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Enhanced Robust Kalman Predictor for Discrete-Time Systems With Uncertain Correlated Noises

Rodrigo Fontes Souto, *Student Member, IEEE,* and João Yoshiyuki Ishihara, *Member, IEEE*

***Abstract*— This paper presents an enhanced robust predictor for uncertain discrete-time systems. Besides uncertainties in both state and output matrices, it is also permitted dynamic and measurement noises to be correlated with unknown correlation covariance. All uncertainties in the proposed model are time-varying and supposed norm-bounded. The filter is obtained minimizing an upper bound of the variance error estimation, that is, the design leads to a guaranteed cost for all allowed uncertainties. Simulation examples are provided to show the performance of the enhanced estimator.**

I. INTRODUCTION

One of the problems with the Kalman predictor is that it may not be robust against modeling uncertainties. The Kalman predictor algorithm is the optimal estimator for a system without uncertainties. In the presence of uncertainties, it is well known that the performance of the estimator is degradated. If a estimator can handle with the uncertainty, *i.e.*, can be used with guaranteed performance for all possible values for the uncertain parameter within a given set, it is called robust estimator (see [1]-[4] and references therein).

The robust estimation problem has been subject of intensive study and many uncertainty modeling and solution directions have been considered along the literature for a wide sort of systems. Robust filters for linear fractional transformation (LFT) uncertain systems are proposed in [5]. A robust Kalman filter for descriptor systems using a deterministic procedure is given in [6]. Robust filtering for bilinear uncertain systems is developed in [7]. Filters for systems with missing measurements are obtained in [8]. Another technique used in robust estimation is the H_∞ estimation, where the noise sources are signals with bounded energy or average power. Although more robust than the Kalman estimator, the H_∞ estimator suffers from the same dependence on the system matrices and therefore robust H_∞ were also studied. In the robust H_∞ estimation is applied similar methodologies used in robust Kalman estimation (see, *e.g.*, [12] and [13]). In [9], robust filtering is used for multisensor fusion estimation for multisensor system with uncertain correlated noise using $\mathcal{H}_\infty$ filtering. Other robust estimators can be found in the references of the aforementioned papers.

The analysis and the design of robust finite horizon Kalman-type estimators for linear dynamic systems with uncertainties have received great attention in recent years. Researchers have been focusing basically on two methodologies to the robust estimation: the linear matrix inequality (LMI) and the Riccati equation approaches. The design of using LMIs is able to deal with norm-bounded or polytopic parameter uncertainty. Current effort has been done to design less conservative estimators, *e.g.*, using parametric Lyapunov functions [10]. Other results include design methods without a limitation on the order of the estimator and the possibility to certify the performance quality [11].

Alternatively, according to [4], a Riccati equation approach can be used with the advantage that the effect of parameter uncertainty on the structure and gain of the estimator is clearly demonstrated, providing useful insights on the problem. One technique is based on the resolution of two discrete Riccati equations ([4], [7], [14], and [15]). Another possibility is the resolution based on one Riccati equation, see [16] and [17]. In the finite horizon case, unlike the classic Kalman predictor, the robust optimal predictor at k may not lead to an optimal state estimation at time $k+1$, see [1]-[4].

The guaranteed cost prediction problem is to design a linear filter to ensure an upper bound on the estimation error variances for all admissible parameter uncertainties. One of the earliest guaranteed cost filtering design was proposed by [19] for continuous time-invariant systems with uncertainties in the state matrix. In [3], it was developed a robust Kalman design for discrete-time systems subject to time-varying norm-bounded parameter uncertainties in both the state and output matrices. A bounded-variance filtered state estimation of linear continuous and discrete-time systems, with an unknown norm-bounded parameter matrix, is considered in [16] for uncertainties allowed in the state dynamics and the output mapping matrices. Necessary and sufficient conditions to the design of robust filters over finite and infinite horizon are given in [14]. Recently, [17] provided a guaranteed cost robust filter for a model with uncertainties in the state and output system matrices and in the covariance noises. The proposed filter, however, have a constraint that both noise signals in state and output equations must be uncorrelated and have the same dimension.

In this paper, we intend to enhance the predictor of [17] in order to allow correlated noise signals with possibly different dimensions and uncertain correlation. The enhancement proposed changes the structure of the uncertainties and adds one more scaling parameter in the filter design, providing a less conservative predictor with better performance.

This paper is organized as follows: in section II, we describe the model and the structure of the uncertainties. The enhanced predictor is obtained in section III. Numerical examples are given in section IV. One comparing the performance of the proposed predictor with the standard Kalman

R. F. Souto and J. Y. Ishihara are with the Department of Electrical Engineering, Brasilia University, CEP 70910-900, Brazil, rsouto@ieee.org; ishihara@ene.unb.br

predictor for the system with correlated noises and another simulation comparing with the robust predictor of [17] for uncorrelated noises. Conclusions are drawn in section V.

Notation: $\Re^n$ represents the n-dimensional Euclidean space, $\Re^{n\times m}$ is the set of real $n\times m$ matrices, $E\{\bullet\}$ denotes the expectation operator, $cov\{\bullet\}$ indicates the covariance operator, $diag\{\bullet\}$ stands for a block-diagonal matrix, $Z^{-1}$and Z^T are the inverse and the transpose of the matrix Z, respectively, $Z > 0$ means that Z is positive-definite, $\hat{x}_{k+1|k}$ denotes the estimated vector x_{k+1} at the time k given the measurements $\{y_0, y_1, \dots y_k\}$ and Z^* indicates a matrix Z that minimizes a cost functional.

