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Robust hybrid source-seeking algorithms based on directional derivatives and their approximations*

Christopher G. Mayhew, Ricardo G. Sanfelice, and Andrew R. Teel †

Abstract— **A family of hybrid control algorithms is developed that steer a nonholonomic autonomous vehicle to the source of a scalar signal present in the environment. In an idealized setting, we develop a general hybrid control scheme that globally asymptotically stabilizes the vehicle position about the source. Pursuing a practical implementation, a series of perturbations to the family of controllers is introduced, resulting in a semi-global practical stability of the vehicle position about the source. An example of a recently developed conjugate direction-based controller fitting into this family is developed and demonstrated by simulation and experiment.**

I. Introduction

We investigate the problem of localizing the source of a scalar signal existing in the environment with an autonomous vehicle. In this scenario, we assume that vehicle position measurements may not be available. Instead, we only assume that measurements of the signal at the current vehicle position are available. To complicate the matter, these measurements may be corrupted by noise and the signal distribution may be slowly varying. Such disturbances are common in a real-world scenario, where sensor noise can corrupt measurements and environmental disturbances can re-distribute the signal strength. Such a situation describes important problems in science and defense, including chemical plume tracing and land mine localization.

A broad spectrum of approaches have been applied to this problem in the recent literature, ranging from gradient descent with a single vehicle [5], to utilizing a sensor network of several vehicles to achieve gradient descent [1] or to conduct the simplex optimization algorithm [4]. In [7], the method of *extremum-seeking* was applied to a nonholonomic vehicle to achieve an average gradient descent by the vehicle. In [3], a method was developed for a point-mass vehicle to lead a group of vehicles to the source by means of nonlinear programming methods. Finally, in [2], a PD-type control law based on directional derivatives was used to steer a nonholonomic vehicle to the source.

In this paper, we develop a family of robust hybrid control algorithms that accomplish this task. The approach in this paper is similar in character to that of [12], [13], where a specific hybrid source-seeking controller was developed for a point-mass vehicle, and a nonholonomic vehicle, respectively. The results in this paper extend previous work by generalizing the hybrid source-seeking framework in [12], [13] to include a wide class of controllers. Moreover, an explicit treatment of noise is made in this paper by utilizing results in robust stability theory for hybrid systems [9].

This paper is organized as follows. Section II states the problem and introduces some convenient notation. Section III provides the necessary hybrid system concepts for this paper. Section IV introduces a family of hybrid controllers that accomplish the source-seeking task given certain assumptions. In Section V, we introduce a series of perturbations to the family of hybrid controllers in Section IV that lift the assumptions to derive a semi-global practical stability result. Finally, Section VI develops a conjugate-direction inspired controller (based upon [12], [13]) that fits into the family of hybrid controllers in Section IV and demonstrates the controller by simulation and experiment.

II. Problem Statement and Vehicle Model

In this paper, we develop a family of hybrid controllers that steer an autonomous vehicle to the source of a scalar signal existing in the vehicle's environment. We assume that this signal is described by a continuously differentiable function $\varphi : \mathbb{R}^2 \to \mathbb{R}$. We assume that for every $c \in \varphi(\mathbb{R}^2)$, the set $L(c) = \{y \in \mathbb{R}^2 : \varphi(y) \leq c\}$ is bounded. We assume that this function has a unique global minima, x^*, and that $\nabla\varphi(x) = 0$ if and only if $x = x^*$ (where $\nabla\varphi(x) = [\ \frac{\partial\varphi}{\partial x_1} \quad \frac{\partial\varphi}{\partial x_2}\ ]^T$). The goal is to locate x^* with an autonomous vehicle and stabilize it about x^*.

We consider the following nonholonomic vehicle model,

$$\dot{x} = \gamma \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \qquad \dot{\theta} = \omega, \tag{1}$$

where $x \in \mathbb{R}^2$ denotes the vehicle position, $\theta \in \mathbb{R}$ denotes the orientation of the vehicle, and γ and ω are control inputs for forward velocity and angular velocity, respectively. When constrained to have a fixed velocity and a minimum turning radius, this unicycle model is sometimes referred to as a *Dubins vehicle* [8] in the literature.

While the model (1) is very common, in this paper, we use an equivalent model, where the vehicle orientation is described by a vector in $\mathbb{S}^1 = \{y \in \mathbb{R}^2 : \|y\|_2 = 1\}$. Then, the autonomous vehicle has a state $(x, \vartheta) \in \mathbb{R}^2 \times \mathbb{S}^1$ that evolves according to

$$\dot{x} = \gamma\vartheta \qquad \dot{\vartheta} = \vartheta \otimes \eta(\omega), \tag{2}$$

† {mayhew,teel}@ece.ucsb.edu, Center for Control Engineering and Computation, Electrical and Computer Engineering Department, University of California, Santa Barbara, CA 93106-9560.

sricardo@mit.edu, Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, MA 02139, Tel (617) 324-8498,

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where, for two vectors $y, z \in \mathbb{R}^2$,

$$z \otimes y = \begin{bmatrix} z_1 y_1 - z_2 y_2 \\ z_2 y_1 + z_1 y_2 \end{bmatrix}.$$

The function $\eta : \mathbb{R} \to \mathbb{R}^2$ maps values according to $\eta(\omega) \mapsto [\ 0 \quad \omega\ ]^T$. In this way, (2) has $[\ \dot{\vartheta}_1 \quad \dot{\vartheta}_2\ ]^T = \omega[\ -\vartheta_2 \quad \vartheta_1\ ]^T$. This representation of the unicycle model makes for a convenient description by allowing the vehicle orientation, ϑ, to remain in a compact set (with the expense of describing it with two states). It is easy to see that $\mathbb{S}^1$ is an invariant set for ϑ by noting that $\langle \vartheta, \vartheta \otimes \eta(\omega) \rangle = 0$. The control input ω simply acts to push ϑ around the unit circle.

