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A Switching Supervisory Control Design for Uncertain Discrete Time-Delay Systems

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Abstract— **This paper presents an adaptive switching supervisory control scheme for highly uncertain discrete-time systems with time-varying state delay and time-varying parameters. The uncertainties appear in the system matrices and the system is assumed to be subject to the external bounded disturbances. It is supposed that a set of stabilizing controllers are available (which are designed off-line) to stabilize the system in the whole uncertain parameter space. To find a supervisory control scheme, it is initially assumed that the system parameters and delay are fixed. A switching algorithm is then proposed to stabilize the system. Next, by modifying the proposed algorithm, the stability analysis of the system with time-varying parameters and time-varying delay is carried out. Furthermore, an upper bound on the permissible rate of change of the system parameters and delay to maintain stability of the closed-loop system is obtained. Simulation results are presented to show the efficacy of the proposed switching scheme.**

I. INTRODUCTION

In numerous control applications such as multi-vehicle coordination, manufacturing systems, spacecraft exploration missions and network control systems, time-delay in system dynamics, if neglected in control design procedure, can lead to instability or poor performance of the system. In fact, in presence of large uncertainties in the magnitude of delay (in addition to uncertainties in the system parameters) it may be difficult (or sometimes impossible) to find a single controller capable of stabilizing the system.

Robust stabilization of uncertain time-delay systems with time-varying delay is well-documented [2], [5], [10], [11], [12], [16], [15]. In most of the existing works in this area, delay-dependent approaches are presented to find a single controller which stabilizes the uncertain system with time-varying delay. However, these works are often unable to effectively handle large uncertainties in both system parameters and delay.

Furthermore, in conventional adaptive control techniques (even for the case of finite-dimensional LTI systems), a number of standard assumptions in the form of *a priori* knowledge (e.g., on the relative degree, non-minimum phase property, and the sign of the high-frequency gain) are required to be made (e.g., see [6]). Furthermore, such techniques are usually inefficient in presence of highly uncertain or rapidly changing parameters. In order to relax the above mentioned limitations of classical adaptive control methods, the supervisory switching control schemes are presented [1], [7], [8], [9], [13], [14]. The main idea of such schemes is to switch among a family of pre-designed and fixed controllers in such a way that adaptive tracking of reference signals is achieved. One of the recent works in this discipline of research is localized-based switching adaptive control proposed in [9], which results in a fast model falsification and an acceptable transient response.

On the other hand, classical adaptive methods and recently developed switched based controllers can stabilize the systems with only large uncertainties on the system parameters. For uncertain time-delay systems there are only a few references that can handle both large uncertainties in system parameters in addition to time-delay in the system dynamic [4]. In [4] a pre-routed switching approach is developed to stabilize a class of uncertain continuous-time system with time-delay while the delays are supposed to be constant and known.

In this work, it is assumed that the system is subject to large parameter uncertainties and time-varying delay in the state with known upper and lower bound on the delay. Since the perturbations could be large, it is aimed to design a set of state feedback gains along with a supervisory algorithm such that the discrete-time system becomes stable. To that end, a decomposition of uncertain parameter space is assumed and a state feedback controller is considered to exist corresponding to each region. In the following, similar to [9], a switching algorithm is proposed with fast falsification property. Based on the properties of the system and designed controllers an upper bound on the rate of the changes on the system parameters and delay is obtained.

Problem formulation is presented in Section II, where some useful definitions are also provided. In Section III, some preliminary results on $\bar{\mu}$- exponential stability of discrete-time systems with time-varying state delay are given. Then, the stability analysis is carried out in two steps. First, it is considered that the uncertain system parameters and delay are fixed and then, in the second subsection, it is assumed that the uncertain parameters and delay are time-varying. A numerical example is presented in Section IV which demonstrates the effectiveness of the proposed adaptive switching controller. Finally, the concluding remarks are given.

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II. PROBLEM FORMULATION

Consider the following uncertain time-varying discrete-time system

$$x(k+1) = A(k)x(k) + A_d(k)x(k-l(k)) + B(k)u(k) + \mu(k) \quad (1)$$

where $x(k) \in \mathbf{R}^n$ is the measurable state vector, $u(k) \in \mathbf{R}^m$ is the control input, $\mu(k) \in \mathbf{R}^n$ is the disturbance vector, $A(k) \in R^{n\times n}$, $A_d(k) \in R^{n\times n}$ and $B(k) \in R^{n\times m}$ are uncertain time-varying system matrices and $l(k)$ is the time-varying delay in state dynamics. It is assumed that the disturbance vector $\mu(k)$ is norm-bounded; i.e. $||\mu(k)|| \le \bar{\mu}$, where, $\bar{\mu}$ is a positive constant. Moreover, the following assumption is made on the size of delay:

$$0 \le \underline{l} \le l(k) \le \bar{l} \quad (2)$$

where $\underline{l}$ and $\bar{l}$ are known non-negative integers. Let the initial condition associated with (1) be given by

$$x(k) = \phi(k), \qquad k \in [-\bar{l}, 0] \quad (3)$$

where $\phi(k)$ is a real valued function on $[-\bar{l}, 0]$. The following definitions for the type of stability of discrete-time systems will prove convenient in the development of the main results.

Definition 1: The uncertain time-delay system (1) with $u(k)$ and $\mu(k)$ both set to zero, $\forall k \in Z$, is said to be exponentially stable if there exist constant scalars $\rho \in (0,1)$ and $M_1 > 0$ such that $||x(k)|| \le M_1\rho^{(k-k_0)} \sup_{k_0-\bar{l}\le v\le k_0} ||x(v)||$, $\forall k \ge k_0$, and for all admissible uncertainties.