II. PROBLEM FORMULATION

Consider the following class of uncertain systems

$$x_{k+1} = (A_k + \Delta A_k)\, x_k + \widetilde{w}_k, \tag{1}$$
$$y_k = (C_k + \Delta C_k)\, x_k + \widetilde{v}_k, \tag{2}$$

where $x_k \in \Re^n$ is the state vector, $y_k \in \Re^m$ is the output vector and $\widetilde{w}_k \in \Re^p$ and $\widetilde{v}_k \in \Re^q$ are noise zero mean signals. The noise signals $\widetilde{w}_k$ and $\widetilde{v}_k$ are allowed to be correlated with uncertain covariance and correlation. We suppose that the noise signals can be written as following

$$\widetilde{w}_k = (B_{w,k} + \Delta B_{w,k})\, w_k + (B_{v,k} + \Delta B_{v,k})\, v_k, \tag{3}$$
$$\widetilde{v}_k = (D_{w,k} + \Delta D_{w,k})\, w_k + (D_{v,k} + \Delta D_{v,k})\, v_k. \tag{4}$$

We assume that the initial conditions $\{x_0\}$ and the noises $\{w_k, v_k\}$ are uncorrelated with the statistical properties

$$E\left\{\left[\, w_k^T \; v_k^T \; x_0^T \,\right]^T\right\} = \left[\, 0 \; 0 \; \overline{x}_0^T \,\right]^T, \tag{5}$$
$$cov\{x_0 - \overline{x}_0\} = X_0, \tag{6}$$
$$E\left\{\left[\, w_k^T \; v_k^T \,\right]^T \left[\, w_j^T \; v_j^T \,\right]\right\} = diag\{W_k\delta_{kj}, V_k\delta_{kj}\}, \tag{7}$$

where W_k, V_k and X_0 denotes the noises and initial state covariance matrices and δ_{kj} is the Kronecker delta function, *i.e.*, δ_{kj} = 1 if $k = j$ and δ_{kj} = 0, otherwise.

Using the definitions (3) and (4), the system (1)-(2) can be rewritten as

$$\begin{aligned} x_{k+1} &= (A_k + \Delta A_k)\, x_k + (B_{w,k} + \Delta B_{w,k})\, w_k \\ &\quad + (B_{v,k} + \Delta B_{v,k})\, v_k, \end{aligned} \tag{8}$$
$$\begin{aligned} y_k &= (C_k + \Delta C_k)\, x_k + (D_{w,k} + \Delta D_{w,k})\, w_k \\ &\quad + (D_{v,k} + \Delta D_{v,k})\, v_k. \end{aligned} \tag{9}$$

Although w_k and v_k are independent, the model (8)-(9) with direct feedthrough is equivalent to one with only one noise vector at the state and output equations with explicit correlation [21]. The predictor proposed in [17] is restricted to the case where $p = q$. In this paper this restriction is relaxed, allowing different dimensions for w_k and v_k. The nominal matrices A_k, $B_{w,k}$, $B_{v,k}$, C_k, $D_{w,k}$ and $D_{v,k}$ are known, time-varying and with appropriate dimensions. The matrices ΔA_k, $\Delta B_{w,k}$, $\Delta B_{v,k}$, ΔC_k, $\Delta D_{w,k}$ and $\Delta D_{v,k}$ represent the associated uncertainties and have the following structure

$$\begin{bmatrix} \Delta A_k & \Delta B_{w,k} & \Delta B_{v,k} \\ \Delta C_k & \Delta D_{w,k} & \Delta D_{v,k} \end{bmatrix} = \begin{bmatrix} H_{1,k} \\ H_{2,k} \end{bmatrix} F_k \left[\, G_{x,k} \; G_{w,k} \; G_{v,k} \,\right],$$

with $H_{1,k} \in \Re^{n\times r}$, $H_{2,k} \in \Re^{m\times r}$, $G_{x,k} \in \Re^{s\times n}$, $G_{w,k} \in \Re^{s\times p}$ and $G_{v,k} \in \Re^{s\times q}$ are known. This structure allows us to use $p \neq q$. The matrix $F_k \in \Re^{r\times s}$ is unknown, time-varying and norm-bounded, *i.e.*,

$$F_k^T F_k \leq I, \quad \forall k \in [0, N]. \tag{10}$$

This paper proposes an enhanced design of a finite horizon robust predictor for state estimation of the uncertain system described by (8)-(9). The predictor has the following structure

$$\widehat{x}_{0|-1} = \overline{x}_0, \tag{11}$$
$$\widehat{x}_{k+1|k} = \Phi_k \widehat{x}_{k|k-1} + K_k\left(y_k - C_k \widehat{x}_{k|k-1}\right). \tag{12}$$

The predictor is intended to ensure an upper limit in the variance error estimation. In other words, there is a sequence of positive-definite matrices $\overline{P}_{k|k-1}$ that, for all allowed uncertainties in $k \in [0, N]$, satisfy

$$cov\left\{x_k - \widehat{x}_{k|k-1}\right\} \leq \overline{P}_{k|k-1}. \tag{13}$$

The matrices Φ_k and K_k are time-varying and can be determined to minimize the $\overline{P}_{k|k-1}$, resulting in a minimal upper bound to the error variance on the state estimation predictor.