We also note that the $\otimes$ operator can act as a vector rotation in the following way. Letting $\alpha \in \mathbb{R}$, $[\ c \quad s\ ]^T = [\ \cos(\alpha) \quad \sin(\alpha)\ ]^T$, and $z \in \mathbb{R}^2$,

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \otimes \begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} c z_1 - s z_2 \\ s z_1 + c z_2 \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

For a given function $\Gamma : X \to X$, we denote Γ^n as the composition of Γ n times, that is, $\Gamma^n = \Gamma \circ \cdots \circ \Gamma$, n times. For some set X, we denote $\operatorname{cl} X$ as the closure of X. Finally, $\mathbb{B} \subset \mathbb{R}^n$ denotes the closed unit ball.

III. Hybrid Systems Preliminaries

A hybrid dynamical system is one in which both continuous and discrete state evolution can occur. Working in the framework of [9], we define the state of a hybrid system as $\zeta \in \mathbb{R}^n$. The continuous evolution is governed by a differential inclusion, $\dot{\zeta} \in F(\zeta)$, and discrete state jumps are governed analogously by a difference inclusion, $\zeta^+ \in G(\zeta)$. Letting $\rightrightarrows$ denote a set-valued mapping, we refer to $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as the *flow map*, and to $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as the *jump map*. Whether or not flows and/or jumps are allowed during certain instances of the state evolution is dictated by set inclusion conditions. We define the sets $C, D \subset \mathbb{R}^n$, as the *flow set* and *jump set*, respectively. The state is allowed to flow continuously when $\zeta \in C$ and allowed to make discrete jumps when $\zeta \in D$. A hybrid system is defined by its data, $\mathcal{H} = (F, G, C, D)$. We write such a hybrid system as

$$\mathcal{H} \begin{cases} \dot{\zeta} \in F(\zeta) & \zeta \in C \\ \zeta^+ \in G(\zeta) & \zeta \in D. \end{cases}$$

In this paper, we will assure that any proposed hybrid system satisfies the *Hybrid Basic Conditions* [9, A0-A3], which are a set of mild regularity conditions on the data of $\mathcal{H}$. These mild conditions allow the use of robust stability theory for hybrid systems fitting this framework.

Following [9], we now define a solution to a hybrid system. Denoting $[0, \infty)$ as $\mathbb{R}_{\geq 0}$ and $\{0, 1, 2, \dots\}$ as $\mathbb{Z}_{\geq 0}$, we define a hybrid time domain as follows. For some number $J \in \mathbb{Z}_{\geq 0}$ and a sequence of times, $0 = t_0 \leq t_1 \leq \cdots \leq t_J$, we call a subset $\mathcal{D} \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ a *compact hybrid time domain* if $\mathcal{D} = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$. We say that $\mathcal{D}$ is a *hybrid time domain* if for all $(T, J) \in \mathcal{D}$, $\mathcal{D} \cap ([0, T] \times \{0, 1, 2, \dots, J\})$ is a compact hybrid time domain. Such a definition allows the "last" interval to be of the form $[t_j, T)$, with T finite or $T = \infty$.

A solution to a hybrid system is given as a mapping $\zeta : \operatorname{dom} \zeta \to \mathbb{R}^n$, where $\operatorname{dom} \zeta$ is a hybrid time domain. To be a solution to $\mathcal{H}$, a ζ must satisfy $\zeta(0, 0) \in C \cup D$, and

1) for all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t, j) \in \operatorname{dom} \zeta$, $\zeta(t, j) \in C$, $\dot{\zeta} \in F(\zeta(t, j))$
2) for all $(t, j) \in \operatorname{dom} \zeta$ such that $(t, j+1) \in \operatorname{dom} \zeta$, $\zeta(t, j) \in D$, $\zeta(t, j+1) \in G(\zeta(t, j))$.

A solution is called *complete* if $\operatorname{dom} \zeta$ is unbounded and *Zeno* if ζ is complete, but the projection of $\operatorname{dom} \zeta$ onto $\mathbb{R}_{\geq 0}$ is bounded. Letting $\mathcal{S}_{\mathcal{H}}(\zeta_0)$ denote the set of solutions to $\mathcal{H}$ with $\zeta(0, 0) = \zeta_0$, a solution is *maximal* if it is not a truncation of some other solution $\zeta' \in \mathcal{S}_{\mathcal{H}}(\zeta_0)$ to some proper subset of $\operatorname{dom} \zeta'$. We say that ζ is *precompact* if $\operatorname{cl} \zeta(\operatorname{dom} \zeta) \subset \mathbb{R}^n$ is compact.

