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Stability of a multi-diamond type family of quasipolynomials

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Abstract— **A reduced stability-testing set for a diamond-like quasipolynomial family is obtained. This family is a generalization of a multivariate diamond-like family, and is the extension to the case of multiple uncertainty bounds. Both delay dependent and delay independent stability conditions are given, based on a finite set of vertices and a set of edges, respectively.**

I. Introduction

In this work the problem of determining the robust stability of linear time-delay systems subjected to parametric uncertainty is adressed in the following way: given an infinite family of quasipolynomials, find a reduced (possibly finite) subset whose stability is equivalent to that of the whole family. This subset is called the testing set. The results for interval monovariate polynomials given in [11], triggered the search for extreme point stability results for several types of dynamical systems, as can be seen in [2]. The vertex results were extended for diamond families of univariate polynomials in [1]. Reduced testing set for multivariate polynomials have been found in [3], [6], [14] . In the case of quasipolynomials, a general result on polytopic families is given in [9], and further criteria for reduction of the testing set were given in [12], [7], [18], but no vertex type results were obtained before [17], where this sort of results were given for an interval-like family of quasipolynomials, and [8] where similar results are stated under additional assumptions on coefficients. The results in [19] attacks a case where the underlying uncertainty structure is the diamond.

In the remainder of this section, the terminology and preliminary results are given. In section II, the polynomial family with multi-diamond uncertainty structure is introduced, followed by the associated quasipolynomial family. Then, stability results are given. A brief example is given in section III, where application of Theorem 1 is ilustrated. In section IV conclusion are given.

A. Preliminaries

1) Time delay systems: The time-delay systems which are studied are of the form

$$\frac{d}{dt}\left[\sum_{\nu=0}^{l} C_\nu x(t-h_\nu)\right] = \sum_{\nu=0}^{l} A_\nu x(t-h_\nu),\ C_0 = E, \quad (1)$$

where $x(t)$ is an n-dimensional vector, E is the identity matrix and A_ν, C_ν, $\nu = 0, 1, \ldots, l$ are real $n \times n$ matrices.

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Definition 1: [4] The system (1) is said to be *exponentially stable* if there exist constants $\alpha > 0$ and $\gamma > 1$ such that the inequality

$$\|x(t)\| \le \gamma \|\varphi\|_H \, e^{-\alpha t} \quad (2)$$

holds for all $t \ge 0$.
In (2), $\|\varphi\|_H = \sup_{\theta\in[-H,0]} \|x(\theta)\|$ is the norm of the *intial condition* of (1), defined as $\varphi(\theta) \in \mathbb{R}^n$ for $-H \le \theta \le 0$, $\|\cdot\|$ is the euclidian norm and $H = \max_{\nu\in\{1,\ldots,l\}}\{h_\nu\}$.

2) Quasipolynomials: The characteristic equation of (1) is

$$f(s) = \det\left(\sum_{\nu=0}^{l}[C_\nu s - A_\nu]e^{-h_\nu s}\right) = 0 \quad (3)$$

The left-hand side of (3) is an entire function called quasipolynomial, which is of the form

$$f(s) = \sum_{i=0}^{n}\sum_{k=0}^{m} a_{ik} s^{n-i} e^{-\tau_k s}, \quad (4)$$

where a_{ik} are the real coefficients, and $0 = \tau_0 < \tau_1 < \cdots < \tau_m$ are shifts which depend on the delays h_ν, $\nu = 1, \ldots, l$. The function (4) has an infinite number of roots (excepting the case $m = 0$), which are distributed through logarithmic chains on the complex plane [4]: if one of these chains goes deeply into the right half plane, the quasipolynomial is said to be of the advanced type, if it has no such chain, but has a vertical strip containing an infinite number of roots, it is of neutral type, and if it has no advanced chains and no such vertical strip, but all of its roots lie in a left half plane, then it is said to be of the retarded type.

Definition 2: The quasipolynomial (4) is said to be *stable* if there exists a number $\sigma > 0$ such that all of its roots have real parts less than $(-\sigma)$.
It is easily seen from Definition 2 that only the retarded and neutral types of quasipolynomials may be stable. The system (1) is stable if and only if the quasipolynomial (4) is stable.

Remark 1: Unlike the case of univariate polynomials, there may be unstable time delay systems corresponding to quasipolynomials with all of their roots on the left half-plane, this instability is due to the presence of chains of roots which asymptotically approach the imaginary axis. This may occur when the quasipolynomial is of neutral type, so special care should be taken when analizing the stability of these quasipolynomials.
The coefficients of the quasipolynomial (4) may be grouped in a *coefficient vector* $\mathbf{a} = (a_{00}, a_{01}, \ldots, a_{nm})$, such that the quasipolynomial (4) is expressed as $f(s) = f(s, \mathbf{a})$. Given a compact, pathwise connected set $\mathrm{Q} \subset \mathbb{R}^M$, where $M = (1+n)(1+m)$, a family of quasipolynomials is defined as the

set $\mathcal{F} = \{f(s, \mathbf{a}) | \mathbf{a} \in \mathrm{Q}\}$. The family $\mathcal{F}$ is said to be stable if it consist only of stable quasipolynomials. Given the complex number $s^{(0)}$, the *value set* of the quasipolynomial family $\mathcal{F}$ evauated at $s^{(0)}$ is the set $\mathcal{V}_{\mathcal{F}}(s^{(0)}) = \{f(s^{(0)}, \mathbf{a}) | \mathbf{a} \in \mathrm{Q}\} \subset \mathbb{C}$. The value set is a key concept on robust stability.