III. ROBUST PREDICTOR DESIGN

In this section, a solution to the robust prediction problem over a finite-horizon $[0, N]$ will be given using the Riccati equation approach. Theorem 1 presents the robust predictor with one more scaling parameter than usually found in literature. The proposed predictor is also a generalization for correlated noise systems.

Theorem 1: A robust predictor with guaranteed cost for the error variance on the state estimation of the model subject to the uncertainties (8)-(9) and to conditions (10)-(7) is given by the recursive in Table I.

We start the proof considering the system (8)-(9) and the struture of the predictor in (12). Then, we define an augmented state as

$$\widetilde{x}_k := \begin{bmatrix} x_k \\ \widehat{x}_{k|k-1} \end{bmatrix}. \tag{14}$$

As a result, the augmented system with the state $\widetilde{x}_k$ is given by

$$\begin{aligned} \widetilde{x}_{k+1} &= \left(\widetilde{A}_k + \widetilde{H}_k F_k \widetilde{G}_{x,k}\right)\widetilde{x}_k \\ &\quad + \left(\widetilde{B}_k + \widetilde{H}_k F_k G_{w,k}\right) w_k \\ &\quad + \left(\widetilde{D}_k + \widetilde{H}_k F_k G_{v,k}\right) v_k, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \widetilde{A}_k &:= \begin{bmatrix} A_k & 0 \\ K_k C_k & \Phi_k - K_k C_k \end{bmatrix}, \\ \widetilde{B}_k &:= \begin{bmatrix} B_{w,k} \\ K_k D_{w,k} \end{bmatrix}, \widetilde{D}_k := \begin{bmatrix} B_{v,k} \\ K_k D_{v,k} \end{bmatrix}, \\ \widetilde{H}_k &:= \begin{bmatrix} H_{1,k} \\ K_k H_{2,k} \end{bmatrix}, \widetilde{G}_{x,k} := \left[\, G_{x,k} \; 0 \,\right]. \end{aligned} \tag{16}$$

TABLE I

ENHANCED ROBUST PREDICTOR ALGORITHM.

Step 0 (Initial conditions):

$$\widehat{x}_{0|-1} = \overline{x}_0,\quad \overline{P}_{0|-1} = X_0.$$

Step 1: Obtain scalar parameters that satisfy

$$\alpha_k^{-1} I - G_{x,k}\overline{P}_{k|k-1}G_{x,k}^T > 0,$$
$$\beta_k^{-1} I - G_{w,k}W_kG_{w,k}^T > 0,$$
$$\gamma_k^{-1} I - G_{v,k}V_kG_{v,k}^T > 0.$$

Step 2: Calculate the corrections due to the presence of uncertainties

$$\overline{P}_{c,k|k-1} := \overline{P}_{k|k-1} + \overline{P}_{k|k-1}G_{x,k}^T \times \left(\alpha_k^{-1}I - G_{x,k}\overline{P}_{k|k-1}G_{x,k}^T\right)^{-1} G_{x,k}\overline{P}_{k|k-1},$$
$$A_{c,k} := A_k + (A_k - K_kC_k)\,\overline{P}_{k|k-1}G_{x,k}^T \times \left(\alpha_k^{-1}I - G_{x,k}\overline{P}_{k|k-1}G_{x,k}^T\right)^{-1} G_{x,k},$$
$$V_{c,k} := V_k + V_kG_{v,k}^T\left(\gamma_k^{-1}I - G_{v,k}V_kG_{v,k}^T\right)^{-1} G_{v,k}V_k,$$
$$W_{c,k} := W_k + W_kG_{w,k}^T\left(\beta_k^{-1}I - G_{w,k}W_kG_{w,k}^T\right)^{-1} G_{w,k}W_k,$$
$$\Delta_{1,k} := \left(\alpha_k^{-1} + \beta_k^{-1} + \gamma_k^{-1}\right) H_{1,k}H_{2,k}^T,$$
$$\Delta_{2,k} := \left(\alpha_k^{-1} + \beta_k^{-1} + \gamma_k^{-1}\right) H_{2,k}H_{2,k}^T,$$
$$\Delta_{3,k} := \left(\alpha_k^{-1} + \beta_k^{-1} + \gamma_k^{-1}\right) H_{1,k}H_{1,k}^T.$$

Step 3: Define augmented matrices

$$B_k := \begin{bmatrix} B_{w,k} & B_{v,k}\end{bmatrix},\quad D_k := \begin{bmatrix} D_{w,k} & D_{v,k}\end{bmatrix},\quad U_{c,k} := \begin{bmatrix} W_{c,k} & 0 \\ 0 & V_{c,k}\end{bmatrix}.$$

Step 4: Calculate the parameters of the predictor

$$\Phi_k = A_{c,k},$$
$$K_k = \left(A_k\overline{P}_{c,k|k-1}C_k^T + B_kU_{c,k}D_k^T + \Delta_{1,k}\right) \times \left(C_k\overline{P}_{c,k|k-1}C_k^T + D_kU_{c,k}D_k^T + \Delta_{2,k}\right)^{-1}.$$