Now we introduce stability and attractivity concepts for hybrid systems. A compact set $\mathcal{A} \subset \mathbb{R}^n$ is *stable* if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall \zeta_0 \in (\mathcal{A} + \delta \mathbb{B}) \cap (C \cup D)$, each solution $\zeta \in \mathcal{S}_{\mathcal{H}}(\zeta_0)$ satisfies $|\zeta(t, j)|_{\mathcal{A}} \leq \epsilon\ \forall (t, j) \in \operatorname{dom} \zeta$. It is *locally attractive* if $\exists \delta > 0$ such that $\forall \zeta_0 \in (\mathcal{A} + \delta \mathbb{B}) \cap (C \cup D)$, every $\zeta \in \mathcal{S}_{\mathcal{H}}(\zeta_0)$ is complete and satisfies $\lim_{t+j \to \infty} |\zeta(t, j)|_{\mathcal{A}} = 0$. The set of points from where maximal solutions are complete and converge to $\mathcal{A}$ is called the *basin of attraction* for $\mathcal{A}$, denoted by $\mathcal{B}_{\mathcal{A}}$. The set $\mathcal{A}$ is called *locally asymptotically stable* if it is both stable and attractive, and *uniformly attractive* if from each compact $M \subset \mathcal{B}_{\mathcal{A}}$ and for every $\epsilon > 0$, there exists $T > 0$ such that $\forall \zeta_0 \in M$, $t + j \geq T \implies |\zeta(t, j)|_{\mathcal{A}} \leq \epsilon$. The set $\mathcal{A}$ is *globally asymptotically stable* if $\mathcal{A}$ is stable and attractive with $\mathcal{B}_{\mathcal{A}} = \mathbb{R}^n$. (Note that by definition, points in $\mathbb{R}^n \setminus (C \cup D)$ belong to the basin of attraction since there are no solutions from those points.)

IV. A General Hybrid Control Algorithm

In this section, we introduce a general hybrid control scheme to accomplish the source localization task. In a somewhat idealized setting, we, for now, assume that measurements of $\langle \nabla \varphi(x), \gamma \vartheta \rangle$ are available for control. We also assume that the controller can force the vehicle orientation to make discrete jumps according to $\vartheta^+ = \vartheta \otimes r$, where $(\vartheta, r) \in \mathbb{S}^1 \times \mathbb{S}^1$ (note that this corresponds to a rotation, so $\mathbb{S}^1$ is still invariant for ϑ). These assumptions will be relaxed in later sections. Similar to [16], we propose the following hybrid controller,

$$\left. \begin{aligned} \gamma &= \bar{\gamma} \\ \begin{bmatrix} \dot{z} \\ \omega \end{bmatrix} &\in \begin{bmatrix} f_z(z) \\ f_\omega(z) \end{bmatrix} \end{aligned} \right\} \quad (x, \vartheta, z) \in C$$
$$\left. \begin{bmatrix} z^+ \\ r \end{bmatrix} \in \begin{bmatrix} g_z(z) \\ g_r(z) \end{bmatrix} \right\} \quad (x, \vartheta, z) \in D, \tag{3}$$

where $0 < |\bar{\gamma}| \leq \gamma^*$,

$$\begin{aligned} C &= \{(x, \vartheta, z) : \langle \nabla \varphi(x), \gamma \vartheta \rangle \leq 0, \vartheta \in \mathbb{S}^1, z \in \Upsilon\} \\ D &= \{(x, \vartheta, z) : \langle \nabla \varphi(x), \gamma \vartheta \rangle \geq 0, \vartheta \in \mathbb{S}^1, z \in \Upsilon\}, \end{aligned} \tag{4}$$

and the set-valued mappings $f_z : \Upsilon \rightrightarrows \mathbb{R}^\kappa$, $f_\omega : \Upsilon \rightrightarrows \mathbb{R}$, $g_z : \Upsilon \rightrightarrows \Upsilon$, $g_r : \Upsilon \rightrightarrows \mathbb{S}^1$, and $\Upsilon \subset \mathbb{R}^\kappa$ satisfy the following:

(A1) f_z and f_ω are nonempty, locally bounded, outer semicontinuous[1] and convex valued on Υ. The set-valued mappings g_z and g_r are nonempty and outer-semicontinuous on Υ.

(A2) Υ is compact.

(A3) The maximal solutions to the continuous-time system $\dot{z} \in f_z(z), z \in \Upsilon$ and the maximal solutions to the discrete-time system $z^+ \in g_z(z), z \in \Upsilon$ are complete.

(A4) $C_0 = \{(x, \vartheta, z) : \langle \nabla\varphi(x), \gamma\vartheta\rangle = 0, \vartheta \in \mathbb{S}^1, z \in \Upsilon\}$. The only complete solutions to

$$\left.\begin{aligned} \dot{x} &= \bar{\gamma}\vartheta \\ \dot{\vartheta} &\in \vartheta \otimes \eta(f_\omega(z)) \\ \dot{z} &\in f_z(z) \end{aligned}\right\} \quad (x, \vartheta, z) \in C_0$$
$$\left.\begin{aligned} x^+ &= x \\ \vartheta^+ &\in \vartheta \otimes g_r(z) \\ z^+ &\in g_z(z) \end{aligned}\right\} \quad (x, \vartheta, z) \in D \tag{5}$$

begin from the set $\{(x, \vartheta, z) : x = x^*\}$.