Lemma 1: [12] Let $\mathcal{F}$ be a quasipolynomial family with at least one stable member. Assume that, for at least one real number $\omega^{(0)}$, the value set of the family does not contain the origin, and the boundary $\partial \mathcal{V}_{\mathcal{F}}(j\omega)$ does not intersect the origin for all $\omega \in \mathbb{R}$. Then, all the members of the family $\mathcal{F}$ are stable polynomials.

Next definition is an extension to the convex stability direction, as defined in [23], to quasipolynomials

Definition 3: [12] A quasipolynomial $g(s)$ is said to be a quasipolynomial convex direction if, for every stable quasipolynomial such that for all $\lambda \in [0, 1]$, the quasipolynomial $f(s) + \lambda g(s)$ has degree n and the fixed maximum τ_0 and minimum τ_m shifts, the stability of $f(s) + g(s)$implies the stability of the whole segment $\{f(s) + \lambda g(s) \mid \lambda \in [0, 1]\}$. A characterization of the convex direction property is given in [12]

Lemma 2: [12] The real quasipolynomial $g(s)$ is a convex direction if and only if the inequality

$$\frac{\partial \arg\{g(j\omega)\}}{\partial \omega} \leq -\frac{\tau_0 + \tau_m}{2} + \left| \frac{\sin\{2\arg[g(j\omega)] + (\tau_0 + \tau_m)\omega\}}{2\omega} \right| \quad (5)$$

holds for all $\omega > 0$ such that $g(j\omega) \neq 0$.

Note that multiplication by any real or imaginary number does not modify the convex direction property.

3) Multivariate polynomials: An m-variate real polynomial is a function

$$p(\mathbf{s}) = p(s_1, \ldots, s_m) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m} s_1^{k_1} \cdots s_m^{k_m}. \quad (6)$$

A root of the polynomial p is a complex vector $\mathbf{s}^{(0)} = (s_1^{(0)}, \ldots, s_m^{(0)})$ such that $p(\mathbf{s}^{(0)}) = 0$. Unlike univariate polynomials, two coprime multivariate polynomials may have a root in common. The coefficients of the multivariate polynomial (6) may be grouped in the *coefficient vector* $\mathbf{b} = (b_{0\ldots0}, b_{1\ldots0}, \ldots, b_{n_1,\ldots n_m}) \in \mathrm{R}^N$, where $N = (n_1+1)(n_2+1)\cdots(n_m+1)$. In this way, the multivariate polynomial may be expresed both as a function of the independent variables and its coefficient vectors $p(\mathbf{s}) = p(\mathbf{s}, \mathbf{b})$. The polynomial (6) may be expressed like a polynomial of any of its variables s_k, for $k = 1, 2, \ldots, m$:

$$p(\mathbf{s}) = \sum_{i=0}^{n_k} b_i^{(k)}(\ldots, s_{k-1}, s_{k+1}, \ldots) s_k^i. \quad (7)$$

The m polynomials in $(m-1)$ variables $b_{n_k}^{(k)}(\ldots, s_{k-1}, s_{k+1}, \ldots)$ are called *main coefficients*. The *degree* of the multivariate polynomial p is the vector $\deg(p) = (n_1, \ldots, n_m)$, its entries being the highest powers of the variables $s_1, \ldots, s_m$. Given a vector with integer entries $\mathbf{n} = (n_1, \ldots, n_m)$, we define the set of constant degree polynomials

$$\mathbf{P_n} = \{p(\mathbf{s}) | \deg(p) = \mathbf{n}\}. \quad (8)$$

Let us define the region

$$\Gamma_m^{(0)} = \{(s_1, \ldots, s_m) \in \mathbb{C}^m | \mathrm{Re}\, s_k \geq 0,\ k = 1, \ldots, m\}, \quad (9)$$

which includes its distinguished boundary

$$\Omega^{(m)} = \{(s_1, \ldots, s_m) \in \mathbb{C}^m | \mathrm{Re}\, s_k = 0,\ k = 1, \ldots, m\}. \quad (10)$$

A multivariate polynomial p is said to be *strict sense stable* (SSS) if $p(\mathbf{s}) \neq 0$ for any $\mathbf{s} \in \Gamma_m^{(0)}$. A disadvantage of this definition of stability is that, in general, it is not preserved under small parametric changes.

Definition 4: [13] An m-variate polynomial of degree $(n_1, \ldots, n_m)$ is said to be *stable* if

1) For $m = 1$, the polynomial is Hurwitz-stable.
2) For $m > 1$,
   a) The polynomial is strict sense stable (SSS);
   b) The main coefficients $b_{n_k}^{(k)}(\ldots, s_{k-1}, s_{k+1}, \ldots)$ are *stable* polynomials of degree $(\ldots, n_{k-1}, n_{k+1}, \ldots)$.

Note the recursive character of this definition: in order to check for stability of a three variate polynomial, one must check for stability of three bivariate polynomials, which in turn needs the Hurwitz stability checking of two monovariate polynomials. This definition of stable polynomial is robust in the sense that, for each such polynomial, there is neighborhood of its coefficient vector such that the coefficient vectors belonging to this neighborhood correspond to stable polynomials. Note also that part 2-b) of Definition 4 guarantees that the distance of any of the roots of p to the distinguished boundary $\Omega^{(m)}$ is greater than zero [16], as it will be seen later, this has strong implications when analizing neutral quasipolynomials. A familiy of multivariate polynomials is defined as follows $\mathcal{P} = \{\ p(\mathbf{s}, \mathbf{b}) \mid \mathbf{b} \in \mathrm{R}\}$, where R is a compact, pathwise connected set. Given a fixed complex m-dimensional vector $\mathbf{s}^{(\mathbf{0})} = (s_1^{(0)}, \ldots, s_m^{(0)})$, the *value set* of the polynomial family *value set* of the family is the set $\mathcal{V}_{\mathcal{P}}(\mathbf{s}^{(0)}) = \{p(\mathbf{s}^{(0)}, \mathbf{b}) | \mathbf{b} \in \mathrm{R}\}$.