Step 5: Update $\left\{\widehat{x}_{k+1|k}, \overline{P}_{k+1|k}\right\}$ as

$$\widehat{x}_{k+1|k} = \Phi_k\widehat{x}_{k|k-1} + K_k\left(y_k - C_k\widehat{x}_{k|k-1}\right),$$
$$\overline{P}_{k+1|k} = A_k\overline{P}_{c,k|k-1}A_k^T + B_kU_{c,k}B_k^T + \Delta_{3,k} - \left(A_k\overline{P}_{c,k|k-1}C_k^T + \Delta_{1,k}\right) \times \left(C_k\overline{P}_{c,k|k-1}C_k^T + D_kU_{c,k}D_k^T + \Delta_{2,k}\right)^{-1} \times \left(A_k\overline{P}_{c,k|k-1}C_k^T + \Delta_{1,k}\right)^T.$$

Consider $\widetilde{P}_{k|k-1} := E\left\{\left(\widetilde{x}_k - E\left\{\widetilde{x}_k\right\}\right)\left(\widetilde{x}_k - E\left\{\widetilde{x}_k\right\}\right)^T\right\}$ and that $\widetilde{x}_k$, w_k and v_k are independent zero mean vectors, such as

$$E\left\{\begin{bmatrix}\widetilde{x}_k\\ w_k\\ v_k\end{bmatrix}\begin{bmatrix}\widetilde{x}_k\\ w_k\\ v_k\end{bmatrix}^T\right\} = \begin{bmatrix}\widetilde{P}_{k|k-1} & 0 & 0\\ 0 & W_k & 0\\ 0 & 0 & V_k\end{bmatrix}. \tag{17}$$

The next lemma give us an upper bound for the covariance matrix of the augmented system (15) and the necessary conditions to its existence.

Lemma 1: An upper limit for the covariance matrix of the augmented system (15) is given by

$$P_{0|-1} = \begin{bmatrix} X_0 & 0\\ 0 & 0\end{bmatrix}, \tag{18}$$

$$\begin{aligned} P_{k+1|k} &= \widetilde{A}_kP_{k|k-1}\widetilde{A}_k^T + \widetilde{B}_kW_k\widetilde{B}_k^T + \widetilde{D}_kV_k\widetilde{D}_k^T \\ &+ \widetilde{A}_kP_{k|k-1}\widetilde{G}_{x,k}^T\left(\alpha_k^{-1}I - \widetilde{G}_{x,k}P_{k|k-1}\widetilde{G}_{x,k}^T\right)^{-1} \\ &\times \widetilde{G}_{x,k}P_{k|k-1}\widetilde{A}_k^T + \widetilde{B}_kW_kG_{w,k}^T \\ &\times \left(\beta_k^{-1}I - G_{w,k}W_kG_{w,k}^T\right)^{-1} G_{w,k}W_k\widetilde{B}_k^T \\ &+ \widetilde{D}_kV_kG_{v,k}^T\left(\gamma_k^{-1}I - G_{v,k}V_kG_{v,k}^T\right)^{-1} G_{v,k}V_k\widetilde{D}_k \\ &+ \left(\alpha_k^{-1} + \beta_k^{-1} + \gamma_k^{-1}\right)\widetilde{H}_k\widetilde{H}_k^T, \end{aligned} \tag{19}$$

where α_k^{-1}, β_k^{-1} and γ_k^{-1} satisfy

$$\alpha_k^{-1}I - \widetilde{G}_{x,k}P_{k|k-1}\widetilde{G}_{x,k}^T > 0, \tag{20}$$
$$\beta_k^{-1}I - G_{w,k}W_kG_{w,k}^T > 0, \tag{21}$$
$$\gamma_k^{-1}I - G_{v,k}V_kG_{v,k}^T > 0. \tag{22}$$

Proof: Given the initial condition (6) and the definition of $\widetilde{P}_{k|k-1}$, it is straightforward that

$$\widetilde{P}_{0|-1} = \begin{bmatrix} X_0 & 0\\ 0 & 0\end{bmatrix}, \tag{23}$$

$$\begin{aligned} \widetilde{P}_{k+1|k} &= \\ &\left(\widetilde{A}_k + \widetilde{H}_kF_k\widetilde{G}_{x,k}\right)\widetilde{P}_{k|k-1}\left(\widetilde{A}_k + \widetilde{H}_kF_k\widetilde{G}_{x,k}\right)^T \\ &+ \left(\widetilde{B}_k + \widetilde{H}_kF_kG_{w,k}\right)W_k\left(\widetilde{B}_k + \widetilde{H}_kF_kG_{w,k}\right)^T \\ &+ \left(\widetilde{D}_k + \widetilde{H}_kF_kG_{v,k}\right)V_k\left(\widetilde{D}_k + \widetilde{H}_kF_kG_{v,k}\right)^T. \end{aligned} \tag{24}$$

Consider the following result defined in Lemma 2 of [18]. *Given matrices A, H, G and F with compatible dimensions and that exists F satisfying $F^TF \leq I$. In addition, consider P a symmetric positive-definite matrix and $\epsilon > 0$ a positive scalar. If $\epsilon^{-1}I - GPG^T > 0$, then*

$$\begin{aligned} (A + HFG)\,P\,(A + HFG)^T &\leq APA^T \\ &+ APG^T\left(\epsilon^{-1}I - GPG^T\right)^{-1} GPA^T + \epsilon^{-1}HH^T. \end{aligned}$$

