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Optimal Risk-sensitive Filtering and Control for Linear Stochastic Systems

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Abstract— **The optimal exponential-quadratic control problem and exponential mean-square filtering problems are considered for stochastic Gaussian systems with polynomial first degree drift terms and intensity parameters multiplying diffusion terms in the state and observations equations. The closed-form optimal control and filtering algorithms are obtained using quadratic value functions as solutions to the corresponding Hamilton-Jacobi-Bellman equations. The performance of the obtained risk-sensitive regulator and filter for stochastic first degree polynomial systems is verified in a numerical example against the conventional linear-quadratic regulator and Kalman-Bucy filter, through comparing the exponential-quadratic and exponential mean-square criteria values. The simulation results reveal strong advantages in favor of the designed risk-sensitive algorithms in regard to the final criteria values.**

I. INTRODUCTION

After the optimal linear stochastic control problem was solved (see [1], [2]), the optimal control theory for nonlinear stochastic systems is based on dynamic programming (Hamilton-Jacobi-Bellman) equation [2] and the maximum principle of Pontryagin [3]. A long tradition of the optimal control design was developed for nonlinear systems with respect to a quadratic Bolza-Meyer criterion (see, for example, [4]-[10], [11]). The optimal control problems with respect to nontraditional criteria were also considered: the stochastic linear exponential-quadratic regulator (LEQR) problem was introduced in [12]. Further connection between the LEQR problem and $H_\infty-$control via a minimum entropy principle was given in [13]. Whittle ([14], [15]) considered problems on a finite-time horizon, using "small-noise" asymptotics. When the process being controlled is governed by stochastic differential equation, the Whittle's formula for the optimal large-derivations rate was obtained using partial differential equation viscosity solution method in [16], [17], [18], [19]. Runolfsson [20], [21] used Ponsker-Varadham-type large-derivations ideas to obtain a corresponding stochastic differential game for which the game payoff is an ergodic (expected average cost per unit time) criterion. In [22], [23], [18], and [19] the risk-averse LEQR optimal control problem for a stochastic system with white Gaussian noises whose intensities depend on parameters was stated and solved using a value function, which is a viscosity solution to the dynamic programming equation (HJB). An advantage of risk-sensitive criteria is the robustness of the obtained solution with respect to noise level. Indeed, since the solution to the classical LQ problem is independent of noise level, it occurs to be too sensitive to parameter variations in noise intensity. On the other hand, the risk-sensitive problem assumes explicit presence of the small parameters in the criteria. This leads to a more robust solution, which correctly responds to parameter variations and results in close criterion values for both, large and small, parameter values. The optimal mean-square filtering theory was initiated by Kalman and Bucy for linear stochastic systems, and then continued for nonlinear systems in a variety of papers (see for example [24] - [28], [29] and for systems with delays see [30]). More than thirty years ago, Mortensen [31] introduced a deterministic filter model which provides an alternative to stochastic filtering theory. In this model, errors in the state dynamics and the observations are modeled as deterministic "disturbance functions," and a mean-square disturbance error criterion is to be minimized. Special conditions are given for the existence, continuity and boundedness of a drift $f(x)$ in the state equation and a linear function $h(x)$ in the observation one. A concept of the stochastic risk-sensitive estimator, introduced more recently by McEneaney [32], in regard to a dynamic system including nonlinear drift $f(x)$, linear observations, and intensity parameters multiplying diffusion terms in both, state and observation, equations. Again, the exponential mean-square (EMS) criterion, introduced in [33] for deterministic systems and in [22] for stochastic ones, is used instead of the conventional mean-square criterion to provide a robust estimate, which is less sensitive parameter variations in noise intensity. This paper presents the explicit closed-form solutions to the optimal exponential-quadratic control problem and exponential mean-square filtering problems for stochastic first degree polynomial (affine) systems including intensity parameters multiplying diffusion terms in both, state and observation, equations. The optimal control and filtering algorithms are derived seeking quadratic value functions as solutions to the corresponding Hamilton-Jacobi-Bellman equations in both problems. Undefined parameters in the value functions are calculated through ordinary differential equations composed by collecting terms corresponding to each power of the state-dependent polynomial in each of the HJB equations. The closed-form risk-sensitive regulator and filter equations are explicitly obtained in the control and filtering problems. The performance of the obtained risk-sensitive regulator and filter for stochastic first degree polynomial systems is

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verified in a numerical example against the conventional linear-quadratic regulator and Kalman-Bucy filter, through comparing the exponential-quadratic and exponential mean-square criteria values for both regulators and both filters, respectively. The simulation results reveal strong advantages in favor of the designed risk-sensitive algorithms in regard to the final criteria values uniformly for all considered values of the intensity parameters multiplying diffusion terms in state and observation equations. Tables of the criteria values and simulation graphs are included.