Definition 5: [17] An m-variate polynomial $q(s)$ is said to be convex direction if the following condition holds: given any stable polynomial $p(\mathbf{s}) \in \mathbf{P_n}$ such that $p(\mathbf{s}) + q(\mathbf{s})$ is also a stable polynomial and $p(\mathbf{s}) + \lambda q(\mathbf{s}) \in \mathbf{P_n}$ for all $\lambda \in [0, 1]$, then $p(\mathbf{s}) + \lambda q(\mathbf{s})$ is stable for all $\lambda \in [0, 1]$.

As in the case of quasipolynomials, there exists a generalization of a condition for a polynomial to be a convex direction; however, this is only a sufficient condition.

Lemma 3: [6] A polynomial $q(\mathbf{s})$ is a convex direction for m variate polynomials if the inequality

$$\sum_{k=1}^{m} |\omega_k| \frac{\partial q(j\boldsymbol{\omega})}{\partial \omega_k} \leq \left| \frac{\sin\{2\arg[q(j\boldsymbol{\omega})]\}}{2} \right|, \quad (11)$$

holds for all $\omega_k \in \mathbb{R}$, $k = 1, \ldots, m$, for which $q(j\boldsymbol{\omega}) \neq 0$.

A straightforward calculation shows that polynomials of the forms

$$(1 \pm s_\nu)(\alpha_0 + \alpha_2 s_k^2 + \cdots + \alpha_{2l} s_k^{2l})(s_1^\alpha s_2^\beta \cdots s_m^\gamma),$$
$$(1 \pm s_\nu)(\alpha_1 + \alpha_3 s_k^2 + \cdots + \alpha_{2l+1} s_k^{2l+1})(s_1^\alpha s_2^\beta \cdots s_m^\gamma), \quad (12)$$

or any of their factors, are convex directions.

Lemma 4: (*Zero exclusion principle [15]*) Let $\mathcal{P}$ a constant degree polynomial family with at least one stable element. Then $\mathcal{P}$ is stable if and only if

1) for $m = 1$: the value set $\mathcal{V}_\mathcal{P}(j\omega)$ does not include the origin for all $\omega \in \mathbb{R}$;
2) for $m > 1$:
   - the value set $\mathcal{V}_\mathcal{P}(j\omega_1, \dots, j\omega_m)$ does not include the origin for all $(j\omega_1, \dots, j\omega_m) \in \Omega^{(m)}$,
   - the sub-families of main coefficients $\mathcal{P}_1, \dots, \mathcal{P}_m$ satisfy the conditions of this lemma.

4) Relation between quasipolynomials and multivariate polynomials: Let us assume that there are r positive numbers $\eta_1, \dots, \eta_r$, called *basic delays* such that the system delays can be expressed as linear combination of them, namely

$$h_\nu = \sum_{i=1}^{r} c_{\nu i} \eta_i, \text{ for nonnegative integers } c_{\nu i}. \quad (13)$$

In this way, each quasipolynomial can be written as

$$f(s) = q(e^{-\eta_1 s}, \dots, e^{-\eta_r s}, s)$$
$$= \det \left\{ \sum_{\nu=0}^{l} [C_\nu s - A_\nu] e^{-c_{\nu 1}\eta_1 s} \cdots e^{-c_{\nu r}\eta_r s} \right\}. \quad (14)$$

Now let us introduce a change of variables

$$e^{-\eta_k s} = \frac{1 - s_k}{1 + s_k}, \text{ for } k = 1, \dots, r; \text{ and } s = s_k, \quad (15)$$

then the quasipolynomial $f(s) = q(e^{-\eta_1 s}, \dots, e^{-\eta_r s}, s)$ is transformed as follows

$$\tilde{p}(s_1, \dots, s_{r+1})$$
$$= (1 + s_1)^{k_1} \cdots (1 + s_r)^{k_r} q\left(\tfrac{1-s_1}{1+s_1}, \dots, \tfrac{1-s_r}{1+s_r}, s_{r+1}\right), \quad (16)$$

where $\deg(\tilde{p}) = (k_1, \dots, k_r, n)$. On the other hand, given an m-variate polynomial of the form (6), may originate a quasipolynomial by applying the mapping

$$f(s) = \mathcal{Q}(p(\mathbf{s}))$$
$$= \prod_{k=1}^{r} \left(\frac{1 + e^{-\eta_k s}}{2}\right)^{n_k} p\left(\frac{1 - e^{-\eta_1 s}}{1 + e^{-\eta_1 s}}, \dots, \frac{1 - e^{-\eta_r s}}{1 + e^{-\eta_r s}}, s\right). \quad (17)$$

Not that the maximal and minimal shifts are $\tau_0 = 0$ and $\tau_m \leq \sum_{k=1}^{r} \eta_k n_k$ It turns out that the coefficient vector of the transformed quasipolynomial $f(s) = \mathcal{Q}(p(\mathbf{s}))$ is in one-to-one relation with the coefficient vector of $p(\mathbf{s})$ by a nonsingular matrix with integer coefficients [21]. There is a useful property relating stability of both functions

Lemma 5: [17] If $p(s_1, \dots, s_{r+1})$ is stable in the sense of Definition 4, then the corresponding quasipolynomial $f(s) = \mathcal{Q}(p(s_1, \dots, s_{r+1}))$ is stable in the sense of Definition 2, for all non-negative values of the basic delays $\eta_1, \dots, \eta_r$.
Not that, from (15)-(17), the distance of a root of q to the imaginary axes grows as the distance of the correspondent root of p to the distinguished boundary grows, this allows to include the delay independent stability of neutral quasipolynomials.