Choose scaling parameters α_k^{-1}, β_k^{-1} and γ_k^{-1} satisfying (20)-(22). Therefore, we have that

$$\begin{aligned} \widetilde{P}_{k+1|k} &\leq \widetilde{A}_k\widetilde{P}_{k,|k-1}\widetilde{A}_k^T + \widetilde{A}_k\widetilde{P}_{k|k-1}\widetilde{G}_{x,k}^T \\ &\times \left(\alpha_k^{-1}I - \widetilde{G}_{x,k}\widetilde{P}_{k|k-1}\widetilde{G}_{x,k}^T\right)^{-1}\widetilde{G}_{x,k}\widetilde{P}_{k|k-1}\widetilde{A}_k^T \\ &+ \alpha_k^{-1}\widetilde{H}_k\widetilde{H}_k^T + \widetilde{B}_kW_k\widetilde{B}_k^T + \widetilde{B}_kW_kG_{w,k}^T \\ &\times \left(\beta_k^{-1}I - G_{w,k}W_kG_{w,k}^T\right)^{-1} G_{w,k}W_k\widetilde{B}_k^T \\ &+ \beta_k^{-1}\widetilde{H}_k\widetilde{H}_k^T + \widetilde{D}_kV_k\widetilde{D}_k^T + \widetilde{D}_kV_kG_{v,k}^T \\ &\times \left(\gamma_k^{-1}I - G_{v,k}V_kG_{v,k}^T\right)^{-1} G_{v,k}V_k\widetilde{D}_k \\ &+ \gamma_k^{-1}\widetilde{H}_k\widetilde{H}_k^T. \end{aligned} \tag{25}$$

If there is a sequence $\left\{P_{k+1|k}\right\}$ given by (19) with initial conditions (18), where α_k^{-1}, β_k^{-1} and γ_k^{-1} satisfying (20)-(22), then $P_{k+1|k}$ is an upper bound of $\widetilde{P}_{k+1|k}$, such that

$\widetilde{P}_{k+1|k} \leq P_{k+1|k}$ for all instants k. The result of the lemma follows applying the Lemma 3.2 of [16]:

For $0 \leq k \leq N$, suppose $X = X^T > 0$, and $s_k(X) = s_k^T(X) \in \Re^{n\times n}$, $h_k(X) = h_k^T(X) \in \Re^{n\times n}$. If there exists $Y = Y^T > X$ such that $s_k(Y) \geq s_k(X)$ and $h_k(Y) \geq h_k(X)$, then the solutions M_k and N_k to the following difference equations

$$M_{k+1} = s_k(M_k), \quad N_{k+1} = h_k(N_k), \quad M_0 = N_0 > 0$$

satisfy $M_k \leq N_k$. ■

Consider $W_{c,k}$ and $V_{c,k}$ as the corrected variance matrices for the uncertain system (8)-(9). Replacing the augmented matrices (16) into (19), the upper bound in (19) can be partitioned as

$$P_{k+1|k} = \begin{bmatrix} P_{11,k+1|k} & P_{12,k+1|k} \\ P_{12,k+1|k}^T & P_{22,k+1|k} \end{bmatrix}, \tag{26}$$

where

$$\begin{aligned} P_{11,k+1|k} &= A_k \left(P_{11,k|k-1} + P_{11,k|k-1} M_k P_{11,k|k-1}\right) A_k^T \\ &\quad + B_{w,k} W_{c,k} B_{w,k}^T + B_{v,k} V_{c,k} B_{v,k}^T + \Delta_{3,k}, \quad (27) \\ P_{12,k+1|k} &= A_k P_{11,k|k-1} C_k^T K_k^T \\ &\quad + A_k P_{11,k,k-1} M_k P_{11,k|k-1} C_k^T K_k^T \\ &\quad + \left(B_{w,k} W_{c,k} D_{w,k}^T + B_{v,k} V_{c,k} D_{v,k}^T + \Delta_{1,k}\right) K_k^T \\ &\quad + A_k M_{1,k} \left(\Phi_k - K_k C_k\right)^T, \quad (28) \\ P_{22,k+1|k} &= K_k C_k \overline{P}_{k|k-1} C_k^T K_k^T + \Phi_k P_{22,k|k-1} \Phi_k^T \\ &\quad + K_k C_k \left(P_{12,k|k-1} - P_{22,k|k-1}\right) \Phi_k^T \\ &\quad + \Phi_k \left(P_{12,k|k-1}^T - P_{22,k|k-1}\right) C_k^T K_k^T \\ &\quad + \Lambda_k M_k \Lambda_k^T + K_k D_{w,k} W_{c,k} D_{w,k}^T K_k^T \\ &\quad + K_k D_{v,k} V_{c,k} D_{v,k}^T K_k^T \\ &\quad + \left(\alpha_k^{-1} + \beta_k^{-1} + \gamma_k^{-1}\right) K_k H_{2,k} H_{2,k}^T K_k^T, \quad (29) \end{aligned}$$

with

$$\begin{aligned} \overline{P}_{k|k-1} &:= P_{11,k|k-1} - P_{12,k|k-1} - P_{12,k|k-1}^T \\ &\quad + P_{22,k|k-1}, \\ M_{1,k} &:= P_{12,k|k-1} + P_{11,k|k-1} M_k P_{12,k|k-1}, \\ M_k &:= G_{x,k}^T \left(\alpha_k^{-1} I - G_{x,k} P_{11,k|k-1} G_{x,k}^T\right)^{-1} G_{x,k}, \\ \Lambda_k &:= \Phi_k P_{12,k|k-1}^T + K_k C_k \left(P_{11,k|k-1} - P_{12,k|k-1}^T\right). \end{aligned}$$

Given $P_{k|k-1} \geq \widetilde{P}_{k|k-1} \geq 0, \forall k$, if we define $\overline{P}_{k|k-1}$ as

$$\overline{P}_{k|k-1} := \begin{bmatrix} I & -I \end{bmatrix} P_{k|k-1} \begin{bmatrix} I & -I \end{bmatrix}^T, \tag{30}$$

thus we have that $\overline{P}_{k|k-1}$ is an upper bound of the error variance on the state estimation.