Definition 2: The uncertain system (1) with $u(k) = 0$ is said to be globally $\bar{\mu}$-exponentially stable, if there exist constant scalars $\rho \in (0,1)$ and $\tilde{M}_1 > 0$, as well as a function $\tilde{M}_2(\cdot) : R^+ \to R^+$ with $\tilde{M}_2(0) = 0$ such that $||x(k)|| \le \tilde{M}_1\rho^{(k-k_0)} \sup_{k_0-\bar{l}\le v\le k_0} ||x(v)|| + \tilde{M}_2(\bar{\mu})$, $\forall k \ge k_0$, and for all admissible uncertainties, where R^+ denotes the set of strictly positive real numbers.

It is desired now to design a switching discrete-time controller under which the system (1) is exponentially $\bar{\mu}$-exponentially stable in the presence of uncertainties and time-varying delay.

III. Main Result

A. Preliminary Results

In this subsection, four lemmas are presented which will be used to develop the main results of the paper. First, the following two definitions are given.

Definition 3: Consider the following LTV discrete-time system with an integer state delay $l(k)$

$$x(k+1) = A_0(k)x(k) + A_1(k)x(k-l(k)) + \mu(k), \quad (4)$$

where $x(k) \in \mathbf{R}^n$ and $\mu(k) \in R^n$ are the state and input of the system, respectively. Assume that $l(k) \in [\underline{l}, \bar{l}]$. $\Lambda(k) \in R^{n(\bar{l}+1)\times n(\bar{l}+1)}$ is defined as

$$\Lambda(k) = \begin{bmatrix} \Lambda_{1,1} & 0_n & \cdots & 0_n & \Lambda_{1,l(k)+1} & 0_n & \cdots & 0_n & 0_n & 0_n \\ I_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & 0_n & 0_n & 0_n \\ \vdots & & & & \vdots & & & & & \vdots \\ 0_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & I_n & 0_n & 0_n \\ 0_n & 0_n & \cdots & 0_n & 0_n & 0_n & \cdots & 0_n & I_n & 0_n \end{bmatrix} \quad (5)$$

where I_n and 0_n denote the $n\times n$ identity and zero matrices, respectively. If $l(k)$ is non-zero,

$$\Lambda_{1,1} = A_0(k), \qquad \Lambda_{1,l(k)+1} = A_1(k)$$

otherwise,

$$\Lambda_{1,1} = \Lambda_{1,l(k)+1} = A_0(k) + A_1(k)$$

Definition 4: For the system (4), let $\phi(k_1, k_2)$ be defined as

$$\Phi(k_1, k_2) = \begin{cases} \Lambda(k_1-1)\Lambda(k_1-2)\ldots\Lambda(k_2), & k_1 > k_2 \\ I, & k_1 = k_2 \end{cases} \quad (6)$$

where $k_1 \ge k_2$.

Lemma 1: Consider the system (4). The state $x(k)$ can be expressed by

$$x(k) = \Psi(k, k_0)z(k_0) + \sum_{p=k_0+1}^{k} \Psi(k,p)E\mu(p-1) \quad (7)$$

for $k > k_0$, where

$$z(k_0) = \begin{bmatrix} x^T(k_0) & x^T(k_0-1) & \cdots & x^T(k_0-\bar{l}) \end{bmatrix}^T$$
$$\Psi(k_1,k_2) = \Xi\Phi(k_1,k_2), \qquad E = \Xi^T \quad (8)$$
$$\Xi = \begin{bmatrix} I_n & 0_n & \cdots & 0_n \end{bmatrix}$$

Note that $\Xi \in R^{n\times n(\bar{l}+1)}$.

Proof: From

$$z(k) = \begin{bmatrix} x^T(k) & x^T(k-1) & \cdots & x^T(k-\bar{l}) \end{bmatrix}^T \quad (9)$$

the equation (4) can be written as

$$\begin{aligned} z(k+1) &= \Lambda(k)z(k) + E\mu(k) \\ x(k+1) &= \Xi z(k+1) \end{aligned} \quad (10)$$

where E and Ξ are given by (8), $\Lambda(k)$ is defined in (5), and $z(k_0)$ is given in (8). By solving the system (10) recursively, one can verify that $x(k)$ satisfies (7). ■

Lemma 2: If the system (4) is exponentially stable for $l(k) \in [\underline{l}, \bar{l}]$, then there exists a constant $\tilde{M} > 0$ such that $\Psi(k, k_0)$ defined in Lemma 1, satisfies

$$\|\Psi(k,k_0)\| \le \tilde{M}\rho^{k-k_0}, \qquad k \ge k_0 \quad (11)$$

Proof: Consider a fixed $k > k_0$. Then, for all k

$$\|\Psi(k,k_0)\| = \sup_{v(k)} \frac{\|\Psi(k,k_0)v(k)\|}{\|v(k)\|} \quad (12)$$

Let $v(k)$ is written in the standard basis $\{e_i\}$, $i = 1, 2, \ldots, n(\bar{l}+1)$; i.e.,

$$v(k) = \sum_{i=1}^{n(\bar{l}+1)} \alpha_i(k) e_i \tag{13}$$

Using (13), (12) is written as following

$$\|\Psi(k, k_0)\| = \sup_{\alpha_i(k)} \frac{\|\sum_{i=1}^{n(\bar{l}+1)} \alpha_i(k)\Psi(k, k_0)e_i\|}{\|\sum_{i=1}^{n(\bar{l}+1)} \alpha_i(k)e_i\|} \tag{14}$$

Thus, it can be concluded that

$$\|\Psi(k, k_0)\| \leq \sup_{\alpha_i(k)} \frac{\sum_{i=1}^{n(\bar{l}+1)} |\alpha_i(k)| \|\Psi(k, k_0)e_i\|}{\|\sum_{i=1}^{n(\bar{l}+1)} \alpha_i(k)e_i\|} \tag{15}$$

In (7), assume that $\mu(k) = 0$ and $z(k_0) = e_i$. Therefore,

$$x(k) = \Psi(k, k_0)e_i \tag{16}$$

Since the system (4) is exponentially stable, according to Definition 1, there exists a constant $M > 0$ such that

$$\|\Psi(k, k_0)e_i\| = \|x(k)\| \leq M\rho^{k-k_0}, \qquad k > k_0 \tag{17}$$

Then, it is resulted that

$$\|\Psi(k, k_0)\| \leq M\rho^{k-k_0} \sup_{\alpha_i(k)} \frac{\sum_{i=1}^{n(\bar{l}+1)} |\alpha_i(k)|}{\|\sum_{i=1}^{n(\bar{l}+1)} \alpha_i(k)e_i\|} \tag{18}$$