5) Families of quasipolynomials and multivariate polynomials: Let us consider a multivariate polynomial family given by

$$\mathcal{P} = \{p(\mathbf{s}, \mathbf{b}) | \, \mathbf{b} \in \mathrm{R}\}, \quad (18)$$

where $\mathrm{R} \subset \mathbb{R}^{N_1}$ with $N_1 = (n_1 + 1)(n_2 + 1) \cdots (n_r + 1)$ is a polytope. One may associate to family $\mathcal{P}$ a family of quasipolynomials by applying the transformation

$$\mathcal{F} = \{\mathcal{Q}(p(\mathbf{s}, \mathbf{b})) \mid p(\mathbf{s}, \mathbf{b}) \in \mathcal{P}\}, \quad (19)$$

which is also a polytopic quasipolynomial family. Consider now the value set of $\mathcal{P}$

$$\mathcal{V}_P(\mathbf{s}^{(0)}) = \left\{ p(\mathbf{s}^{(0)}) \mid p(\mathbf{s}) \in \mathcal{P} \right\}, \quad (20)$$

and the value set of $\mathcal{F}$

$$\mathcal{V}_\mathcal{F}(s^{(0)}) = \left\{ f(s^{(0)}) \mid f(s) \in \mathcal{F} \right\}, \quad (21)$$

where $\mathbf{s}^{(0)} = \left(s_1^{(0)}, \dots, s_{r+1}^{(0)}\right)$ and $s^{(0)}$ are a complex vector and a complex number, respectively. Then

$$\mathcal{V}_\mathcal{F}(s^{(0)}) = \varrho e^{j\varphi} \cdot \mathcal{V}_\mathcal{P}(j\omega_1, \dots, j\omega_{r+1}), \quad (22)$$

where

$$\varrho e^{j\varphi} = \prod_{k=1}^{r} \left(\frac{1 + e^{-j\eta_k \omega}}{2}\right)^{n_k}. \quad (23)$$

The value set $\mathcal{V}_\mathcal{F}(j\omega)$ has the same shape as $\mathcal{V}_\mathcal{P}(j\boldsymbol{\omega})$, but scaled by a factor ϱ and rotated an angle φ.

Remark 2: It is worth to notice that the boundary of $\mathcal{V}_\mathcal{F}(j\omega)$ is directly determined by the boundary of $\mathcal{V}_\mathcal{P}(j\boldsymbol{\omega})$, this implies that $\mathcal{Q}$ tranforms polynomials generating vertices and edges of $\mathcal{V}_\mathcal{P}(j\boldsymbol{\omega})$ into vertices and edges of $\mathcal{V}_\mathcal{F}(j\omega)$

II. Main results

A. Multivariate polynomial family

The polynomial family studied in this work is

$$\mathcal{D} = \left\{ \sum_{\nu=0}^{n_{r+1}} p_\nu(s_1, \dots, s_r) \cdot s_{r+1}^\nu \;\middle|\; p_\nu(s_1, \dots, s_\nu) \in \mathcal{D}_\nu^1, \; \nu = 0, 1, \dots, n_{r+1} \right\}, \quad (24)$$

the $1 + n_{r+1}$ polynomial ordinary diamond families $\mathcal{D}_\nu^1$ are in turn

$$\mathcal{D}_\nu^1 = \left\{ \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} b_{k_1 \cdots k_r \nu} s_1^{k_1} \cdots s_r^{k_r} \;\middle|\; \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \left| b_{k_1 \cdots k_r \nu} - b_{k_1 \cdots \nu}^{(0)} \right| \leq \varrho_\nu \right\}, \quad (25)$$

where $\varrho_\nu > 0$, $\nu = 0, 1, \ldots, n_{r+1}$. The coefficients $b^{(0)}_{k_1\ldots k_r\nu}$ correspond to *central* polynomial of the ν-th diamond

$$p^{(0)}_\nu(s_1,\ldots,s_r) = \sum_{k_1=0}^{n_1}\cdots\sum_{k_r=0}^{n_r} b^{(0)}_{k_1\ldots k_r\nu} s_1^{k_1}\cdots s_r^{k_r}. \quad (26)$$

The shape of each value set $\mathcal{V}_{\mathcal{D}^1_\nu}(j\omega_1,\ldots,j\omega_r)$ is a rhombus with horizontal and vertical axis, and its vertex polynomials are of the form

$$p^{(0)}_\nu(s_1,\ldots,s_r) \pm \varrho_\nu s_\alpha^{k_\alpha} s_\beta^{k_\beta}\cdots s_\gamma^{k_\gamma}, \quad (27)$$

the selection of exponents in (27) depends on the region of $\Omega^{(m)}$ in which the value set $\mathcal{V}^1_{\mathcal{D}_\nu}(j\omega_1,\ldots,j\omega_r)$ is computed. It turns out that the value set $\mathcal{V}_\mathcal{D}(j\omega_1,\ldots,j\omega_{r+1})$ is an eight-sided convex polygon with simmetry axis parallel to the real and the imaginary axis of the complex plane [21]. The structure of $\mathcal{D}$ determines that the value set is the sum of several value sets, each rotated by a multiple of $\frac{\pi}{2}$:

$$\begin{aligned}
&\mathcal{V}_\mathcal{D}(j\omega_1,\ldots,j\omega_{r+1})\\
&\quad= \Bigg\{ \sum_{\nu=0}^{n_{r+1}} p_\nu(j\omega_1,\ldots,j\omega_r)\cdot(j\omega_{r+1})^\nu \\
&\qquad\qquad \Bigg|\ p_\nu(j\omega_1,\ldots,j\omega_r) \in \mathcal{V}_{\mathcal{D}^1_\nu},\ \nu = 0,1,\ldots,n_{r+1}\Bigg\}\\
&\quad= \sum_{\nu=0}^{n_{r+1}} \mathcal{V}_{\mathcal{D}^1_\nu}(j\omega_1,\ldots,j\omega_r)\cdot(j\omega_{r+1})^\nu
\end{aligned} \quad (28)$$