The data $(f_z, f_\omega, g_z, g_r, \Upsilon)$ defines a source-seeking controller if it fits into (3) and (4). We say that a source-seeking controller $\mathcal{S} = (f_z, f_\omega, g_z, g_r, \Upsilon)$ has the *continual search* property if it satisfies assumptions **(A1)-(A4)**.

Applying (3) to (2) results in the closed-loop system

$$\left.\begin{aligned} \dot{x} &= \bar{\gamma}\vartheta \\ \dot{\vartheta} &\in \vartheta \otimes \eta(f_\omega(z)) \\ \dot{z} &\in f_z(z) \end{aligned}\right\} \quad (x, \vartheta, z) \in C$$
$$\left.\begin{aligned} x^+ &= x \\ \vartheta^+ &\in \vartheta \otimes g_r(z) \\ z^+ &\in g_z(z) \end{aligned}\right\} \quad (x, \vartheta, z) \in D. \tag{6}$$

Then, (6), on the state space $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^\kappa$, satisfies the Hybrid Basic Conditions [9, A0-A3].

For convenience, we group the variables (x, ϑ, z) into a single variable ξ and, letting $n = 4 + \kappa$, denote the right hand sides of (6) as $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and define our closed loop as $\mathcal{H} = (F, G, C, D)$. Conveniently, we define the set $B = \mathbb{R}^2 \times \mathbb{S}^1 \times \Upsilon$ so that we can re-write $C = \{\xi \in B : \langle \nabla\varphi(x), \gamma\vartheta\rangle \le 0\}$ and $D = \{\xi \in B : \langle \nabla\varphi(x), \gamma\vartheta\rangle \ge 0\}$ in terms of B. We define the closed-loop hybrid system $\mathcal{H} = (F, G, C, D)$.

Lemma 4.1: Suppose that a hybrid source-seeking controller has the continual search property. Then, solutions to (6) *exist everywhere in $C \cup D$ and every maximal solution is complete.*

Theorem 4.2: Suppose that the proposed hybrid source-seeking controller (3), (4) *has the continual search property. Then, the set $\mathcal{A} = \{x^*\} \times \mathbb{S}^1 \times \Upsilon$ is globally asymptotically stable for $\mathcal{H}$.*

[1]A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if for all $x \in \mathbb{R}^n$ and all sequences $\{x_i\}_{i=1}^\infty$, $\{y_i\}_{i=1}^\infty$, such that $y_i \in F(x_i)$, $x_i \to x$, and $y_i \to y$ as $i \to \infty$, we have $y \in F(x)$.

Proof: (Sketch) Using $V(\xi) = \varphi(x) - \varphi(x^*)$ as a Lyapunov-like function, it is easily seen that this function is non-increasing along flows and jumps. This follows from the structure of the flow set and the fact that $x^+ = x$ during every jump. Since V has compact level sets, is positive definite on $C \cup D$ with respect to $\mathcal{A}$, and since $\mathcal{A}$ is compact, $\mathcal{A}$ is stable, by [15, Theorem 7.6]). Attractivity is shown by computing invariant sets where V is constant along solutions (this involves using assumption **A4**) and invoking an invariance principle for hybrid systems [15, Theorem 4.7]. ∎

V. PRACTICALITY THROUGH PERTURBATION

In this section, we remove the assumptions that $\langle \nabla\varphi(x), \gamma\vartheta\rangle$ is available for measurement and that ϑ can be updated during jumps. We shall address these issues through a series of perturbations to the proposed hybrid controllers. The first issue can be addressed by collecting several values of φ along the vehicle's trajectory and using this data to estimate the directional derivative. To obviate the need for updating the vehicle orientation at jumps, it is possible to execute open-loop maneuvers.

The main result of this section is a semi-global practical asymptotic stability theorem which asserts robustness to the perturbations to the idealized algorithm.

A. Temporal Regularization

The first step in estimating $\langle \nabla\varphi(x), \gamma\vartheta\rangle$ is to introduce functionality into the controller which allows a minimum amount of information to be collected about φ along the current search direction. We implement this functionality into the controller through a technique known as *temporal regularization*, meant to eliminate Zeno solutions in hybrid systems by enforcing a small amount of flow between jumps. Temporal regularization has been discussed in the recent literature, with emphasis on simulation [11], [10], and perturbations to hybrid systems [9].

We define the temporal regularization parameter, $\delta_\tau \in \mathbb{R}_{\ge 0}$, and introduce a new timer state, $\tau \in \mathbb{R}$, into our controller in the following way. Let the new state variable be $\tilde{\xi} = (\xi, \tau) \in \mathbb{R}^{n+1}$ and $\widetilde{B} = B \times [0, K] \subset \mathbb{R}^{n+1}$, the new dynamics are given by

$$\left.\begin{aligned} \dot{\xi} &= F(\xi) \\ \dot{\tau} &= K - \tau \end{aligned}\right\} \quad (\xi, \tau) \in \widetilde{C}_{\delta_\tau}$$
$$\left.\begin{aligned} \xi^+ &\in G(\xi) \\ \tau^+ &= 0 \end{aligned}\right\} \quad (\xi, \tau) \in \widetilde{D}_{\delta_\tau}, \tag{7}$$