On the other hand,

$$\|v(k)\|_1 = \sum_{i=1}^{n(\bar{l}+1)} |\alpha_i(k)| \tag{19a}$$

$$\|v(k)\| = \|\sum_{i=1}^{n(\bar{l}+1)} \alpha_i(k)e_i\| \tag{19b}$$

It is known that in finite-dimensional space, all the vector norms are equivalent; i.e.,

$$\|v(k)\|_1 \leq c\|v(k)\| \tag{20}$$

where c is a constant. By defining $\hat{M} = cM$, it is followed that

$$\|\Psi(k, k_0)\| \leq \hat{M}\rho^{k-k_0}, \qquad k > k_0$$

Let $\tilde{M} = \max\{1, \hat{M}\}$, then (11) is resulted. ■

Lemma 3: Suppose that the system

$$x(k+1) = A(k)x(k) + A_d(k)x(k - l(k)) + B(k)u(k) \tag{21}$$

is exponentially stable under the feedback control $u(k) = Kx(k)$ for all $l(k) \in [\underline{l}, \bar{l}]$. Then the system

$$x(k+1) = A(k)x(k) + A_d(k)x(k - l(k)) + B(k)u(k) + \mu(k)$$

where $0 \leq \|\mu(k)\| \leq \bar{\mu}$, is $\bar{\mu}$-exponentially stable under the control law $u(k) = Kx(k)$.

Proof: Substituting $u(k) = Kx(k)$ in (21), the following equation is obtained for $k > k_0$

$$x(k+1) = (A(k) + B(k)K)\, x(k) + A_d(k)x\,(k - l(k)) + \mu(k) \tag{22}$$

According to Lemma 1, $x(k)$ can be written as (7), with $A_0(k) = A(k) + B(k)K$, and $A_1(k) = A_d(k)$. Since the system obtained in (22) is exponentially stable for all $l(k) \in [\underline{l}, \bar{l}]$,

$$\|\Psi(k, k_0)z(k_0)\| \leq M\rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| \tag{23}$$

Applying triangular inequality to (7) leads to

$$\|x(k)\| \leq \|\Psi(k, k_0)z(k_0)\| + \sum_{p=k_0+1}^{k} \|\Psi(k, p)E\mu(p-1)\| \tag{24}$$

Using (23), it follows that

$$\begin{aligned} \|x(k)\| \leq & M\rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| \\ & + \sum_{p=k_0+1}^{k} \|\Psi(k, p)\|\|E\|\bar{\mu} \end{aligned} \tag{25}$$

From Lemma 2, it can be concluded that

$$\|x(k)\| \leq M\rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \sum_{p=k_0+1}^{k} \tilde{M}\rho^{k-p}\bar{\mu} \tag{26}$$

Thus,

$$\|x(k)\| \leq M\rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \tilde{M}\bar{\mu}\frac{1-\rho^{k-k_0}}{1-\rho} \tag{27}$$

Define

$$\tilde{M}_2(\bar{\mu}) := \tilde{M}\bar{\mu}\frac{1}{1-\rho} \tag{28}$$

It is clear that

$$\tilde{M}_2(\bar{\mu}) : R^+ \to R^+, \qquad \tilde{M}_2(0) = 0 \tag{29}$$

Therefore, for $k > k_0$,

$$\|x(k)\| \leq M\rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \tilde{M}_2(\bar{\mu})$$

Let $\tilde{M}_1$ denote $\max\{1, M\}$; it can now be inferred from Definition 2 that (21) is $\bar{\mu}$-exponentially stable under the feedback $u(k) = Kx(k)$. ■

Lemma 4: Consider the following system

$$x(k+1) = A(k)x(k) + A_d(k)x(k - l(k)) + B(k)u(k) + \mu(k) \tag{30}$$

where $l_m \leq l(k) \leq l_M$, and $\mu(k)$ is a bounded disturbance $(0 \leq \|\mu(k)\| \leq \bar{\mu})$. If (30) is exponentially stable under the feedback law $u(k) = Kx(k)$, $\forall l(k) \in [l_m, l_M]$, then there exist constants $\tilde{M} > 0$, $\rho \in (0, 1)$, and a function $\hat{M}(\cdot) : \mathbf{R}^+ \to \mathbf{R}^+$ with $\hat{M}(0) = 0$ such that for every $\bar{l} \geq l_M$, and $k \geq k_0$

$$\sup_{k-\bar{l} \leq v \leq k} \|x(v)\| \leq \tilde{M}\rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \hat{M}(\bar{\mu}) \tag{31}$$

Proof: Since (30) is exponentially stable under the feedback $u(k) = Kx(k)$, it is $\bar{\mu}$-exponentially stable as well.

Thus, for any $k \geq k_0$, there exist a positive constant M_1 and a function $\hat{M}_1$ such that

$$\|x(k)\| \leq M_1\rho^{k-k_0} \sup_{k_0-l_M \leq v \leq k_0} \|x(v)\| + \hat{M}_1(\bar{\mu}) \quad (32)$$

This means that

$$\|x(k)\| \leq M_1\rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \hat{M}_1(\bar{\mu}) \quad (33)$$

If $k > k_0 + \bar{l}$, then for $0 \leq i \leq \bar{l}$

$$\|x(k-i)\| \leq M_1\rho^{k-i-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \hat{M}_1(\bar{\mu}), \quad (34)$$

or equivalently,

$$\|x(k-i)\| \leq M_1\rho^{-i}\rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \hat{M}_1(\bar{\mu}) \quad (35)$$