Since those sets are convex, this value set is also convex; furthermore, its vertices are obtained from the sum of vertices of each $\mathcal{V}_\mathcal{D}(j\omega_1,\ldots,j\omega_r)$, i.e., polynomials of the form (27) evaluated at $(s_1,\ldots,s_r) = (j\omega_1,\ldots,j\omega_r)$ and multiplied by a power of $j\omega_{r+1}$. In order to determine whether a certain polynomial produces, when evaluated at $(s_1,\ldots,s_r) = (j\omega_1,\ldots,j\omega_r)$, it is necessary determine: which of the first r components of $(j\omega_1,\ldots,j\omega_r,j\omega_{r+1})$ that are equal or less than 1, let l be this number. Let $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_r)$ the index vector for these first r components with entries defined as

$$\mu_k = \begin{cases} 0 & \text{if } |\omega_k| \le 1 \\ 1 & \text{if } |\omega_k| > 1 \end{cases} \quad (29)$$

There are 2^r regions, each corresponding to an index vector μ. A further clasification follows from the definition of *dominant variable*. Given a $\boldsymbol{\mu}$, a variable ω_i is said to be dominant if the following equation holds

$$|\omega_i|^{(-1)^{\mu_i}} > |\omega_k|^{(-1)^{\mu_k}}, \quad (30)$$

for $k = 1,2,\ldots,r$ and $k \neq i$. Let $\sigma_1,\ldots,\sigma_l$ the indices corresponding to the components of $\boldsymbol{\mu}$ that are greater than zero.

For each subregion $\Omega_i^{\boldsymbol{\mu}}$ of the distinguished boundary $\Omega^{(r+1)}$, the vertex polynomials of the family are

$$\begin{aligned}
&p^{(0)}(\mathbf{s}) \pm \xi_1 \pm \xi_2,\\
&p^{(0)}(\mathbf{s}) \pm \xi_1 \pm \xi_3,\\
&p^{(0)}(\mathbf{s}) \pm \xi_3 \pm \xi_4,\\
&\quad \text{if } \mu_i = 0,
\end{aligned} \quad (31)$$

as well as

$$\begin{aligned}
&p^{(0)}(\mathbf{s}) \pm \delta_1 \pm \delta_2,\\
&p^{(0)}(\mathbf{s}) \pm \delta_1 \pm \delta_3,\\
&p^{(0)}(\mathbf{s}) \pm \delta_3 \pm \delta_4,\\
&\quad \text{if } \mu_i = 1.
\end{aligned} \quad (32)$$

The combination of the signs of a particular combination $\omega_1,\ldots,\omega_r,\omega_{r+1}$, determines the eight polynomials of the twelve (31) or (32) which correspond to the vertices of $\mathcal{V}_\mathcal{D}(j\omega_1,\ldots,j\omega_r,j\omega_{r+1})$. From 2^r different subregions $\Omega^{\boldsymbol{\mu}}$, each one with r distinct values of i, we conclude that $\Omega^{(r)}$ may be partitioned into $r\cdot 2^r$ distinct subregions $\Omega_i^{\boldsymbol{\mu}}$. This yields the total $12\cdot r\cdot 2^r$ vertex polynomials.

In (31) and (32) $p^{(0)}(\mathbf{s})$ stands for the central polynomial of the family $\mathcal{D}$:

$$p^{(0)}(s_1,\ldots,s_r,s_{r+1}) = \sum_{\nu=0}^{n_{r+1}} p^{(0)}_\nu(s_1,\ldots,s_r)\cdot(s_{r+1})^\nu, \quad (33)$$

and

$$\begin{aligned}
\delta_1 &= \sum_k \varrho_{2k} s_{r+1}^{2k}\cdot(s_{\sigma_1}^{n_{\sigma_1}}\cdots s_{\sigma_l}^{n_{\sigma_l}})\\
\delta_2 &= \sum_\nu \varrho_{2\nu+1} s_{r+1}^{2\nu+1}\cdot(s_{\sigma_1}^{n_{\sigma_1}}\cdots s_{\sigma_l}^{n_{\sigma_l}})s_i\\
\delta_3 &= \sum_\nu \varrho_{2\nu+1} s_{r+1}^{2\nu+1}\cdot(s_{\sigma_1}^{n_{\sigma_1}}\cdots s_{\sigma_l}^{n_{\sigma_l}})\\
\delta_4 &= \sum_k \varrho_{2k} s_{r+1}^{2k}\cdot(s_{\sigma_1}^{n_{\sigma_1}}\cdots s_{\sigma_l}^{n_{\sigma_l}})s_i
\end{aligned} \quad (34)$$

and

$$\begin{aligned}
\xi_1 &= \sum_k \varrho_{2k} s_{r+1}^{2k}\cdot(s_{\sigma_1}^{n_{\sigma_1}}\cdots s_{\sigma_l}^{n_{\sigma_l}})\\
\xi_2 &= \sum_\nu \varrho_{2\nu+1} s_{r+1}^{2\nu+1}\cdot(s_{\sigma_1}^{n_{\sigma_1}}\cdots s_{\sigma_l}^{n_{\sigma_l}})s_i^{-1}\\
\xi_3 &= \sum_\nu \varrho_{2\nu+1} s_{r+1}^{2\nu+1}\cdot(s_{\sigma_1}^{n_{\sigma_1}}\cdots s_{\sigma_l}^{n_{\sigma_l}})\\
\xi_4 &= \sum_k \varrho_{2k} s_{r+1}^{2k}\cdot(s_{\sigma_1}^{n_{\sigma_1}}\cdots s_{\sigma_l}^{n_{\sigma_l}})s_i^{-1}
\end{aligned} \quad (35)$$