This paper is organized as follows. The optimal risk-sensitive stochastic control problem for linear polynomial systems with an exponential-quadratic criterion is stated in Section 2, and Section 3 gives the optimal solution. The dual filtering problem for linear polynomial systems with an exponential mean-square criterion is stated in Section 4, and Section 5 provides the optimal solution. A numerical example is simulated for the risk-sensitive and L-Q optimal control algorithms and the risk-sensitive and Kalman-Bucy filtering algorithms in Section 6. Section 7 presents conclusions to this study.

II. Optimal Risk-Sensitive Stochastic Control Problem

The following stochastic risk-sensitive problem is given with state dynamics:

$$\begin{aligned} dX_t &= f(t, X_t, u_t)dt + \sqrt{\frac{\epsilon}{2\gamma^2}}dW_t \qquad (1)\\ X_s &= x, \end{aligned}$$

with the exponential-quadratic cost criterion

$$I(s, X, u) = \epsilon log E_{s,X}\{exp\{\frac{1}{\epsilon}[\int_s^T L(t, X_t, u_t)dt + \psi(X_T)]\}. \qquad (2)$$

where $X_t = X(t)$ is the state at time t, $X_t \in \mathbf{R}^n$, X is the initial state at time $s \geq 0$, $f(t, X_t, u_t)$ is a nonlinear function, which represents the nominal dynamics with control u_t taking values in $U \in \mathbf{R}^l$ and $\{W, F\}$ is an m-dimensional Brownian motion on the probability space (Ω, F, P) The parameter ϵ is a measure of the risk-sensitivity and scales the diffusion term in (1) above so that the cost remains bounded (for each X as a function of ϵ), $0 \leq s \leq T < \infty, T$ is a fixed terminal time, $L(t, X_t, u_t)$ is the quadratic running cost, and $\psi(X_T)$ is the quadratic terminal cost. Define:

$$A(s, X, u, \omega) = \int_s^T L(t, X_t, u)dt + \psi(X_T),$$

and

$$J(s, X, u) = E_{s,X} exp[\frac{1}{\epsilon}A(s, X, u, \omega)], \qquad (3)$$

so that

$$I(s, X, u) = \epsilon log J(s, X, u) = \epsilon log E_{s,X}\{exp[\frac{1}{\epsilon}A(s, X, u, \omega)]\}$$

Taking into account that the controller u_t is minimizing, the following value function is considered:

$$V(s, X) = inf_{u \in A_{s,\nu}} I(s, X, u) \qquad (4)$$

where $A_{s,v}$ is the set of progressively measurable controls with values in U. It is shown in [23] that under certain conditions, if $f(t, X_t, u_t)$ is a nonlinear function, V is a viscosity solution of the dynamical programming equation

$$\begin{aligned} 0 &= V_s + \frac{\epsilon}{2\gamma^2}\sum V_{X_iX_j} + min_{u\in U}\{f(t, X_t, u_t) \times \\ &\quad \nabla_x V + L(t, X_t, u_t) + \frac{1}{2\gamma^2}\nabla V^T \nabla V\} \qquad (5)\\ V(X_t, T) &= \psi(X) \end{aligned}$$

This paper shows that if $f(t, X_t, u_t) = A_t + A_{1t}X_t + B_t u_t$, a viscosity solution V of the dynamical programming equation (5) can be explicitly found. The optimal control problem is to find explicitly a viscosity solution V to the dynamic programming equation (5) when $f(t, X_t, u_t)$ is linear, and to find the optimal control which minimizes the exponential-quadratic criterion I and the optimal trajectory X^*, substituting u^* into the state equation. The conditions cited in [23] for f, L, ψ, U are supposed throughout the paper, which remain true for $f(t, X_t, u_t) = A_t + A_{1t}X_t + B_t u_t$. U is a compact subset of $\mathbf{R}^n$.