Using the definitions (26) and (30), the matrix $\overline{P}_{k+1|k}$ can be written as

$$\begin{aligned} \overline{P}_{0|-1} &= X_0, & (31) \\ \overline{P}_{k+1|k} &= \left(R_k P_{22,k|k-1} - T_k P_{12,k|k-1}\right) R_k^T \\ &\quad - \Theta_k T_k^T + \Theta_k M_k \Theta_k^T \\ &\quad + \left(B_{w,k} - K_k D_{w,k}\right) W_{c,k} \left(B_{w,k} - K_k D_{w,k}\right)^T \\ &\quad + \left(B_{v,k} - K_k D_{v,k}\right) V_{c,k} \left(B_{v,k} - K_k D_{v,k}\right)^T \\ &\quad + \left(\alpha_k^{-1} + \beta_k^{-1} + \gamma_k^{-1}\right) \\ &\quad \times \left(H_{1,k} - K_k H_{2,k}\right) \left(H_{1,k} - K_k H_{2,k}\right)^T, & (32) \end{aligned}$$

where

$$\begin{aligned} \Theta_k &:= R_k P_{12,k|k-1}^T - T_k P_{11,k|k-1}, & (33) \\ R_k &:= \Phi_k - K_k C_k, & (34) \\ T_k &:= A_k - K_k C_k. & (35) \end{aligned}$$

We note that (32) is valid for any Φ_k and K_k. Calculating the first and second order partial derivatives of (32) with respect to Φ_k and K_k and making

$$\frac{\partial}{\partial \Phi_k} \overline{P}_{k+1|k} = 0, \tag{36}$$

$$\frac{\partial}{\partial K_k} \overline{P}_{k+1|k} = 0, \tag{37}$$

then we find the expressions $\Phi_k = \Phi_k^*$ and $K_k = K_k^*$ that minimize the upper bound error variance on the state estimation, $\overline{P}_{k+1|k}$, as

$$\Phi_k^* = A_k + \left(A_k - K_k C_k\right) \left(Z_k - I\right) \tag{38}$$

and

$$\begin{aligned} K_k^* &= \left(\left(A_k \Gamma_k - \Phi_k^* \Theta_k\right) C_k^T + \Psi_{1,k}\right) \\ &\quad \times \left(C_k \left(\overline{P}_{k|k-1} + \Xi_k\right) C_k^T + \Psi_{2,k}\right)^{-1}, \quad (39) \end{aligned}$$

where

$$\begin{aligned} Z_k &:= M_{1,k} M_{2,k}^{-1}, & (40) \\ M_{2,k} &:= P_{22,k|k-1} + P_{12,k|k-1}^T M_k P_{12,k|k-1}, & (41) \\ \Psi_{1,k} &:= B_{w,k} W_{c,k} D_{w,k}^T + B_{v,k} V_{c,k} D_{v,k}^T + \Delta_{1,k}, & (42) \\ \Psi_{2,k} &:= D_{w,k} W_{c,k} D_{w,k}^T + D_{v,k} V_{c,k} D_{v,k}^T + \Delta_{2,k}, & (43) \\ \Gamma_k &:= P_{11,k|k-1} + P_{11,k|k-1} M_k P_{11,k|k-1} - M_{1,k}, & (44) \\ \Theta_k &:= P_{12,k|k-1}^T \left(I + P_{11,k|k-1}\right) - M_{2,k}, & (45) \\ \Xi_k &:= \left(P_{11,k|k-1} - P_{12,k|k-1}^T\right) M_k \\ &\quad \times \left(P_{11,k|k-1} - P_{12,k|k-1}\right). & (46) \end{aligned}$$

Replacing (38) and (39) in (28), (29) and $P_{12,k+1|k}^T$, and after some algebra, it is straightforward that

$$\begin{aligned} &P_{12,k+1|k} = P_{12,k+1|k}^T = P_{22,k+1|k} \\ &= \left(A_k S_k C_k^T + \Psi_{1,k}\right) \left(C_k S_k C_k^T + \Psi_{2,k}\right)^{-1} \\ &\times \left(A_k S_k C_k^T + \Psi_{1,k}\right)^T + A_k M_{1,k} M_{2,k}^{-1} M_{1,k}^T A_k^T, \quad (47) \end{aligned}$$

where

$$S_k := P_{11,k|k-1} + P_{11,k|k-1} M_k P_{11,k|k-1} - M_{1,k} M_{2,k}^{-1} M_{1,k}^T.$$