where

$$\begin{aligned} \widetilde{C}_{\delta_\tau} &= \{(\xi, \tau) \in \widetilde{B} : \xi \in C \text{ or } \tau \in [0, \delta_\tau]\} \\ \widetilde{D}_{\delta_\tau} &= \{(\xi, \tau) \in \widetilde{B} : \xi \in D \text{ and } \tau \in [\delta_\tau, K]\} \end{aligned} \tag{8}$$

and $K > \delta_\tau$. We denote this system by

$$\mathcal{H}_{\delta_\tau} \begin{cases} \dot{\tilde{\xi}} = \widetilde{F}(\tilde{\xi}) & \tilde{\xi} \in \widetilde{C}_{\delta_\tau} \\ \tilde{\xi}^+ \in \widetilde{G}(\tilde{\xi}) & \tilde{\xi} \in \widetilde{D}_{\delta_\tau}, \end{cases} \tag{9}$$

with $\widetilde{F}(\tilde{\xi}) = \begin{bmatrix} F(\xi)^T & K - \tau \end{bmatrix}^T$ and $\widetilde{G}(\tilde{\xi}) = \begin{bmatrix} G(\xi)^T & 0 \end{bmatrix}^T$.

This change in dynamics can be seen as *requiring* flow as long as the timer, τ, has not reached its limit, δ_τ. Surely, this eliminates any possible Zeno behavior brought on by shorter flow times as the vehicle approaches x^*. Moreover, as in [9], we note that as $\delta_\tau \to 0$, the ξ solutions of (7) approach the ξ solutions of (6) in a graphical sense and with $\delta_\tau = 0$, the ξ solutions of (7) are exactly the same as those of (6). In this direction, we denote a "nominal" (i.e. $\delta_\tau = 0$) system with temporal regularization, by $\widetilde{\mathcal{H}} = (\widetilde{F}, \widetilde{G}, \widetilde{C}, \widetilde{D})$, where $\widetilde{C}$ and $\widetilde{D}$ are defined by evaluating $\widetilde{C}_{\delta_\tau}$ and $\widetilde{D}_{\delta_\tau}$ with $\delta_\tau = 0$. In preparation for the following section, we make the following observation, which follows from the arguments presented in [9, Example 6.8].

Corollary 5.1: Under the assumptions in Theorem 4.2, the set $\widetilde{\mathcal{A}} = \mathcal{A} \times [0, K]$ is globally asymptotically stable for the system $\widetilde{\mathcal{H}}$.

B. Directional Derivative Estimation

In this section, we model the estimation error of $\langle \nabla\varphi(x), \gamma\vartheta \rangle$ with perturbations to the sets C and D. Due to length constraints, we do not attempt to elaborate on the details of estimating $\langle \nabla\varphi(x), \gamma\vartheta \rangle$. Instead, we note that temporal regularization and sample-and-hold can be used to collect and store information about φ for estimation of the derivative along the vehicle trajectory.

We assume that the algorithm is equipped with a corrupted measurement. We model the measurement by perturbing the flow and jump sets, C and D, to capture the many trajectories that can result from such error-ridden measurements. Letting $\hat{\delta} = (\delta_1, \delta_2) \in \mathbb{R}^2_{>0}$, we define

$$\widehat{C}_{\hat{\delta}} = \{\xi \in B : \exists \hat{x} \in x + \delta_1 \mathbb{B}, \text{ s.t. } \langle \nabla\varphi(\hat{x}), \gamma\vartheta \rangle \leq \delta_2\}$$
$$\widehat{D}_{\hat{\delta}} = \{\xi \in B : \exists \hat{x} \in x + \delta_1 \mathbb{B}, \text{ s.t. } \langle \nabla\varphi(\hat{x}), \gamma\vartheta \rangle \geq -\delta_2\}.$$

Defining $\delta = (\delta_\tau, \hat{\delta})$ and adding temporal regularization to the perturbed system as before in (7) and (8), we have

$$\mathcal{H}_\delta \begin{cases} \dot{\tilde{\xi}} \in \widetilde{F}(\tilde{\xi}) & \tilde{\xi} \in C_\delta \\ \tilde{\xi}^+ \in \widetilde{G}(\tilde{\xi}) & \tilde{\xi} \in D_\delta, \end{cases} \tag{10}$$

where

$$\begin{aligned} C_\delta &= \{(\xi, \tau) \in \widetilde{B} : \xi \in \widehat{C}_{\hat{\delta}} \text{ or } \tau \in [0, \delta_\tau]\} \\ D_\delta &= \{(\xi, \tau) \in \widetilde{B} : \xi \in \widehat{D}_{\hat{\delta}} \text{ and } \tau \in [\delta_\tau, K]\}. \end{aligned} \tag{11}$$

C. Perturbation and Practical Stability

Analyzing the errors introduced to our idealized algorithm (6) by derivative approximation and temporal regularization requires invoking the existing robustness theory for hybrid systems in [9]. In this section, we will define a family of perturbed hybrid systems and assert a semi-global practical stability property of the perturbed system.