As in [23], first consider the "cut off" problem, where the possibly unbounded functions f, L and ψ are replaced by bounded counterparts. We obtain analogous results for this "cut off" problem and then take a limit to obtain the desired result. It is proved [23] that V^k is the unique, bounded, classical solution to (5), considering that $f(t, X_t, u_t)$ is nonlinear.

III. Optimal Risk-Sensitive Regulator

Taking into account that $f(t, X_t, u_t) = A + A_{1t}X_t + B_t u_t$ and substituting it in (1), the following state equation is obtained:

$$\begin{aligned} dX_t &= (A_t + A_{1t}X_t + B_t u_t)dt + \sqrt{\frac{\epsilon}{2\gamma^2}}dW_t, \qquad (6)\\ X_s &= x, \end{aligned}$$

where $X_t, \quad A_t \in \mathbf{R}^n, A_{1t} \in M_{nxn}$, M_{nxn} denotes the field of matrices of dimension nxn, and W_t is as in (1). If $L(t, X_t^\epsilon, u) = X_t^T G X_t + u_t^T R u_t$, the exponential-quadratic cost criterion has the form:

$$\begin{aligned} I^k(s, X, u) &= \epsilon log E_{s,X}\{exp[\frac{1}{\epsilon}[\int_s^T (X_t^T G X_t + \qquad (7)\\ &\quad u_t^T R u_t)dt + X_T^T \psi X_T]]\} \end{aligned}$$

Theorem 1: The solution to the stochastic control problem for the dynamical system (6) with criterion (7) takes the form:

$$\begin{aligned} &\dot{P} = P^T(-\frac{B_t B_t^T}{2} - \frac{1}{\gamma^2})P - A_{1t}^T P - \qquad (8)\\ &P_s A_{1t} - 2G, \quad \dot{C} = A_{1t}^T C + 2C^T G P^{-1} + A_t \end{aligned}$$

with terminal conditions: $P(T) = \psi, C(T) = 0$. The optimal control law that minimizes the exponential-quadratic criterion (7) is given by:

$$u_t^* = -\frac{1}{2}PB_t^T R^{-1}(X - C) \qquad (9)$$

Proof: The value function is proposed:

$$V(s, X) = \frac{1}{2}(X_t - C)^T P(X_t - C) + r. \qquad (10)$$

(C, P, r are functions of $s \in [0,T], C \in \mathbf{R}^n, P$ is a symmetric matrix of dimension $n \times n$ and r is a scalar function) as a viscosity solution of the PDE:

$$0 = V_s + \frac{\epsilon}{2\gamma^2}\sum V_{x_i x_j} + min_{u\in U}\{(A + A_{1t}X_t + B_t u_t)\nabla_x V + X_t^T G X_t + u_t^T R u_t + \frac{1}{2\gamma^2}\nabla V^T \nabla V\}, V(X_t,T) = X_T^T \psi X_T, \tag{11}$$

where V_s, V_x are the partial derivatives of V with respect to s, x, respectively, G, ψ are non-negative symmetric matrices, R is a positive definite symmetric matrix, and ∇V is the gradient of V. Then, the partial derivatives of V are given by:

$$\begin{aligned} V_s &= \frac{1}{2}(X-C)^T \dot{P}(X-C) + \dot{r} - \frac{1}{2}\dot{C}^T P(X - C) - \frac{1}{2}(X-C)^T P\dot{C} \qquad (12)\\ V_x &= \frac{1}{2}P(X-C) + \frac{1}{2}(X-C)^T P,\\ V_{xx} &= P \end{aligned}$$

Substituting (12) to the HJB-PDE (11); yields

$$\begin{aligned} 0 = {} & \frac{1}{2}(X-C)^T \dot{P}(X-C) + \dot{r} - \frac{1}{2}(X-C)^T \times P\dot{C} - \frac{1}{2}\dot{C}^T P(X-C) + \frac{\epsilon}{2\gamma^2}\sum P + (A + A_1 X)P(X-C) - \frac{1}{4}(X-C)^T P^T (B_t R^{-1} \times B_t^T)P(X-C) + X_t^T G X_t + \frac{1}{2\gamma^2}(X - C)^T P^2 (X-C). \end{aligned} \tag{13}$$

Collecting the second degree terms, the first equation of (8) is obtained. Collecting the first degree terms the second equation of (8) is obtained. Doing the same for independent terms of X, the following equation is obtained. $\dot{r} = -C^T G C - \frac{\epsilon}{2\gamma^2}\sum_{i,j=1}^n p_{ij}$, where p_{ij} are the elements of the symmetric matrix P.