Since $P_{12,k+1|k} = P^T_{12,k+1|k} = P_{22,k+1|k}$, for any symmetric $P_{k|k-1}$, if we start with a matrix $P_{n|n-1}$ satisfying $P_{12,n|n-1} = P^T_{12,n|n-1} = P_{22,n|n-1}$ for some $n \geq 0$, then we can conclude that $P_{12,k+1|k} = P^T_{12,k+1|k} = P_{22,k+1|k}$ is valid for any $k \geq n$. At this point, we can conclude that $\alpha_{x,k}$ shall now satisfy

$$\alpha^{-1}_{x,k} I - G_{x,k}\overline{P}_{k|k-1}G^T_{x,k} > 0. \qquad (48)$$

Using $P_{12,k+1|k} = P^T_{12,k+1|k} = P_{22,k+1|k}$ and (48), we can simplify the expressions for Φ^*_k, K^*_k and $\overline{P}_{k+1|k}$. Using this simplification, we can define $A_{c,k}$ given in Table I: $A_{c,k} := \Phi^*_k$. The simplified expression for the predictor gain is given by

$$K^*_k = \left(A_k\overline{P}_{c,k|k-1}C^T_k + \Psi_{1,k}\right)\left(C_k\overline{P}_{c,k|k-1}C^T_k + \Psi_{2,k}\right)^{-1}.$$

The matrix $\overline{P}_{c,k|k-1}$ can be interpreted as a correction of the variance matrix due to the presence of uncertainties in the model. The expression for the Riccati equation is

$$\begin{aligned}\overline{P}_{k+1|k} &= \left(A_k - K^*_kC_k\right)\overline{P}_{c,k|k-1}\left(A_k - K^*_kC_k\right)^T \\ &+ \left(B_{w,k} - K^*_kD_{w,k}\right)W_{c,k}\left(B_{w,k} - K^*_kD_{w,k}\right)^T \\ &+ \left(B_{v,k} - K^*_kD_{v,k}\right)V_{c,k}\left(B_{v,k} - K^*_kD_{v,k}\right)^T \\ &+ \left(\alpha^{-1}_k + \beta^{-1}_k + \gamma^{-1}_k\right) \\ &\times \left(H_{1,k} - K^*_kH_{2,k}\right)\left(H_{1,k} - K^*_kH_{2,k}\right)^T. \qquad (49)\end{aligned}$$

Replacing K^*_k into (49) we obtain the Riccati equation given in Table I.

Remark 1: The expression (27) gives the covariance matrix recursion of the state vector.

Remark 2: Considering the model (8)-(9) without uncertainties, the parameters of the predictor and the Riccati equation present at Table I are the same as those presented in [21].

IV. NUMERICAL EXAMPLES

In this section, we provide two numerical examples. In the first simulation, the system presents correlated noises in the dymanic and output equation. The simulation compares the results using the enhanced predictor and the usual Kalman predictor for systems with correlated noise described in [21]. The other simulation compares the performance of the enhanced design with another robust predictor proposed recently in [17] using a system with uncorrelated noises.

A. Correlated Noise Simulation

Consider the following model with correlated noise and subject to uncertainties in every matrices

$$\begin{aligned}x_{k+1} &= \begin{bmatrix} 0.01\delta_{1,k} & -0.5 + 0.003\delta_{2,k} \\ 1 + \delta_{3,k} & 1 + 0.3\delta_{4,k}\end{bmatrix} x_k \\ &+ \begin{bmatrix} -6 - 0.02\delta_{5,k} \\ 1 - 2\delta_{6,k}\end{bmatrix} w_k \\ &+ \begin{bmatrix} -2 - 0.01\delta_{7,k} & -1 - 0.03\delta_{9,k} \\ 1 - 1\delta_{8,k} & -0.1 - 3\delta_{10,k}\end{bmatrix} v_k, \qquad (50)\\ y_k &= \begin{bmatrix} -100 - 0.1\delta_{11,k} & 10 - 0.03\delta_{12,k}\end{bmatrix} \\ &+ \left(1 + 0.2\delta_{13,k}\right) w_k \\ &+ \begin{bmatrix} 0.5 + 0.1\delta_{14,k} & 0.2 + 0.3\delta_{15,k}\end{bmatrix} v_k, \qquad (51)\end{aligned}$$

where $\delta_{n,k}$ varies randomly at each step and $|\delta_{n,k}| < 1$, for $n = 1, ..., 15$. We also use $W_k = 1$ and $V_k = I$ with initial conditions $\overline{x}_0 = 0$ and $X_0 = I$. Moreover,

$$\begin{aligned}&H_{1,k} = \begin{bmatrix} 0.1 \\ 10 \end{bmatrix}, \quad H_{2,k} = -1, \quad G_{x,k} = \begin{bmatrix} 0.1 & 0.03 \end{bmatrix}, \\ &G_{w,k} = 2, \quad G_{v,k} = \begin{bmatrix} -0.1 & -0.3 \end{bmatrix}. \qquad (52)\end{aligned}$$