Let us form the convex combination of polynomials corresponding to two adjacent vertex of the value set, e.g., $p^{(0)}(\mathbf{s}) + \delta_1 + \delta_2$ and $p^{(0)}(\mathbf{s}) + \delta_1 + \delta_3$:

$$\begin{aligned}
(1-\lambda)(p^{(0)}(\mathbf{s}) + \delta_1 + \delta_2) + \lambda(p^{(0)}(\mathbf{s}) + \delta_1 + \delta_3)&\\
= p^{(0)}(\mathbf{s}) + \delta_1 + \delta_2 + \lambda(\delta_3 - \delta_2).&
\end{aligned} \quad (36)$$

Let us repeat the procedure with polynomials $p^{(0)}(\mathbf{s})+\delta_1-\delta_2$ and $p^{(0)}(\mathbf{s}) + \delta_1 + \delta_3$:

$$\begin{aligned}
(1-\lambda)(p^{(0)}(\mathbf{s}) + \delta_1 - \delta_2) + \lambda(p^{(0)}(\mathbf{s}) + \delta_1 + \delta_3)&\\
= p^{(0)}(\mathbf{s}) + \delta_1 - \delta_2 + \lambda(\delta_3 + \delta_2).&
\end{aligned} \quad (37)$$

It is interesting to analyze the form of the polynomial which appears multiplied by λ, since this may lead to a reduction of the testing set:

$$\delta_3 - \delta_2 = (1 - s_i)\cdot(s_{\sigma_1}^{n_{\sigma_1}},\ldots,s_{\sigma_l}^{n_{\sigma_l}})\sum_\nu \varrho_{2\nu+1} s_{r+1}^{2\nu+1} \quad (38)$$

in view of (12) it is easily seen that these polynomials are convex stability directions. Thus in order to asses the stability of the set of polynomials which, when evaluated at $\mathbf{s} = (j\omega_1, \ldots, j\omega_{r+1})$, generate an edge of the value set $\mathcal{V}_{\mathcal{D}}(j\omega_1, \ldots, j\omega_{r+1})$, it suffices to verify the stability of the two vertex polynomials.

Theorem 1: The diamond polynomial family (24) is stable if and only if the $12 \cdot r \cdot 2^r$ vertex polynomials are stable.

B. Quasipolynomial family

Given the nonnegative basic delays $\eta_1, \ldots, \eta_r$, a quasipolynomial family is obtained from the diamond polynomial family as follows

$$\mathcal{F} = \mathcal{Q}(\mathcal{D}). \tag{39}$$

The vertex quasipolynomials are

$$\begin{aligned} &f^{(0)}(s) \pm \beta_1 \pm \beta_2, \\ &f^{(0)}(s) \pm \beta_1 \pm \beta_3, \\ &f^{(0)}(s) \pm \beta_3 \pm \beta_4, \end{aligned} \tag{40}$$

and

$$\begin{aligned} &f^{(0)}(s) \pm \pi_1 \pm \pi_2, \\ &f^{(0)}(s) \pm \pi_1 \pm \pi_3, \\ &f^{(0)}(s) \pm \pi_3 \pm \pi_4, \end{aligned} \tag{41}$$

where $f^{(0)}(s) = \mathcal{Q}(p(\mathbf{s}))$ is the *central quasipolynomial*, which is the image of the central polynomial of the family $\mathcal{D}$ under the mapping $\mathcal{Q}$. the terms are

$$\beta_i = \mathcal{Q}(\xi_1), \quad \pi_i = \mathcal{Q}(\delta_i), \quad i = 1, 2, 3, 4. \tag{42}$$

Some properties of convex combinations from the vertex quasipolynomials may be inferred from the way they are constructed:

$$\begin{aligned} &(1-\lambda)\big(f^{(0)}(s) + \pi_1 + \pi_2\big) + \lambda\big(f^{(0)}(s) + \pi_1 + \pi_3\big) \\ &\quad = f^{(0)}(s) + \pi_1 + \pi_2 + \lambda(\pi_3 - \pi_2), \\ &(1-\lambda)\big(f^{(0)}(s) + \pi_1 - \pi_2\big) + \lambda\big(f^{(0)}(s) + \pi_1 + \pi_3\big) \\ &\quad = f^{(0)}(s) + \pi_1 - \pi_2 + \lambda(\pi_3 + \pi_2). \end{aligned} \tag{43}$$

As in the case of polynomials, it is interesting to search for convex directions:

$$\begin{aligned} \pi_3 - \pi_2 &= \mathcal{Q}(\delta_3 - \delta_2) \\ &= \prod_{k=1}^{r} \left(\frac{1+e^{-\eta_k s}}{2}\right)^{n_k} \left(\frac{2e^{-\eta_i s}}{1+e^{-\eta_i s}}\right) \cdot \\ &\qquad \cdot \prod_{w=1}^{l} \left(\frac{1-e^{-\eta_{\sigma_w} s}}{1+e^{-\eta_{\sigma_w} s}}\right)^{n_{\sigma_w}} \sum_{\nu} \varrho_{2\nu+1} s^{2\nu+1} \end{aligned} \tag{44}$$

a straightforward calculation shows that

$$\frac{\partial}{\partial\omega} \arg\{[\pi_3 - \pi_2](j\omega)\} = -\sum_{k=1}^{r} \frac{\eta_k n_k}{2} - \frac{\eta_i}{2} < -\frac{\tau_0 + \tau_m}{2}, \tag{45}$$