We define a sequence $\{\delta_i\}_{i=1}^\infty = \{(\delta_\tau^i, \delta_1^i, \delta_2^i)\}$, with $\bar{\delta}_i = \max\{\delta_\tau^i, \delta_1^i, \delta_2^i\}$, such that for all $i \in \mathbb{Z}_{>0}$, $\bar{\delta}_{i+1} < \bar{\delta}_i$ and $\bar{\delta}_i \to 0$ as $i \to \infty$. We then define a sequence of perturbed hybrid systems by $\mathcal{H}_{\delta_i} = (\widetilde{F}, \widetilde{G}, C_{\delta_i}, D_{\delta_i})$. Due to space constraints, we omit the proof of the following lemma; however, we note that it follows from basic set convergence arguments in [14, Exercise 4.3].

Lemma 5.2: The sequences of sets $\{C_{\delta_i}\}_{i=1}^\infty$ and $\{D_{\delta_i}\}_{i=1}^\infty$ converge and their limits are given by $\widetilde{C}$ and $\widetilde{D}$, respectively.

The following corollary relates $\mathcal{H}_\delta$ to $\mathcal{H}_{\delta_\tau}$. Defining $\mathcal{H}_0$ as the system $\mathcal{H}_\delta$ with $\delta = 0$, we see that $\mathcal{H}_0$ is identical to $\widetilde{\mathcal{H}}$. The following corollary follows from Corollary 5.1.

Corollary 5.3: For the system $\mathcal{H}_0$, the set $\widetilde{\mathcal{A}} = \mathcal{A} \times [0, K]$ is globally asymptotically stable.

Our main result of this section is stated next. We say that a continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ if, for each fixed t, $\beta(s, t)$ is zero at zero and strictly increasing with respect to s, and for each fixed s, $\beta(s, t)$ is non-increasing with respect to t and converges to zero as $t \to \infty$. Following [9, Theorem 6.6], we can restate Corollary 5.3 with a $\mathcal{KL}$ function.

Theorem 5.4: For the system $\mathcal{H}_\delta$, with $\delta = 0$, there exists $\beta \in \mathcal{KL}$ such that all solutions $\tilde{\xi}$ satisfy

$$|\tilde{\xi}(t, j)|_{\mathcal{A}} \leq \beta(|\tilde{\xi}(0, 0)|_{\mathcal{A}}, t + j) \qquad \forall (t, j) \in \operatorname{dom} \tilde{\xi}. \tag{12}$$

Moreover, using [9, Theorem 6.6], the following result holds.

Theorem 5.5: (Semi-global practical stability) For the system $\mathcal{H}_\delta$, with $\delta = 0$, there exists $\beta \in \mathcal{KL}$ such that (12) *holds and for every compact set $M \subset \mathcal{B}_{\widetilde{\mathcal{A}}}$ and each $\epsilon > 0$, there exists $\delta^* > 0$ such that for each δ with $\bar{\delta} \in (0, \delta^*]$, the solutions $\tilde{\xi}_\delta$ of $\mathcal{H}_\delta$ starting from M satisfy, for all $(t, j) \in \operatorname{dom} \tilde{\xi}_\delta$,*

$$|\tilde{\xi}_\delta(t, j)|_{\mathcal{A}} \leq \beta(|\tilde{\xi}_\delta(0, 0)|_{\mathcal{A}}, t + j) + \epsilon. \tag{13}$$

D. Open-loop Turning Maneuvers

In this section, we remove the assumption that ϑ can be updated during controller jumps by $\vartheta^+ = \vartheta \otimes g_r(z)$. Since measurements of (x, ϑ) may not be available for control, we integrate a family of *open-loop* maneuvers into our controller.

Given $r \in \mathbb{S}^1$, the family of open-loop control laws, $\Theta : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ must satisfy the following.

(A5) $\forall r \in \mathbb{S}^1$, $\exists T_m(r) \geq 0$ such that the system,

$$\begin{aligned} \dot{x} &= \gamma\vartheta & \dot{\vartheta} &= \vartheta \otimes \eta(\omega) \\ \dot{\tau}_m &= 1 & \begin{bmatrix} \gamma & \omega \end{bmatrix} &= \Theta(\tau_m, r), \end{aligned} \tag{14}$$

with

$$\begin{bmatrix} x(0) & \vartheta(0) & \tau_m(0) \end{bmatrix}^T = \begin{bmatrix} x_0 & \vartheta_0 & 0 \end{bmatrix}^T, \tag{15}$$

satisfies $x(T_m(r)) = x_0$ and $\vartheta(T_m(r)) = \vartheta \otimes r$ (and $\tau_m = T_m(r)$). Moreover, we assume that $\sup_{r \in \mathbb{S}^1} T_m(r) \leq K_T$ and $|\gamma| \leq \gamma^*$.