The optimal control law (9) that minimizes the exponential-quadratic criterion (7) is obtained from: $min_{u\in U}\{f^k(t, X_t, u_t)\nabla_x V + L^k(t, X_t, u_t) + \frac{1}{2\gamma^2}\nabla V^T \nabla V\} \diamond$.

IV. Optimal Risk-Sensitive Filtering Problem.

Consider the following stochastic model, X_t satisfies the diffusion model given by:

$$dX_t = f(X_t)dt + \sqrt{\frac{\epsilon}{2\gamma^2}}dW_t \tag{14}$$

where $f(X_t)$ represents the nominal dynamics, and W is a Brownian motion,and the observation process Y_t satisfies the equation:

$$dY_t = h(X_t)dt + \sqrt{\frac{\epsilon}{2\gamma^2}}d\tilde{W}_t, \quad Y_0 = 0, \tag{15}$$

here, ϵ is a parameter and W and $\tilde{W}$ are independent Brownian motions, which are also independent of the initial state X_0. X_0 has probability density $k_\epsilon exp(-\epsilon^{-1}\phi(x_0))$ for a certain constant k_ϵ. The mean-square cost criterion to be minimized is given by:

$$J = \epsilon log E\{\exp\frac{1}{\epsilon}\int_0^T H(X_t, m_t, t)dt/Y_t\}, \tag{16}$$

where $H(X_t, m_t, t) = e^T h e$ and $e = (X_t - m_t)$, m_t is the estimate of the state X_t. In the rest of the paper the assumptions (A1)-(A4) from [18] hold. Let $q(T,x)$ denote the unnormalized conditional density of X_T, given observations Y_t for $0 \le t \le T$. It satisfies the Zakai stochastic PDE, in a sense made precise, for instance in [7], sec. 7. Since the normalizing constant k_ϵ above is unimportant for q, it is assumed that

$$\begin{aligned} q(0,x) &= exp(-\epsilon^{-1}\phi(x)) \qquad (17)\\ q(s,x) &= p(s,x)exp[\epsilon^{-1}Y_t \cdot h(x)] \end{aligned}$$

where $p(s,x)$ is called pathwise unnormalized filter density. Then p, satisfies the following linear second-order parabolic PDE with coefficients depending on Y_T.

$$\frac{\partial p}{\partial s} = (L(s))^* p + \frac{K}{\epsilon}p,$$

where, for every $g \in \mathbf{R}^n$, let

$$\begin{aligned} Lg &= \frac{\epsilon}{2}tr(g_{xx}) + f \cdot g_x, \qquad (18)\\ K(t,x) &= \frac{1}{2}(Y_t \cdot h)_x \cdot (Y_t \cdot h)_x - L(Y_t \cdot h) - \frac{1}{2}|h|^2, \end{aligned}$$

and L denote the differential generator of the Markov diffusion X_t in (14). By assumptions (A1) and (A3) in [18], K is bounded and continuous. Since $Y_0 = 0, p(0,x) = q(0,x)$. The initial condition for (18) is given by (17). We rewrite (18) as follows:

$$\frac{\partial p}{\partial s} = \frac{1}{2}tr(p_{xx}) + A \cdot p_x + \frac{B}{\epsilon}p, \tag{19}$$

where

$$\begin{aligned} A &= -f(x) + (Y_t \cdot h(x))_x \qquad (20)\\ B(t,x) &= -\epsilon div[f(x) - (Y_t \cdot h(x))_x] + K(t,x) \end{aligned}$$

Taking log transform: $Z(T,x) = \epsilon log p(T,x)$, the nonlinear parabolic PDE is obtained

$$\frac{\partial Z}{\partial s} = \frac{\epsilon}{2}tr(Z_{xx}) + A \cdot Z_x + \frac{1}{2}Z_x \cdot Z_x + B, \tag{21}$$

with initial data $Z_x(0,x) = -\phi(x)$. The risk-sensitive optimal filter problem consists in finding the estimate C_T of the state x_t, verifying that

$$Z(s,x) = \frac{1}{2}(x-C)^T Q(x-C) + \rho - Y_t \cdot h(x_t), \tag{22}$$

is a viscosity solution of (21). The notation for all the variables is $X(t) = X_t, x_t \in R^n, w_t \in R^m, y_t, v_t \in R^p, f, h \in R^n$ where f_x, h_x are assumed bounded. Here, h_x is the matrix of partial derivatives of h. The same notation holds for Z_x.