thus, according to (5), $\pi_3 - \pi_2$ is a convex direction. Similarly, for the other quasipolynomial

$$\begin{aligned} \pi_3 + \pi_2 &= \mathcal{Q}(\delta_3 + \delta_2) \\ &= \prod_{k=1}^{r} \left(\frac{1+e^{-\eta_k s}}{2}\right)^{n_k} \left(\frac{2}{1+e^{-\eta_i s}}\right) \cdot \\ &\qquad \cdot \prod_{w=1}^{l} \left(\frac{1-e^{-\eta_{\sigma_w} s}}{1+e^{-\eta_{\sigma_w} s}}\right)^{n_{\sigma_w}} \sum_{\nu} \varrho_{2\nu+1} s^{2\nu+1}, \end{aligned} \tag{46}$$

this time, a new calculation shows

$$\begin{aligned} \arg[\pi_3 + \pi_2](j\omega) &= -\sum_{k=1}^{r} \frac{\eta_k n_k \omega}{2} + \frac{\eta_i \omega}{2} + n_1 \frac{\pi}{2}, \\ &\text{for some integer } n_1, \text{ and} \\ \frac{\partial}{\partial\omega} \arg[\pi_3 + \pi_2](j\omega) &= -\sum_{k=1}^{r} \frac{\eta_k n_k}{2} + \frac{\eta_i}{2} > \frac{\tau_0 + \tau_m}{2}, \end{aligned} \tag{47}$$

replacing the above results in (5), the second term of the right hand side of the inequality becomes

$$\left|\frac{\sin(\eta_i \omega)}{2\omega}\right|, \tag{48}$$

and, for $\eta_i > 0$, there is an infinite number of roots of this function; this implies that $\pi_3 + \pi_2$ is not a convex direction and, in order to asess the stability of the set

$$\big\{ (1-\lambda)\big(f^{(0)}(s) + \pi_1 - \pi_2\big) + \lambda\big(f^{(0)}(s) + \pi_1 + \pi_3\big) \mid \lambda \in [0,1] \big\},$$

the stability of the extreme points is not enough and the whole family should be verified.

1) Stability results:

Theorem 2: The quasipolynomial family (39) is stable if the $12 \cdot r \cdot 2^r$ vertex polynomials (31) and (31) of the family $\mathcal{D}$ are stable.

PROOF. By Theorem 1 the stability of the $12 \cdot r \cdot 2^r$ vertex polynomials implies the stability of the family $\mathcal{D}$, then applying Lemma 5, the result is established Q.E.D.

The following is the delay dependent stability result

Theorem 3: The quasipolynomial family (39) is stable if and only if $12 \cdot r \cdot 2^r$ uniparametric families of quasipolynomials, corresponding to edges of the value set, are stable.

PROOF. The necessity is obvious, since the $12 \cdot r \cdot 2^r$ segments of quasipolynomials belong to the family. To prove sufficiency, first note that all of the vertex quasipolynomial are included in these families, quasipolynomial of the form $\pm(\beta_i - \beta_k)$ and $\pm(\pi_i - \pi_k)$ are convex direction, while those of the form $\pm(\beta_i + \beta_k)$ and $\pm(\pi_i + \pi_k)$ are not, therefore, the families corresponding to these quasipolynomials need to be checked for stability. Finally, apply the zero exclusion principle. Q.E.D.

III. CASE STUDY

Let $a/(b+cs)$ the open-loop, first-order transfer function of a system with delay in the output, let us introduce a proportional integral derivative controller $(K+Ds+I/s)$. The characteristic quasipolynomial is

$$f(s) = cs^2 + bs + Das^2e^{-\eta s} + Kase^{-\eta s} + aIe^{-\eta s}. \quad (49)$$

Quasipolynomial (49) is a quasipolynomial with the basic delay η, $(r=1)$ and the second degree with respect to s. The polynomial which corresponds to (49) according to (16) is

$$\begin{aligned} p(s_1,s_2) &= aI - aIs_1 + \big((b+Ka)+(b-Ka)s_1\big)s_2 \\ &\quad + \big((c+Da)+(c-Da)s_1\big)s_2^2. \end{aligned} \quad (50)$$

Now define the polynomial

$$p^{(0)}(s_1,s_2) = 1 + s_1 + s_2 + s_1s_2 + s_2^2 + s_1s_2^2, \quad (51)$$

along with inequalities

$$\begin{aligned} |aI-1| + |aI+1| &\le \varrho_0 \\ |b+Ka-1| + |b-Ka-1| &\le \varrho_1 \\ |c+Da-1| + |c-Da-1| &\le \varrho_2, \end{aligned} \quad (52)$$

define a family $\mathcal{D}$. Note that the coefficent variations are not independent, therefore interval coefficient is not a suitable model for this uncertainty. The vertex polynomials corresponding to this family are

$$\begin{aligned} &p^{(0)}(s_1,s_2) \pm (\varrho_0 - \varrho_2 s_2^2) \pm \varrho_1 s_1 s_2 \\ &p^{(0)}(s_1,s_2) \pm (\varrho_0 - \varrho_2 s_2^2) \pm \varrho_1 s_2 \\ &p^{(0)}(s_1,s_2) \pm (\varrho_0 s_1 - \varrho_2 s_1 s_2^2) \pm \varrho_1 s_2 \\ &p^{(0)}(s_1,s_2) \pm (\varrho_0 s_1 - \varrho_2 s_1 s_2^2) \pm \varrho_1 s_1 s_2. \end{aligned} \quad (53)$$

Note that, when forming the set of vertex polynomials , the low degree of the family causes that some of them appear more than once, thus instead of $12 \cdot 1 \cdot 2^1 = 24$, there are only 16 different vertex polynomials. By Theorem 2, given the values ϱ_0, ϱ_1 and ϱ_2, the family is stable independently of the value of the delay $\eta \ge 0$, if the polynomials (53) are stable. In particular the family is stable for the bounds $\varrho_0 = 0.5$, $\varrho_1 = 0.5$ and $\varrho_2 = 0.5$.