The parameters α^{-1}_k, β^{-1}_k and γ^{-1}_k are calculated as

$$\alpha^{-1}_k = \sigma_{max}\left\{G_{x,k}\overline{P}_{k|k-1}G^T_{x,k}\right\} + \epsilon_x, \qquad (53)$$

$$\beta^{-1}_k = \sigma_{max}\left\{G_{w,k}W_kG^T_{w,k}\right\} + \epsilon_w, \qquad (54)$$

$$\gamma^{-1}_k = \sigma_{max}\left\{G_{v,k}V_kG^T_{v,k}\right\} + \epsilon_v, \qquad (55)$$

where $\sigma_{max}\{\bullet\}$ indicates the maximum singular value of a matrix. Numerical simulations show that, in general, smaller values of ϵ_x, ϵ_w and ϵ_v result in lower upper bounds. However, too small values can lead to ill conditioned inverses. In this example, we have chosen $\epsilon_x = \epsilon_w = \epsilon_v = 0.1$. Also, the upper bounds are sensitive to the values of the uncertainty model matrices H and G. If we consider all parameters constant but use $cH_{1,k}$ and $\frac{1}{c}G_{x,k}$ (or likewise, $\frac{1}{c}H_{2,k}$, $cG_{w,k}$ and $cG_{v,k}$), then in general, lower bounds are obtained with smaller values of the adjustment constant c. The state estimation errors, over $N = 500$ finite horizon experiments, of the proposed predictor and the classic predictor using the nominal model are shown at Table II.

TABLE II

APPROXIMATED ACTUAL ERROR VARIANCES.

Predictors	State 1	State 2
Enhanced Predictor	19.13dB	22.68dB
Kalman	19.56dB	24.21dB

The actual error variances were approximated using the ensemble-average, used in [6] and [22]. The ensemble-average is defined as

$$var\left\{e_{i,k}\right\} \approx \frac{1}{N}\sum_{j=1}^{N}(e^{(j)}_{i,k})^2, \qquad (56)$$

where $e^{(j)}_{i,k}$ is the i-th component of the estimation error vector $e^{(j)}_k$ at the experiment j defined as

$$e^{(j)}_k = x^{(j)}_k - \widehat{x}^{(j)}_{k|k-1}. \qquad (57)$$

The performance of the proposed predictor in this paper is better than the usual Kalman predictor in the presence of modeling errors. The actual estimation error variances for the states of the proposed predictor are always below their upper bounds, *i.e.*, 21.60dB and 27.20dB, respectively.

B. Uncorrelated Noise Simulation

The next simulation compares the performance of the enhanced predictor with another robust predictor recently proposed in [17]. Since the predictor in [17] is used for uncorrelated systems and is restricted to $dim\{w_k\} = dim\{v_k\}$, we consider the following uncertain model

$$x_{k+1} = \begin{bmatrix} 0 & -0.5 \\ 1 & 1 + 0.3\delta_{1,k}\end{bmatrix} + \begin{bmatrix} -6 \\ 1 + 0.01\delta_{2,k}\end{bmatrix} w_k, \qquad (58)$$

$$y_k = \begin{bmatrix} -1 & 1 + 1.5\delta_{3,k}\end{bmatrix} x_k + 100\delta_{4,k}v_k, \qquad (59)$$

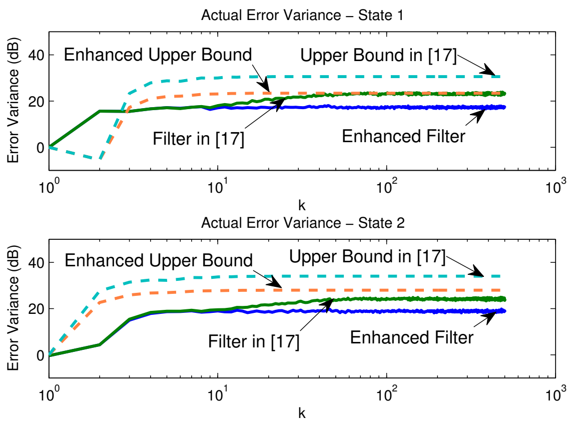

Fig. 1. Actual error variances for uncorrelated noise simulation.

where $\delta_{n,k}$ varies randomly at each step and $|\delta_{n,k}| < 1$, for $n = 1, ..., 4$. We also use $W_k = 0.01$ and $V_k = 1$ with initial conditions $\overline{x}_0 = 0$ and $X_0 = I$. The matrices associated to the uncertainties are given by

$$H_{1,k} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \quad H_{2,k} = 50, \quad G_{x,k} = \begin{bmatrix} 0 & 0.03 \end{bmatrix}, \\ G_{w,k} = 0.001, \quad G_{v,k} = 2. \tag{60}$$

Fig. 1 presents the actual error variances for the two states using the proposed predictor and the predictor developed in [17]. Fig. 1 also shows that the actual error variance of the enhanced predictor is lower for both states. The enhanced predictor provided a less conservative design, mainly due to the presence of an additional scalar parameter in the predictor design. Tables III and IV summarize these results.

TABLE III

PREDICTORS PERFOMANCE FOR STATE 1.

Predictors	Actual Error Variance	Upper Bound
Enhanced Predictor	17.02dB	23.40dB
Predictors in [17]	22.76dB	30.57dB

TABLE IV

PREDICTORS PERFOMANCE FOR STATE 2.

Predictors	Actual Error Variance	Upper Bound
Enhanced Predictor	18.56dB	27.95dB
Predictor in [17]	24.04dB	34.01dB

V. CONCLUSIONS

This paper has developed a robust Kalman predictor for finite horizon state-space estimation with correlated noises and subject to norm-bounded and time-varying uncertainties in every system matrices. The paper provides an enhancement over a recent guaranteed cost predictor by using an additional scaling parameter. The proposed predictor is suited for systems with unknown correlated dynamical and measurement noises, which is a very common situation in practice. Numerical simulations confirm the performance enhancement.

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