A. Risk-sensitive optimal filter

Taking $f(X_t) = A_t + A_{1t}X_t$, $h(X_t) = E_t + E_{1t}X_t$, with $A_t \in R^n, A_{1t}, \in M_{n\times n}, E_t \in R^p, E_{1t} \in M_{n\times p}$ where $M_{i\times j}$ denotes the field of matrices of dimension $i \times j$. The following stochastic equations system is obtained:

$$\begin{aligned} dX_t &= A_t + A_{1t}X_t + \sqrt{\tilde{\epsilon}}dW_t \\ dY_t &= E_t + E_{1t}X_t + \sqrt{\tilde{\epsilon}}d\tilde{W}_t, \end{aligned} \quad (23)$$

where $\tilde{\epsilon} = \frac{\epsilon}{2\gamma^2}$.

Theorem 2: The solution to the filtering problem, for the system (23) with mean-square criterion (16) takes the form:

$$\begin{aligned} \dot{C} &= A_t + A_{1t}^T C - Q^{-1}E_1(dy - E_{1t}C - E_t), \\ \dot{Q} &= -A_{1t}^T Q - QA_{1t} + Q^T Q - E_{1t}^T E_{1t}. \end{aligned} \quad (24)$$

Proof: The value function is proposed: $Z(s, X) = \frac{1}{2}(X_t - C)^T Q(X_t - C) + \rho - Y_t \cdot (E_t + E_{1t}x_t)$, $Z_x(0, X) = -\phi(X)$, (C, Q, ρ are functions of $s \in [0, T], C \in \mathbf{R}^n, Q$ is a symmetric matrix of dimension $n \times n$ and ρ is a scalar function) as a viscosity solution of the nonlinear parabolic PDE (21), where Z_x, Z_{xx} are the partial derivatives of Z respect to x, and ∇Z is the gradient of Z. Then the partial derivatives of Z are given by:

$$\begin{aligned} \frac{\partial Z}{\partial s} &= \frac{1}{2}(X_t - C)^T \dot{Q}(X_t - C) + \dot{\rho} - \frac{1}{2}\dot{C}^T Q(X_t \\ &\quad -C) - \frac{1}{2}(X_t - C)^T Q\dot{C} - dY_t \cdot (E_t + E_{1t}X_t) \\ \frac{\partial Z}{\partial x} &= \frac{1}{2}Q(X - C) + \frac{1}{2}(X - C)^T Q - Y_t E_{1t}, \\ \frac{\partial^2 Z}{\partial x \partial x} &= Q. \end{aligned} \quad (25)$$

Substituting (25) and the expressions for A, B in (21), collecting the second degree terms, equalizing them to zero, and doing it again for the terms with coefficient of first degree, the filtering equations (24) are obtained. Similarly to the case of the risk-sensitive control, collecting the independents terms, the equation for ρ is obtained. $\diamond$

Here Q_T is a symmetric negative definite matrix, and the initial condition $Q_0 = q_0$ is derived from initial conditions for Z. If $\phi(X_t) = X_t^T K X_t$, $Q(0) = -K$.

V. EXAMPLE

A. Risk-sensitive optimal stochastic control

Consider the following linear stochastic state equation:

$$\begin{aligned} dX_{1t} &= X_{2t}dt + \sqrt{\frac{\epsilon}{2\gamma^2}}dW_{1t}, \\ dX_{2t} &= 1 + u_{1t}dt + \sqrt{\frac{\epsilon}{2\gamma^2}}dW_{2t}, \end{aligned} \quad (26)$$

where $A_t \in R^2$, $A_{1t} \in M_{2x2}$, $\gamma = 2$, $X_0 = x_o$ The quadratic cost criterion takes the form:

$$\begin{aligned} I(X_t, u_t) &= \epsilon log E_{s,X}\{exp(\frac{1}{\epsilon}\int_0^T (X_t^T G X_t + u_t^T R u_t)dt \\ &\quad +X_T^T \psi X_T)\} \end{aligned} \quad (27)$$