IV. CONCLUSIONS

A. Conclusions

A new family of diamond-type quasipolynomial presenting general reduced testing set stability conditions was introduced. This family is derived from a generalized diamond polynomial structure through a linear operator. Both necessary and sufficient stability conditions are given, which depend on a reduced subset of the edge members of the family. This conditions are obtained from the convex direction property possesed by the vertex quasipolynomial. The theorems presented here paralell the results for the families described in [17], [19].

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REFERENCES

[1] B.R. Barmish, R. Tempo, C.V. Hollot and H.I. Kang, "An extreme point results for robust stability of a diamond of polynomials", *IEEE Trans. on Automatic Control*, vol. AC-37, 1992, pp. 1460–1462.
[2] B.R. Barmish *New Tools for Robustness of Linear Systems, 1994,* MacMillan Publishing Co.
[3] S. Basu, "On boundary implications of stability and positivity properties of multidimensional systems", *Proceedings of IEEE*, vol. 78, 1990, pp. 614–626.
[4] R. Bellman and K. L. Cooke, *Differential-Difference Equations, 1963,* New York: Academic.
[5] N.K. Bose, *Applied Multidimensional Systems Theory*, 1982, Van Nostrand Reinhold Company.
[6] N.K. Bose, "Edge property from end points for scattering hurwitz polynomials", *Automatica*, vol. 32, no. 4, 1996, pp. 655–657.
[7] J. Chen, S.-I. Niculescu, "Robust stability of quasipolynomials: frequency sweeping conditions "*Proceedings of 42nd IEEE conference on Decision and Control*, 2003, pp 5048–5053.
[8] J. Chen, S.-I. Niculescu, "Robust stability of quasipolynomials: vertex type tests and extensions " *Proceedings of 2004 American Control Conference* 2004, pp 4159–4164.
[9] M. Fu, A.W. Olbrot and M.P. Polis, "Robust stability of time-delay systems: The edge theorem and graphical tests", *IEEE Trans. on Automatic Control*, vol. 40 (1989), pp. 813–820.
[10] F. Gantmacher, *The Theory of Matrices,* vol. 2, 1959, Chelsea.
[11] V.L. Kharitonov, "Asymptotic stability of an equilibrium position of a family of systems of linear differential equations", *Differentsial'nie Uravneniya,* vol. 14, 1978, pp. 2086–2088.
[12] V. L. Kharitonov and A. Zhabko "Robust Stability of Time-Delay Systems," *IEEE Trans. on Automatic Control,* vol. 39, no. 12, 1994, pp. 2388-2397.
[13] V.L. Kharitonov and J.A. Torres Muñoz, "Robust Stability of Multivariate Polynomials Part 1: small coefficient perturbations," *Multidimensional Systems and Signal Processing,* vol.10, 1999, pp. 7–20.
[14] V.L. Kharitonov, J.A. Torres Muñoz and M.I. Ramirez-Sossa, "Robust Stability of Multivariate Polynomials Part 2: Polytopic coefficient variations," *Multidimensional Systems and Signal Processing,* vol.10, 1999, pp. 21–32.
[15] V.L. Kharitonov, J.A. Torres-Muñoz and M.I. Ramírez-Sossa. "Robust stability of multivariate polynomials, part 3: Frequency domain approach.," *Multidimensional Systems and Signal Processing,* vol.11, 2000, pp. 213–231.
[16] V.L. Kharitonov and J.A. Torres Muñoz, "Recent Results on the Stability of Multivariate Polynomials," *IEEE Trans. on Circuits and Systems 1: Fundamental Theory and Applications, vol. 49, no. 6, 2002, pp. 715–724.*
[17] V.L. Kharitonov, J.A. Torres Muñoz and M.B. Ortiz-Moctezuma, "Polytopic families of Quasi-Polynomials: Vertex-type stability conditions," *IEEE Trans. on Circuits and Systems 1: Fundamental Theory and Applications, vol. 50, no. 11, 2003, pp. 1413–1420.*
[18] K.D. Kim and N.K. Bose. "Vertex implications of stability for a class of delay differential interval system, "% *IEEE Trans. on Automatic Control, vol. 37, no. 7, 1990*
[19] J.A. Torres-Muñoz, V.L. Kharitonov and M.B. Ortiz-Moctezuma "Reduced stability testing set for a diamond-type family of quasipolynomials, *Multidimensional System and Signal Processing* March 2008. DOI: 10.1007/s11045-008-0052-5.
[20] M. Mansour, F.J. Kraus and B.D.O. Anderson, "Strong Kharitonov theorem for discrete systems," *Proc. of the IEEE Conference on Decision and Control*, Austin, Texas, 1988.
[21] M.B. Ortiz-Moctezuma,"Análisis de estabilidad robusta de quasipolynomios y polinomios multivariables," *Ph.D. Dissertation,* In Spanish*,* Cinvestav, México, september 2005.
[22] L. Pontryagin, "On the Zeros of Some Elementary Trascendental Functions," English Translation, *Amer. Math. Soc. Translation.*, vol. 2, 1955, pp. 95-110.
[23] A. Rantzer, "Stability conditions for polytopes of polynomials" *IEEE Trans. on Automatic Control*, vol. 37, no. 1, pp. 79–89, Jan. 1992.