*Lemma 5.6: Suppose that Θ satisfies **(A5)**. Then, for every $r \in \mathbb{S}^1$, the x-component of the solution to* (14) *satisfies $\|x(t) - x(0)\| \leq |\gamma| K_T$ for all $t \in [0, T_m(r)]$.*

Proof: $\|x(t) - x(0)\| = \left\| \int_0^t \gamma\vartheta dt \right\| \leq \gamma^* K_T$. ■

We integrate this functionality into the controller in the following way. We first introduce two additional states: a logic state $m \in \{0, 1\}$ and a timer for the open-loop control law $\tau_m \in [0, K_T]$. Let the new state be $\xi_\Theta = (\tilde{\xi}, \tau_m, m) \in \mathbb{R}^{n+3}$. Letting $B_\Theta = \widetilde{B} \times [0, K_T] \times \{0, 1\}$, we write the new perturbed closed-loop as

$$\left.\begin{aligned} \dot{x} &= \gamma\vartheta \\ \dot{\vartheta} &= \vartheta \otimes \eta(\omega) \\ \dot{z} &\in f_z(z)\,\mathrm{toggle}(m) \\ \dot{\tau} &= (K-\tau)\,\mathrm{toggle}(m) \\ \dot{\tau}_m &= m \\ \dot{m} &= 0 \\ \begin{bmatrix}\gamma \\ \omega\end{bmatrix} &\in \begin{cases} \begin{bmatrix}\bar{\gamma} & f_\omega(z)\end{bmatrix}^T & m = 0 \\ \Theta(\tau_m, g_r(z)) & m = 1 \end{cases} \end{aligned}\right\} \xi_\Theta \in C_\Theta^\delta \tag{16}$$

$$\left.\begin{aligned} x^+ &= x \\ \vartheta^+ &= \vartheta \\ z^+ &\in \begin{cases} z & m = 0 \\ g_z(z) & m = 1 \end{cases} \\ \tau^+ &= 0 \\ \tau_m^+ &= 0 \\ m^+ &= \mathrm{toggle}(m) \end{aligned}\right\} \xi_\Theta \in D_\Theta^\delta.$$

Here, we have $\mathrm{toggle}(s) = 1 - s$ and

$$C_\Theta^\delta = \{(\tilde{\xi}, \tau_m, m) \in B_\Theta : \tilde{\xi} \in C_\delta \text{ and } m = 0 \text{ or } \tau_m \in [0, T_m(g_r(z))] \text{ and } m = 1\} \tag{17}$$

$$D_\Theta^\delta = \{(\tilde{\xi}, \tau_m, m) \in B_\Theta : \tilde{\xi} \in D_\delta \text{ and } m = 0 \text{ or } \tau_m = T_m(g_r(z)) \text{ and } m = 1\}. \tag{18}$$

For compactness, we write (16) as $\mathcal{H}_\Theta^\delta = \big(F_\Theta, G_\Theta, C_\Theta^\delta, D_\Theta^\delta\big)$. where F_Θ and G_Θ are defined as above in (16). Additionally, we define a nominal system with open-loop turns as $\mathcal{H}_\Theta = (F_\Theta, G_\Theta, C_\Theta, D_\Theta)$ where

$$C_\Theta = \{(\tilde{\xi}, \tau_m, m) \in B_\Theta : \tilde{\xi} \in \widetilde{C} \text{ and } m = 0 \text{ or } \tau_m \in [0, T_m(g_r(z))] \text{ and } m = 1\} \tag{19}$$

$$D_\Theta = \{(\tilde{\xi}, \tau_m, m) \in B_\Theta : \tilde{\xi} \in \widetilde{D} \text{ and } m = 0 \text{ or } \tau_m = T_m(g_r(z)) \text{ and } m = 1\}. \tag{20}$$

The following corollary asserts a stability property for the perturbed closed-loop system $\mathcal{H}_\Theta^\delta$. This result follows examining the structure of C_Θ^δ and Lemma 5.2. D_Θ^δ

Corollary 5.7: The sequences of sets $\{C_\Theta^{\delta_i}\}_{i=1}^\infty$ and $\{D_\Theta^{\delta_i}\}_{i=1}^\infty$ converge and their limits are given by C_Θ and D_Θ, respectively.

We now make a convergence and stability claim for the nominal system, $\mathcal{H}_\Theta$. The proof is omitted due to space constraints, but we note that it requires comparing solutions of $\mathcal{H}_\Theta$ to those of (9) for proving uniform attractivity, then invoking [6, Theorem 1] to prove the existence of an asymptotically stable set.

Theorem 5.8: For $\mathcal{H}_\Theta$, for every $\sigma > 0$, the set

$$\mathcal{A}'_\Theta = x^* + (\gamma^* K_T + \sigma)\mathbb{B} \times \mathbb{S}^1 \times \Upsilon \times [0, K] \times [0, K_T] \times \{0, 1\}$$

is uniformly attractive from any compact set $M \subset C_\Theta \cup D_\Theta$. Moreover, there exists a globally asymptotically stable set $\mathcal{A}_\Theta \subset \mathcal{A}'_\Theta$.

Theorem 5.9: For the system $\mathcal{H}_\Theta$, there exists $\beta \in \mathcal{KL}$ such that all solutions ξ_Θ satisfy

$$|\xi_\Theta(t, j)|_\mathcal{A} \le \beta(|\xi_\Theta(0, 0)|_\mathcal{A}, t + j) \qquad \forall (t, j) \in \mathrm{dom}\,\xi_\Theta.$$

Then, for every compact set $M \subset \mathcal{B}_{\mathcal{A}_\Theta}$ and every $\epsilon > 0$, there exists $\delta^* > 0$ such that for each δ with $\bar{\delta} \in (0, \delta^*]$, the solutions ξ_Θ of $\mathcal{H}_\Theta^\delta$ starting from M satisfy, for all $(t, j) \in$ $\mathrm{dom}\,\xi_\Theta$,

$$|\xi_\Theta(t, j)|_\mathcal{A} \le \beta(|\xi_\Theta(0, 0)|_\mathcal{A}, t + j) + \epsilon.$$

VI. Source Localization with a Conjugate Direction Algorithm

We give an algorithm that fits into the framework presented in this paper and satisfies the continual search property. This algorithm is based upon the *Recursive Smith-Powell* algorithm reported in [12], [13], which utilizes the efficiency of *conjugate directions* in the search for x^*. We implement the algorithm as follows.