We suppose that there exists solution $V(s, X)$ of (11) given by: $V(s, X) = \frac{1}{2}(X - C)^T P(X - C) + r$. Substituting the values of A_t, A_{1t} into the equations (8) and (9), the following equations for the risk-sensitive optimal control are obtained, where p_{ij} are the components of the P matrix, and C_i are the components of the vector C.

$$\begin{aligned} \frac{dp_{11}}{dt} &= -2 + (p_{11}^2 + p_{12}^2)(\frac{1}{2} - \frac{1}{\gamma^2}), \\ \frac{dp_{12}}{dt} &= -p_{11}(p_{11}p_{12} + p_{12}p_{22})(\frac{1}{2} - \frac{1}{\gamma^2}), \\ \frac{dp_{22}}{dt} &= -2 + (p_{11}^2 + p_{22}^2)(\frac{1}{2} - \frac{1}{\gamma^2}) - 2p_{12}, \\ \frac{dC_1}{dt} &= 2\frac{C_1 p_{22} - C_2 p_{11}}{p_{22}p_{11} - p_{12}^2}, \\ \frac{dC_2}{dt} &= 1 + C_1 + 2\frac{C_2 p_{11} - C_1 p_{21}}{p_{22}p_{11} - p_{12}^2}, \\ u_1^* &= -\frac{1}{2}(p_{12}(X_1 - C_1) + p_{22}(X_2 - C_2)), u_2^* = 0. \end{aligned} \quad (28)$$

With terminal conditions: $p_{11}(0.5) = 1$, $p_{12}(0.5) = 0$, $p_{22}(0.5) = 1$, $C_1(0.5) = 0$, $C_2(0.5) = 0$. The system (28), is stable, if $|\gamma| \geq 1.40$. Solving this system of equations (28), we can obtain the values of the optimal control law u^* and the optimal value of X^*, as the solution of the equation:

$$\begin{aligned} dX_t &= (A_t + A_{1t}X_t - \frac{1}{2}B_t P_t B_t^T R^{-1}(X - C))dt + \\ &\quad \sqrt{(\epsilon/2\gamma^2)}dW_t. \end{aligned} \quad (29)$$

The initial conditions are given by $X(0) = 0$. The value of the exponential-quadratic criterion to be minimized is obtained through Monte Carlo method for time $T = 0.5$ The simulation is made in MatLab7.

B. Linear quadratic stochastic control

The optimal linear quadratic control takes the form: $u^*(t) = -R^{-1}B^T(Q(t)X(t)+p)$, where $Q(t)$ is the solution of the gain equation:

$$\frac{dQ}{dt} = -Q(t)A_t - A_t^T Q(t) + L - Q(t)BR^{-1}B^T Q, \quad (30)$$

$Q(T) = I$ and p is the solution of the differential equation:

$$\dot{p} = -Q(t)A_{1t} - (p^T A_t)^T - Q(t)BR^{-1}B^T p; \; p(T) = 0, \quad (31)$$

Taking into account the state equations (26), the following equations for the components of the gain matrix Q and the vector p are obtained:

$$\begin{aligned} \dot{q}_{11} &= 2 - \frac{q_{12}^2}{2} \\ \dot{q}_{12} &= -q_{11} - \frac{q_{12}q_{22}}{2} \\ \dot{q}_{22} &= 2 - \frac{q_{22}^2}{2} - 2q_{12} \\ \dot{p}_1 &= -q_{11} - \frac{q_{12}p_2}{2} \\ \dot{p}_2 &= -p_1 - q_{12} - \frac{q_{22}p_2}{2} \\ u_1^* &= -q_{21}X_1 - q_{22}X_2 - p_2 \\ u_2^* &= 0. \end{aligned} \quad (32)$$

With terminal conditions: $q_{11}(0.5) = -2$, $q_{12}(0.5) = 0, q_{22}(0.5) = -2$, $p_1(0.5) = 0$, $p_2(0.5) = 0$ The optimal trajectory satisfies the equation: $dX_t = (A_t + A_{1t}X_t - B_t R^{-1}B_t^T(Q(t)X(t) + p))dt + \sqrt{\frac{\epsilon}{2\gamma^2}}dW_t$. The quadratic criterion is the same as in the risk- sensitive optimal control problem. The results of the simulation show better performance for the linear exponential-quadratic control for all

ϵ	I(r-s control)	I(trad. control)
0.01	0.4055	2.0913
0.1	0.4051	2.0835
1	0.4038	2.059
10	0.401	116.9832
100	0.4048	1.7597
1000	0.5436	1.2167

TABLE I

COMPARISON OF EXPONENTIAL QUADRATIC CRITERION (16) FOR R-S AND LQ CONTROL

values of ϵ. The graphs of the state, the optimal control, the criterion I for both cases can be observed in Figures 1 and 2.

