}}$ in (19), are $p_1 = p_2 = z - \frac{1}{2}$. On the other hand, controllability indices of the given (A, B), are easily obtained as $\kappa_1 = 3$ and $\kappa_2 = 1$. Clearly, this set of $\{p_1, p_2, \kappa_1, \kappa_2\}$ violates the inequalities in (11). In fact, application of Rosenbrock's theorem (in which the set of conditions is both necessary and sufficient) implies that there is no value for K such that similarity in (16) holds for any choice of Y. Therefore, given the set of interpolation data (18), interpolants with spectral-zero dynamics similar to $A_{\hat{F}}$ in (19) cannot be obtained via Theorem 1. □

To sum up, the first conclusion to be drawn in multivariable interpolation is that corresponding to any spectral-zero dynamics whose invariant polynomials satisfy (11), there exists a solution of McMillan degree $n - m$. Remark 1 summarizes the construction of this class of interpolants. Further, there may exist solutions of degree $n - m$ whose spectral-zero dynamics do not satisfy the set of inequalities in (11). The existence of such solutions depends on a particular H in interpolation data (e.g., as in Example 1). Therefore in general, there is no systematic way to find solutions whose spectral-zero dynamics do not fall in the class specified by Theorem 1.

V. Interpolants and their Jordan structure

Contrary to the scalar case [4], [5], the Jordan structure of the spectral-zero dynamics is not sufficient to specify an interpolant uniquely. This is to be expected since the same zero dynamics may be contributed by different channels. However, if the interpolation is sought in a fractional form as in [1], then the choice of the "numerator" of the spectral factor in a likewise factorized form may be sufficient to ensure uniqueness (modulo multiplication by a unitary matrix). The next example demonstrates the first point, followed by a discussion of uniqueness in the context of the trigonometric moment problem in the spirit of [1].

Example 2: Consider

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, H = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix},$$

and let the desired Jordan structure for spectral-zero dynamics of solutions be

$$A_z = \begin{bmatrix} -1/2 & 0 \\ 0 & -1/3 \end{bmatrix}, \quad (20)$$

which falls in the class specified by Corollary 1. It can be shown that there exist more than one K that give rise to interpolants with Jordan structure (20) as their spectral-zero dynamics. To see this, we follow the construction in Remark 1 with two choices of K as

$$K_1 = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \end{bmatrix}, K_2 = \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{bmatrix}.$$

These lead to $\hat{F}_1$ and $\hat{F}_2$ with minimal realizations

$$\hat{F}_1 : \left\{ \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.34 & 0 \\ 0 & -0.25 \end{bmatrix}, \begin{bmatrix} 0.28 & 0 \\ 0 & 0.24 \end{bmatrix} \right\},$$
$$\hat{F}_2 : \left\{ \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -0.25 & 0 \\ 0 & -0.34 \end{bmatrix}, \begin{bmatrix} 0.24 & 0 \\ 0 & 0.28 \end{bmatrix} \right\},$$

respectively. Therefore, the spectral-zero dynamics of the corresponding interpolants F_1 and F_2 share the same Jordan structure. □

Theorem 1 provides a rather clear picture of the interpolants of degree $n-m$ for the special case of trigonometric moment problem. The relevant statement is given below while the proof can be found in [26].

Corollary 2: Consider the trigonometric moment problem with data (A, B, R) as in (12) and (13) and $R \succ 0$. Then, every interpolant F of McMillan degree $n-m$ can be obtained (as described in Remark 1) for a unique choice of K which satisfies

(i) $A - BK$ is Hurwitz,
(ii) rank$(A - BK) = n - m$,

and determines the spectral-zero dynamics of $F(z)$.

Further, any K as in Corollary 2 gives rise to an interpolant of McMillan degree less than or equal to $n-m$. Although in the multivariable setting, the solution corresponding to a given Jordan structure as spectral-zero dynamics is not unique, in the case of trigonometric matrix functions we can establish a bijective correspondence between solutions and the parameter K. This implies that corresponding to any K which satisfies the conditions in Corollary 2, there exists a unique interpolant of McMillan degree $n-m$.

Remark 2: Interpolation of trigonometric matrix functions with a degree constraint has been studied long ago in [1], as the rational covariance extension problem. Because of the uniformly distribution of controllability indices in this problem, solutions are fully parameterized via Theorem 1. Our results herein are compatible with earlier results in [1]. Further, we showed that there is no bijective correspondence between interpolants and Jordan structure of spectral-zero dynamics. However, if we fix the spectral factor of the solution with a specific K in (9), there exists a corresponding unique interpolant.

VI. Concluding Remarks

Analytic interpolation with degree constraint is motivated by control and signal processing applications. Earlier studies for scalar interpolation led to a parametrization of solutions in terms of the so-called spectral zeros. Our contribution in this paper is to study the spectral-zero dynamics for analytic interpolation with $m \times m$ matrix-valued functions and to characterize corresponding invariant subspaces and Jordan structures that are permissible for (generically minimal) McMillan degree $n-m$ interpolants (n being the size and rank of the corresponding Pick matrix). While interpolants of McMillan degree lower than $n-m$ may be possible, depending on the data, the class of solutions of McMillan degree $n-m$ is always non-empty and represents a natural generic family of interpolants of low complexity. An obstruction in assigning invariant subspaces for the spectral zero dynamics is related to the type of obstruction in assigning poles via state feedback, as characterized by the celebrated Rosenbrock's theorem.

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