Let $\tilde{M}_1$ be equal to $\frac{M_1}{\rho^{\bar{l}}}$. Then, it can be concluded that

$$\sup_{k-\bar{l} \leq v \leq k} \|x(v)\| \leq \tilde{M}_1\rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \hat{M}_1(\bar{\mu}). \quad (36)$$

Consider now the case when $k \leq k_0 + \bar{l}$. Since $A(k)$, $A_d(k)$ and $B(k)$ are norm bounded and K is a constant matrix, one can find a positive constant M_2 and a function $\hat{M}_2(\cdot)$ such that

$$\sup_{k-\bar{l} \leq v \leq k} \|x(v)\| \leq M_2 \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \hat{M}_2(\bar{\mu}), \quad (37)$$

or equivalently,

$$\sup_{k-\bar{l} \leq v \leq k} \|x(v)\| \leq M_2\rho^{k_0-k}\rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \hat{M}_2(\bar{\mu}) \quad (38)$$

Let $\tilde{M}_2$ be equal to $\frac{M_2}{\rho^{\bar{l}}}$. Then, it can be concluded that

$$\sup_{k-\bar{l} \leq v \leq k} \|x(v)\| \leq \tilde{M}_2\rho^{k-k_0} \sup_{k_0-\bar{l} \leq v \leq k_0} \|x(v)\| + \hat{M}_2(\bar{\mu}). \quad (39)$$

Define $\tilde{M}$ and $\hat{M}$ as

$$\begin{aligned} \tilde{M} &= \max_{i=1,2} \tilde{M}_i \\ \hat{M}(\bar{\mu}) &= \max_{i=1,2} \hat{M}_i(\bar{\mu}), \qquad \forall \bar{\mu} \geq 0 \end{aligned} \quad (40)$$

It can be deduced that for a constant $\tilde{M}$ and a function $\hat{M}(\cdot): R^+ \rightarrow R^+$, $\hat{M}(0) = 0$ defined above, the inequality (31) holds. ■

B. A Time-Invariant System with Fixed Delay

In this subsection, it is assumed that the system matrices are uncertain nevertheless time invariant, and that the delay is an unknown, bounded constant. Moreover, no single controller is assumed to exist with the property that it stabilizes the system within the whole uncertain parameter space pertaining to A, A_d and l. The following assumption is essential for developing the main results.

Assumption 1: Suppose that the uncertain parameter space associated with A, A_d and l, denoted by Ω, is compact and can be decomposed into a finite cover $\{\Omega_i\}_1^L$, for which the following conditions hold

i) $\Omega_i \subset \Omega, \quad \Omega_i \neq \{\}, \ i = 1, \ ..., \ L$

ii) $\bigcup_{i=1}^L \Omega_i = \Omega$

iii) For any $i \in \{1, ..., L\}$, there exist $(A_i, A_{di}, B_i) \in \Omega_i$ (center) and K_i (control gain) such that for any $l \in [l_{m_i}, l_{M_i}]$, the controller $u(k) = K_i x(k)$ exponentially stabilizes the system (1) for all (A, A_d, B) satisfying

$$||A - A_i|| \leq \alpha_i, \quad ||A_d - A_{di}|| \leq \beta_i, \quad ||B - B_i|| \leq \delta_i \quad (41)$$

It is to be noted that, l_{m_i} and l_{M_i} in condition (iii) above represent the lower and upper bounds of l in Ω_i, respectively. Note that $\underline{l} \leq l_{m_i}$, and $l_{M_i} \leq \bar{l}$ and (41) holds for all $(A, A_d, B) \in \Omega_i$ and $\forall l \in [l_{mi}, \ l_{Mi}]$.

In Conditions (i) and (ii) given above, it is supposed that the uncertain set Ω can be constructed from the union of the sets Ω_i, $i = 1, ..., L$. In the next step, a condition is presented based on which a supervisory control scheme is obtained to $\bar{\mu}$-exponentially stabilize the system (1). Suppose that the uncertain plant (1) with any time-delay $l \in [l_{m_p}, l_{M_p}]$ lies in Ω_p. Then, there exists a delay $\hat{l} \in [l_{m_p}, l_{M_p}]$ that is equal to the plant delay (i.e., $l = \hat{l}$) which can be interpreted as the nominal delay corresponding to Ω_p. Condition (iii), on the other hand, implies that

$$\begin{aligned} ||x(k) - A_p x(k-1) - A_{dp} x(k - \hat{l} - 1) - B_p u(k-1)|| \leq \\ \alpha_p ||x(k-1)|| + \beta_p ||x(k - \hat{l} - 1)|| \\ + \delta_p ||u(k-1)|| + \bar{\mu} \end{aligned} \quad (42)$$

The above inequality provides the core falsifying criterion for the switching rule proposed in this work. If this inequality is violated for $\forall \hat{l} \in [l_{m_p}, l_{M_p}]$, it implies that the plant is not in Ω_p, i.e. $i(k) \neq p$, where $i(k)$ denotes the index of the controller.

The following algorithm is proposed to design supervisory control based on (42).

Algorithm 1

1) Let $k = k_0$, $k_0 > 0$
   $P = \{1, \ ..., \ L\}$
   $Q_p = \{m_p, \ ..., \ M_p\}$, $\forall p \in P$
   $H(k_0) = \{(p, \ l_q) \mid p \in P$ and $q \in Q_p\}$
   Choose $i(k_0) \in P$
2) $k = k + 1$
3) $\hat{H}(k) = \left\{(p, \ l_q) \mid (42) \text{ holds } p \in P, \ \hat{l} = l_q, q \in Q_p\right\}$
4) $H(k) = H(k-1) \cap \hat{H}(k)$
5) **if** $\exists \ q \in Q_{i(k-1)}$ such that $(i(k-1), l_q) \in H(k)$,
   then $i(k) = i(k-1)$. Go to step 2
   else
   $i(k)$= any entry of P such that $(p, \ l_q) \in H(k)$. Go to step 2

The above algorithm can be summarized as follows: In Step 1 of the algorithm, the controller is initialized and the set $H(k_0)$, which includes all the regions in the parameter space of Ω, is formed. The set $\hat{H}(k)$ is then constructed in Step 3. This set consists of all ordered pairs (p, l_q) which are not falsified by (48) at time k. The falsified pairs (p, l_q) prior to time k are also omitted form $\hat{H}(k)$ in the next step (Step 4). Finally, if the current controller index belongs to $H(k)$ obtained in Step 4, no switching from the current controller is required. Otherwise, a new controller index is chosen from $H(k)$, and therefore the falsification procedure will be repeated from Step 2 again.