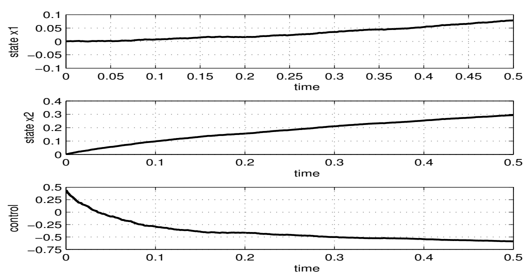

Fig. 1. Graphs of the optimal state variable x_t, optimal control u^* and exponential-quadratic criterion I for traditional L-Q control, with $\epsilon = 1, \gamma = 2$.

Table 1 presents values of the exponential-quadratic criterion to be minimized for different values of ϵ, comparing the equations of risk-sensitive control and LQ traditional regulator.

C. Risk-sensitive optimal filter

For the dynamical system (23), if $x_t \in R^2,\ y_t \in R,\ u_t \in R$, the following stochastic state and output equations are considered:

$$\begin{aligned} dX_{1t} &= (1 - X_1 + X_2)dt + \sqrt{\epsilon}dW_1, \quad (33)\\ dX_2 &= -X_2dt + \sqrt{\epsilon}dW_2, X_{i0} = x_i \\ dY_t &= X_{1t}dt + \sqrt{\epsilon}d\bar{W}_t \end{aligned}$$

where $W_1,\ W_2,\ d\bar{W}_t$ are independent Brownian motions, which are also independents of $x_{i0} = x_{io}$. ϵ is a varing parameter. Proposing (22) as a viscosity solution of (21), getting the derivatives $Z_x,\ Z_{xx},\ \frac{\partial Z}{\partial T}$ of (22), and substituting them into (21), the following equations are obtained for the estimate C_T and the symmetric matrix Q_T, upon substituting the corresponding values into (24):

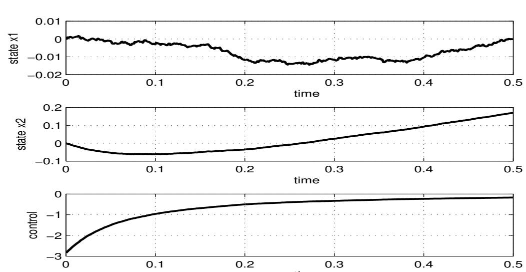

Fig. 2. Graphs of the state variable x_t, optimal control u^*, and exponential-quadratic criterion I for risk-sensitive control, with $\epsilon = 1, \gamma = 2$.

$$\begin{aligned} \dot{C}_1 &= (1 - C_1 + C_2) - \frac{q_{22}}{q_{22}q_{11} - q_{12}^2}(\dot{Y}_t - C_1) \\ \dot{C}_2 &= -C_2 - \frac{q_{12}}{q_{22}q_{11} - q_{12}^2}(\dot{Y}_t - C_1), \quad (34) \end{aligned}$$

where q_{12}, q_{22}, q_{11} are the solutions of the following Riccati matrix equation :

$$\begin{aligned} \dot{q_{11}}_T &= 2q_{11} - 2q_{12} + q_{11}^2 + q_{12}^2 - 1 \quad (35) \\ \dot{q_{12}}_T &= 2q_{12} - q_{22} + q_{11}q_{12} + q_{12}q_{22} \\ \dot{q_{22}}_T &= 2q_{22} + q_{12}^2 + q_{22}^2 \end{aligned}$$

The last equations (34) are simulated using Simulink in $MatLab7$. The initial conditions for the simulation are $x_0 = 0,\ q_{11}(0) = -2.9,\ q_{12}(0) = -1.7598,\ q_{22}(0) = -2,\ C_1 = 10,\ C_2 = 10,\ T = 5$. The graphs of the difference between the state x_t, and the estimate C_T, that is, $e_i = |x_i - C_i|$, for $i = 1, 2$ are shown in Figure 3.