Remark 1: Note that the controller gain K_i will not be replaced in step 5 unless all potential values for delay, i.e. $l_i = l_{m_i},\ \ldots,\ l_{M_i}$ are falsified.

Remark 2: Algorithm 1 guarantees that all elements of the compact set Ω (uncertain parameter space) with any plant delay corresponding to $l = \underline{l}, \ldots, \bar{l}$ is examined. It also guarantees the existence of a control law that stabilizes the plant (1), leading to the convergence of the switching sequence $i(k_0),\ i(k_0+1),\ \ldots$.

Theorem 1: Consider the system (1) and suppose that the conditions of Assumption 1 hold. Then, using the proposed switching algorithm the resultant closed-loop system is $\bar{\mu}$-exponentially stable.

Proof: Consider the finite set $\{k_1,\ \ldots,\ k_f\}$ as the sequence of switching instants. Consider also two consecutive instants k_s and k_{s+1} (it is known that such switching instants exist for some s). From (1), the dynamics of the closed-loop system (with time-invariant parameters and fixed delay) for $k \in [k_s, k_{s+1})$ can be presented by

$$\begin{aligned} x(k+1) =&(A + BK_{i(k_s)})x(k) + A_d x(k-l) + \mu(k) \\ =&(A_{i(k_s)} + B_{i(k_s)}K_{i(k_s)})x(k) \\ &+ A_{di(k_s)}x(k-\hat{l}(k)) \\ &+ (A + BK_{i(k_s)})x(k) + A_d x(k-l) + \mu(k) \\ &- (A_{i(k_s)} + B_{i(k_s)}K_{i(k_s)})x(k) \\ &- A_{di(k_s)}x(k-\hat{l}(k)) \end{aligned} \tag{43}$$

where $\hat{l}(k)$ is the nominal delay that satisfies (42) for $k \in [k_s,\ k_{s+1})$. Define $\psi(k)$ as

$$\begin{aligned} \psi(k) =&(A + BK_{i(k_s)})x(k) + A_d x(k-l) \\ &- (A_{i(k_s)} + B_{i(k_s)}K_{i(k_s)})x(k) \\ &- A_{di(k_s)}x(k-\hat{l}(k)) + \mu(k) \end{aligned} \tag{44}$$

Thus, on substituting (44) in (43) one will obtain

$$\begin{aligned} x(k+1) =&(A_{i(k_s)} + B_{i(k_s)}K_{i(k_s)})x(k) \\ &+ A_{di(k_s)}x(k-\hat{l}(k)) + \psi(k) \end{aligned} \tag{45}$$

where $||\psi(k)|| \leq \alpha_{i(k_s)}||x(k)|| + \beta_{i(k_s)}||x(k-\hat{l}(k))|| + \gamma_{i(k_s)}||u(k)|| + \bar{\mu}$, as (45) is not violated in the switching interval $[k_s,\ k_{s+1})$. Introducing fictitious matrices and parameter ΔA, ΔA_d, ΔB and $\hat{\mu}(k)$, it can be concluded from the structure of (45) that

$$\begin{aligned} x(k+1) =&(A_{i(k_s)} + \Delta A + B_{i(k_s)}K_{i(k_s)} + \Delta BK_{i(k_s)})x(k) \\ &+ (A_{di(k_s)} + \Delta A_d)x(k-\hat{l}(k)) + \hat{\mu}(k) \end{aligned} \tag{46}$$

where $||\hat{\mu}(k)|| \leq \bar{\mu}$, $||\Delta A|| \leq \alpha_{i(k_s)}$, $||\Delta A_d|| \leq \beta_{i(k_s)}$ and $||\Delta B|| \leq \gamma_{i(k_s)}$.

Since $K_{i(k)} \in \{K_i\}_{i=1}^{L}$, and $i(k) \in P$, (P is a finite set), there exist a finite positive constant M_0 and a function $\gamma_0 : R^+ \to R^+$, with $\gamma_0(0) = 0$, satisfying

$$\begin{aligned} &\sup_{k_s-\bar{l}\leq v\leq k_s} ||x(v)|| \leq \\ &\qquad M_0 \sup_{k_s-\bar{l}-1\leq v\leq k_s-1} ||x(v)|| + \gamma_0(\bar{\mu}) \end{aligned} \tag{47}$$

and since (46) behaves like an exponentially stable system, for any $k \in [k_s,\ k_{s+1}-1]$, it follows from Lemma 4 that there exist constants $M_{i(k_s)}$, $\rho_{i(k_s)}$, and a function $\hat{M}_{i(k_s)}(\cdot)$ with $\hat{M}_{i(k_s)}(0) = 0$ such that

$$\begin{aligned} \sup_{k-\bar{l}\leq v\leq k} ||x(v)|| \leq& M_{i(k_s)}\rho_{i(k_s)}^{k-k_s} \sup_{k_s-\bar{l}\leq v\leq k_s} ||x(v)|| \\ &+ \hat{M}_{i(k_s)}(\bar{\mu}) \end{aligned} \tag{48}$$

By substituting (47) in (48) one will obtain

$$\begin{aligned} \sup_{k-\bar{l}\leq v\leq k} ||x(v)|| \leq& M_0 M_{i(k_s)}\rho_{i(k_s)}^{k-k_s} \sup_{k_s-\bar{l}-1\leq v\leq k_s-1} ||x(v)|| \\ &+ \hat{M}_{i(k_s)}(\bar{\mu}) + M_{i(k_s)}\rho_{i(k_s)}^{k-k_s}\gamma_0(\bar{\mu}) \end{aligned} \tag{49}$$

Now, since the closed-loop system is exponentially stable in the time interval $k \in [k_{s-1}, k_s)$, it can be concluded from Lemmas 3 and 4 that

$$\begin{aligned} &\sup_{k_s-\bar{l}-1\leq v\leq k_s-1} ||x(v)|| \leq \\ &\qquad M_{i(k_{s-1})}\rho_{i(k_{s-1})}^{k_s-k_{s-1}-1} \sup_{k_{s-1}-\bar{l}\leq v\leq k_{s-1}} ||x(v)|| \\ &\qquad + \hat{M}_{i(k_{s-1})}(\bar{\mu}) \end{aligned} \tag{50}$$

Define

$$\rho := \max_{1\leq i(k)\leq L} \rho_{i(k)}$$

$$M := \max_{1\leq i(k)\leq L} M_{i(k)}$$

Substitute (50) in (49) to obtain

$$\begin{aligned} \sup_{k-\bar{l}\leq v\leq k} ||x(v)|| \leq& M^2 M_0 \rho^{k-k_{s-1}-1} \sup_{k_{s-1}-\bar{l}\leq v\leq k_{s-1}} ||x(v)|| \\ &+ \bar{M}_{i(k_s)}(\bar{\mu}) \end{aligned} \tag{51}$$

for $k \in [k_s, k_{s+1})$, where

$$\begin{aligned} \bar{M}_{i(k_s)}(\bar{\mu}) =&\hat{M}_{i(k_s)}(\bar{\mu}) \\ &+ M_{i(k_s)}\rho^{k-k_s}[M_0\hat{M}_{i(k_{s-1})}(\bar{\mu}) + \gamma_0(\bar{\mu})] \end{aligned}$$

On the other hand, like (49) one can verify that

$$\sup_{k_{s-1}-\bar{l}\leq v\leq k_{s-1}} ||x(v)|| \leq M_0 \sup_{k_{s-1}-\bar{l}-1\leq v\leq k_{s-1}-1} ||x(v)|| + \gamma_0(\bar{\mu}) \tag{52}$$

Therefore, from the inequalities (51) and (52), the following can be obtained for $k \in [k_s, k_{s+1})$

$$\begin{aligned}\sup_{k-\bar{l}\leq v\leq k} ||x(v)|| \leq & \\ & M^2 M_0^2 \rho^{k-k_{s-1}-1} \sup_{k_{s-1}-\bar{l}-1\leq v\leq k_{s-1}-1} ||x(v)|| \\ & + \bar{M}_{i(k_s)}(\bar{\mu}) + M^2 M_0 \rho^{k-k_{s-1}-1}\gamma_0(\bar{\mu})\end{aligned} \tag{53}$$

Using the above inequality iteratively yields

$$\begin{aligned}||x(k)|| \leq & M^f M_0^f \rho^{k-k_0-f} \sup_{k_0-\bar{l}\leq v\leq k_0} ||x(v)|| \\ & + \bar{M}_{i(k_f)}(\bar{\mu}) + M^f M_0^{f-1}\rho^{k-k_0-f}\gamma_0(\bar{\mu})\end{aligned} \tag{54}$$

for $k \geq k_f$, where

$$\begin{aligned}\bar{M}_{i(k_f)}(\bar{\mu}) = & \hat{M}_{i(k_f)}(\bar{\mu}) + M\rho^{k-k_f}[M_0\hat{M}_{i(k_{f-1})}(\bar{\mu}) \\ & + \gamma_0(\bar{\mu})]\end{aligned}$$

Define $\tilde{M}_1 = (\frac{MM_0}{\rho})^f$ and $\tilde{M}_2(\bar{\mu}) = \bar{M}_{i(k_f)}(\bar{\mu}) + M^f M_0^{f-1}\rho^{k-k_0-f}\gamma_0(\bar{\mu})$. Note that $\tilde{M}_1$ is a bounded positive constant and $\tilde{M}_2(\bar{\mu})$ is a function of $\bar{\mu}$ such that $\tilde{M}_2(\cdot): R^+ \to R^+$ with $\tilde{M}_2(0) = 0$. Therefore (54) can be written as

$$||x(k)|| \leq \tilde{M}_1\rho^{k-k_0} \sup_{k_0-\bar{l}\leq v\leq k_0} ||x(v)|| + \tilde{M}_2(\bar{\mu})$$

Hence, it follows from Definition 2, that the system (1) is $\bar{\mu}$-exponentially stable. ∎

C. A Time-Varying System with Time-Varying Delay

It is now assumed that the uncertain system (1) can have infrequent parameter jumps. The following assumption is made on the maximum allowable speed of parameter jumps.

Assumption 2: The number of jumps in system parameters in (1) (i.e. $A(k)$, $A_d(k)$, $B(k)$ and $l(k)$) for any time interval $[k,\ k + \sigma N + \bar{l}]$ cannot exceed σ, where $\bar{l}$ is the maximum bound on the delay, and σ, N are strictly positive constants.

Algorithm 1 can still be used in the case of a system with slowly vary parameters satisfying Assumption 2, after modifying step 4 as follows [9]

$$H(k) = \begin{cases} H(k-1)\cap\hat{H}(k), & \text{if } H(k-1)\cap\hat{H}(k) \neq \{\} \\ \hat{H}(k), & \text{otherwise}\end{cases} \tag{55}$$

This modification allows the algorithm to recheck the falsified items. In the following theorem, conditions for $\bar{\mu}$-exponential stability of the time-varying systems are presented.

Theorem 2: Consider the system (1) and let the condition of Assumption 2 holds for some σ and N (strictly positive). Then the closed system is globally $\bar{\mu}$-exponentially stable if $\tilde{M}_1\rho^N < 1$, where $\tilde{M}_1$ and ρ are constant scalars defined in Definition 2.