D. Kalman-Bucy optimal filter.

Applying the Kalman-Bucy optimal filter algorithms [34] to the state equations (33), the equations for the estimate vector $m(t)$ and symmetric covariance matrix $P(t)$ are obtained:

$$\begin{aligned} dm_1(t) &= (-m_1(t) + m_2(t) + 1)dt + \\ &\quad p_{21}(dY_t - m_1(t)dt) \\ dm_2(t) &= -m_2(t)dt + p_{12}(dY_t - m_1(t)dt) \\ \dot{p}_{11}(t) &= -2p_{11}(t) + 2p_{12}(t) - \frac{p_{11}^2(t)}{\epsilon} + \epsilon \\ \dot{p}_{12}(t) &= -2p_{12}(t) + p_{22}(t) - \frac{p_{11}(t)p_{12}(t)}{\epsilon} \\ \dot{p}_{22}(t) &= -2p_{22}(t) - \frac{p_{12}^2(t)}{\epsilon} + \epsilon \end{aligned}$$

This system of equations is simulated with the initial conditions: $m_{1,2}(0) = 10,\ p_{11}(0) = 0.73059,\ p_{12}(0) = -0.639269,\ p_{22}(0) = 1.059360$. The graph of the value of the difference between state x_i, and the estimate $m_i(t)$, that is: $e_i = |x_i - m_i|$, for $\hat{\epsilon} = 1$ and $\gamma = 2$ can be observed in Figure 4.

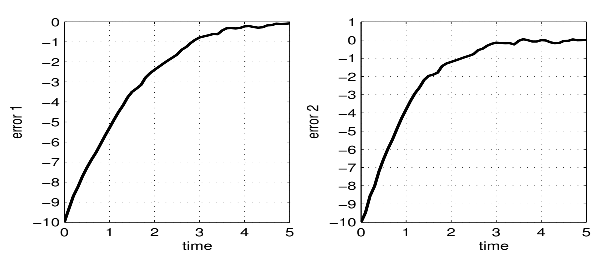

Fig. 3. Graphs of the absolute values of the difference between the state x_t and the risk-sensitive estimate C_T, for $\tilde{\epsilon} = 1$.

Table 2 presents some values of the risk-sensitive and Kalman-Bucy mean-square criterion values, it can be observed that the J_{r-s} values are uniformly less that the J_{K-B} values.

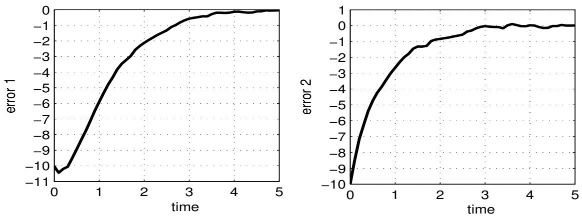

Fig. 4. Graphs of the absolute values of the difference between the x_t and the Kalman-Bucy estimate m, for $\tilde{\epsilon} = 1$.

$\frac{\epsilon}{2\gamma^2}$ *value*	J_{R-S}	J_{K-B}
1	128.2699	129.082
10	116.7305	168.764
100	90.6002	376.683
1000	111.5718	4,165.163

TABLE II

COMPARISON OF MEAN-SQUARE CRITERION (16) FOR R-S FILTER AND K-B FILTER.

VI. CONCLUSIONS

This paper presents the optimal solutions to the risk-sensitive optimal control and filtering problems for stochastic first degree polynomial systems with Gaussian white noises, an exponential-quadratic criterion to be minimized, and intensity parameters multiplying the white noises, using quadratic value functions as solutions to the corresponding Hamilton-Jacobi-Bellman equations. Numerical simulations are conducted for both cases to compare performance of the obtained risk-sensitive regulator and filter algorithms against the conventional linear-quadratic regulator and Kalman-Bucy filter, through comparing the exponential-quadratic and exponential mean-square criteria values. The simulation results reveal strong advantages in favor of the designed risk-sensitive algorithms in regard to the final criteria values uniformly for all considered values of the intensity parameters multiplying diffusion terms in state and observation equations. The tables of the criteria values in both cases are included.

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