Proof: Consider the behavior of system (1) in the interval $[k,\ k + \sigma N + \bar{l}]$, where the number of parameter jumps is assumed to be less than σ. Let h denote the number of switchings carried out by the controller in the above interval. Consider the constant ρ and the function $\hat{M}_f(\cdot)$ in Definition 2, where $\rho \in (0,1)$ and the function $\hat{M}_f(\cdot) > 0$, $\hat{M}_f(0) = 0$. For any interval $[k,\ k + \sigma N + \bar{l}]$ one can use (54) to obtain

$$\begin{aligned}||x(k+\sigma N+\bar{l})|| \leq & (\frac{MM_0}{\rho})^h\rho^{\sigma N+\bar{l}} \sup_{k-\bar{l}\leq v\leq k} ||x(v)|| \\ & + \hat{M}_f(\bar{\mu})\end{aligned} \tag{56}$$

Denote with f the maximum number of switchings that can be made by the controller applied to the plant with unknown time-invariant parameters and fixed unknown time delay. Then,

$$h \leq \sigma f + 1$$

which implies that

$$\begin{aligned}||x(k+\sigma N+\bar{l})|| \leq & (\frac{MM_0}{\rho})((\frac{MM_0}{\rho})^f\rho^N)^\sigma\rho^{\bar{l}} \\ & \times \sup_{k-\bar{l}\leq v\leq k} ||x(v)|| + \hat{M}_f(\bar{\mu})\end{aligned} \tag{57}$$

Note that if $(\frac{MM_0}{\rho})((\frac{MM_0}{\rho})^f\rho^N)^\sigma \leq 1$, then

$$\sup_{k+\sigma N\leq v\leq k+\sigma N+\gamma} ||x(v)|| \leq a \sup_{k-\bar{l}\leq v\leq k} ||x(v)|| + \hat{M}_f(\bar{\mu}) \tag{58}$$

where a and γ are scalar constants such that $0 < a < 1$ and $\gamma \in \{0,\ ...,\ \bar{l}\}$, respectively.

It can be easily verified that the inequality $(\frac{MM_0}{\rho})((\frac{MM_0}{\rho})^f\rho^N)^\sigma \leq 1$, pointed out above is equivalent to

$$(\frac{MM_0}{\rho})^f\rho^N \leq (\frac{MM_0}{\rho})^{\frac{-1}{\sigma}} \tag{59}$$

Using the same argument as in [9], and noting that $(\frac{MM_0}{\rho})^{\frac{-1}{\sigma}}$ is increasing in σ and $\lim_{\sigma\to\infty}(\frac{MM_0}{\rho})^{\frac{-1}{\sigma}} = 1$, the inequality $(\frac{MM_0}{\rho})^f\rho^N < 1$ will be hold for sufficiently large N which guarantees the existence of a finite σ such that (59) holds. Note that it can be concluded from (58) that

$$\begin{aligned}\sup_{k_0+k(\sigma N)+(k-1)\bar{l}\leq v\leq k_0+k(\sigma N+\bar{l})} ||x(v)|| \leq & \\ a^k \sup_{k_0-\bar{l}\leq v\leq k_0} ||x(v)|| + \sum_{i=1}^{i=k} a^{i-1}\hat{M}_f(\bar{\mu}) &\end{aligned} \tag{60}$$

for $\forall k \in \mathbf{N}$. Since $0 < a < 1$ and $\sum_{i=1}^{k} a^{i-1} < \infty$, it is easy to notice from Definition 2, that $||x(k_0+k(\sigma N+\bar{l}))||$ is $\bar{\mu}$-exponentially stable for $\forall k \in \mathbf{N}$. Now, it is straightforward to observe that $\forall j \in \{1, \ ..., \ \sigma N+\bar{l}-1\}$ and $\forall k \in \mathbf{N}$, there exist positive constants G_1 and G_2 such that

$$\begin{aligned} &||x(k_0+k(\sigma N+\bar{l})+j)|| \leq \\ &G_1 \sup_{k_0+k\cdot(\sigma N)+(k-1)\cdot\bar{l} \leq v \leq k_0+k\cdot(\sigma N+\bar{l})} ||x(v)|| + G_2 \end{aligned} \tag{61}$$

This implies $\bar{\mu}$-exponential stability of the system (1) with time-varying parameters and time-varying delay satisfying Assumption 2. ■

IV. Numerical Example

Example 1: Consider the following discrete-time system with time-delay:

$$\begin{aligned} x(k+1) = A(k)x(k) + A_d(k)x(k-l(k)) \\ + B(k)u(k) + \mu(k) \end{aligned} \tag{62}$$

where $l(k) \in \{0, 1, ..., 24\}$, $\mu(k)$ is the uniformly bounded disturbance such that $||\mu(k)|| < 0.1$, $\forall k \in \mathbf{N}$. Furthermore, the uncertain matrices $A(k)$, $A_d(k)$ and $B(k)$ can switch between the following 7 sets of matrices

$$\begin{aligned} 1)\ \ A_1 &= \begin{bmatrix} 0.8 & 0 \\ 0.1 & 0.9 \end{bmatrix} + \Delta A_1, \ B_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + \Delta B_1, \\ A_{d1} &= \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix} + \Delta A_{d1} \end{aligned} \tag{63a}$$

$$\begin{aligned} 2)\ \ A_2 &= \begin{bmatrix} 0.3 & 1 \\ 0.1 & 0.6 \end{bmatrix} + \Delta A_2, \ B_2 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + \Delta B_2, \\ A_{d2} &= \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix} + \Delta A_{d2} \end{aligned} \tag{63b}$$

$$\begin{aligned} 3)\ \ A_3 &= \begin{bmatrix} 1 & 1 \\ 0.2 & 0.7 \end{bmatrix} + \Delta A_3, \ B_3 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} + \Delta B_3, \\ A_{d3} &= \begin{bmatrix} 0.4 & 0 \\ 0.8 & 0.4 \end{bmatrix} + \Delta A_{d3} \end{aligned} \tag{63c}$$

$$\begin{aligned} 4)\ \ A_4 &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \Delta A_4, \ B_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \Delta B_4, \\ A_{d4} &= \begin{bmatrix} 0.5 & 0 \\ -0.5 & 0.3 \end{bmatrix} + \Delta A_{d4} \end{aligned} \tag{63d}$$

$$\begin{aligned} 5)\ \ A_5 &= \begin{bmatrix} 1 & 1 \\ -1 & -0.1 \end{bmatrix} + \Delta A_5, \ B_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \Delta B_5, \\ A_{d5} &= \begin{bmatrix} -0.5 & 0 \\ 0.1 & 0.03 \end{bmatrix} + \Delta A_{d5} \end{aligned} \tag{63e}$$

$$\begin{aligned} 6)\ \ A_6 &= \begin{bmatrix} -0.2 & 0.1 \\ -0.5 & 1 \end{bmatrix} + \Delta A_6, \ B_6 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \Delta B_6, \\ A_{d6} &= \begin{bmatrix} -0.1 & 0 \\ 0.1 & -0.5 \end{bmatrix} + \Delta A_{d6} \end{aligned} \tag{63f}$$

$$\begin{aligned} 7)\ \ A_7 &= \begin{bmatrix} -0.1 & 1 \\ 0.1 & -1 \end{bmatrix} + \Delta A_7, \ B_7 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \Delta B_7, \\ A_{d7} &= \begin{bmatrix} -0.4 & 0.4 \\ -0.1 & 0.1 \end{bmatrix} + \Delta A_{d7} \end{aligned} \tag{63g}$$

where $||\Delta A_i|| \leq 0.05$, $||\Delta A_{di}|| \leq 0.05$, and $||\Delta B_i|| \leq 0.05$ for $i = 1, ..., 7$, such that $||\cdot||$ denotes the 2-norm. Using the method developed in [5], 50 controllers are designed to cover the whole parameter space corresponding to the uncertain time-delay and state-space matrices given above. Switching sequence between different subsystems in (63a)-(63g) along with time-delay profile are depicted in Fig. 1. Using the 50 controllers mentioned above and following Algorithm 1, the state trajectories sketched in Fig. 2 are obtained. These trajectories clearly show that the system is stable (or more precisely, $\bar{\mu}$-exponentially stable) in the presence of the uncertain time-delay and state-space matrices. The switching instances between different controllers are given in Fig. 3.

V. Conclusions

An adaptive switching control algorithm is developed for uncertain discrete-time systems with time-varying delay, in the presence of disturbances. A set of discrete-time controllers are designed with the property that at least one of them can stabilize the system with adaptive switching algorithm is then established which a stabilizing controller through fast model falsification. The proposed switching scheme is convergent and guarantees the stability of the closed-loop system. Simulation results elucidate the effectiveness of the method in stabilizing a highly uncertain time-delay system with a relatively large upper bound on delay.

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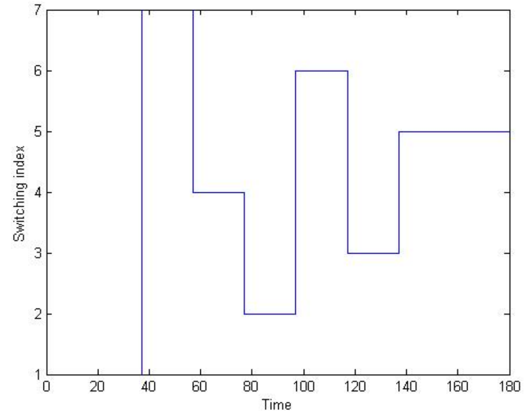
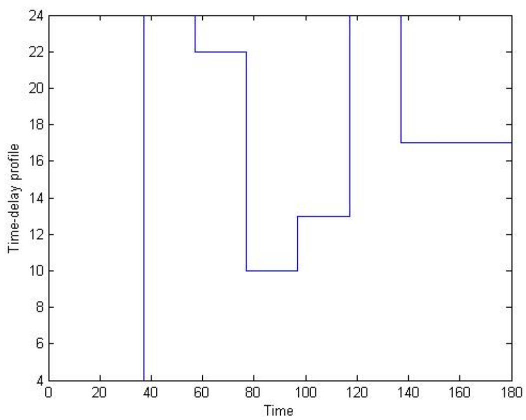


(a)



(b)

Fig. 1. (a) Switching sequence between different subsystems in (63a)-(63g); (b) time-delay profile.

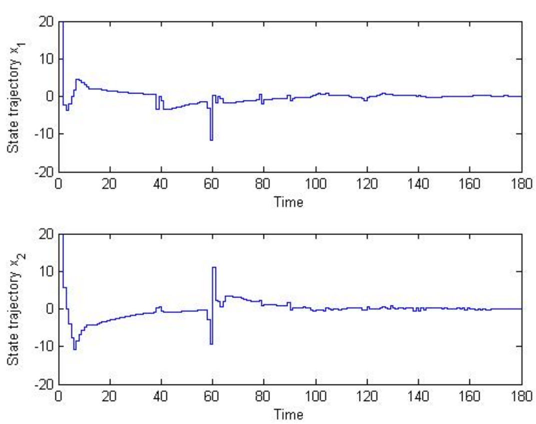


Fig. 2. State trajectories

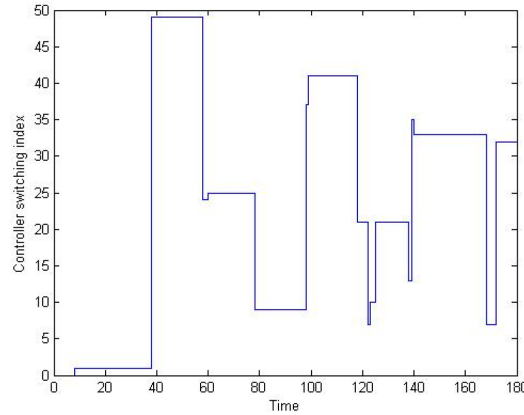


Fig. 3. Controller index at switching instances

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