The state z consists of several elements, $\overline{z} := (\lambda_1, \lambda_2, v, p, k)$, where $\lambda_1, \lambda_2 \in h\mathbb{B}$, $v \in \mathbb{S}^1$, $p \in \{-1, 1\}$, $k \in \{0, 1, 2\}$. The value h is a large positive constant. The purpose of these variables is similar to that in [12]: the states λ_1 and λ_2 store the vectors traveled by the vehicle, v stores the current search direction, p is a logic variable which coordinates the line minimization, and k stores the current algorithm mode.

Then, the state $\overline{z}$ evolves in

$$\overline{\Upsilon} = (h\mathbb{B})^2 \times \mathbb{S}^1 \times \{-1, 1\} \times \{0, 1, 2\}. \tag{21}$$

To completely define the controller, we must define f_z, f_ω, g_z, and g_r. During flows, the vehicle is driven in a straight line, which corresponds to setting $\overline{f}_\omega(\overline{z}) := 0$. While driving the vehicle in a straight line, an open-loop integration of the vehicle's movement is stored in λ_1, while keeping the other states constant. To ensure that λ_1, λ_2 remain in $\overline{\Upsilon}$, we design the flow map to stop this integration when λ_1 approaches the boundary of $\overline{\Upsilon}$. Letting $0 < \epsilon_h \ll h$, we define

$$\begin{aligned} \overline{\Upsilon}_\epsilon^{\text{in}} &= ((h - \epsilon_h)\mathbb{B})^2 \times \mathbb{S}^1 \times \{-1, 1\} \times \{0, 1, 2\} \\ \overline{\Upsilon}_\epsilon^{\text{out}} &= \mathrm{cl}\,(h\mathbb{B} \setminus (h - \epsilon_h)\mathbb{B})^2 \times \mathbb{S}^1 \times \{-1, 1\} \times \{0, 1, 2\}. \end{aligned} \tag{22}$$

Then, we define $\overline{f}_z$ as,

$$\dot{\overline{z}} = \begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{v} \\ \dot{p} \\ \dot{k} \end{bmatrix} = \overline{f}_z(\overline{z}) := \begin{cases} \begin{bmatrix} \bar{\gamma} v \\ 0 \end{bmatrix} & \overline{z} \in \overline{\Upsilon}_\epsilon^{\text{in}} \\ \overline{\mathrm{con}} \left\{ \begin{bmatrix} \bar{\gamma} v \\ 0 \end{bmatrix}, 0 \right\} & \overline{z} \in \overline{\Upsilon}_\epsilon^{\text{in}} \cap \overline{\Upsilon}_\epsilon^{\text{out}} \\ 0 & \overline{z} \in \overline{\Upsilon}_\epsilon^{\text{out}}, \end{cases} \tag{23}$$

where $\overline{\mathrm{con}}$ denotes taking the closed, convex hull.

During jumps, the controller uses the stored information to generate new search directions. Letting $d_{\min}$ be some small number (perhaps, with temporal regularization, $d_{\min} = \bar{\gamma}\delta_\tau + \epsilon$), we define the following sets,

$$D_1 = \{z \in \Upsilon : p = 1 \text{ and } \|\lambda_1\| \leq d_{\min}\} \tag{24}$$
$$D_2 = \{z \in \Upsilon : p = 1 \text{ and } \|\lambda_1\| \geq d_{\min} \text{ or } p = -1\} \tag{25}$$

(note that $D_1 \cup D_2 = \Upsilon$). We also define

$$g_1(z) = \begin{bmatrix}\lambda_1 & \lambda_2 & \mathcal{R}_\pi v & -p & k\end{bmatrix}^T \tag{26}$$
$$g_2(z) = \begin{bmatrix}0 & \lambda_1 & V(z) & 1 & (k+1) \bmod 3\end{bmatrix}^T \tag{27}$$

where the $\mathcal{R}_\alpha$ operator denotes a vector rotation by α radians (so that for some $v \in \mathbb{S}^1$, $\mathcal{R}_\pi v = -v$),

$$V(z) \in \begin{cases}\mathcal{R}_{\pi/2} v & k = 0\\ \mathcal{R}_{-\pi/2} v & k = 1\\ \Phi(\lambda_1, \lambda_2, v) & k = 2\end{cases} \tag{28}$$

$$\Phi(\lambda_1, \lambda_2, v) \in \begin{cases}\frac{\lambda_1+\lambda_2}{\|\lambda_1+\lambda_2\|} & \|\lambda_1+\lambda_2\| > \sqrt{2}d_{\min}\\ \left\{\frac{\lambda_1+\lambda_2}{\|\lambda_1+\lambda_2\|}, \Pi(v)\right\} & \|\lambda_1+\lambda_2\| = \sqrt{2}d_{\min}\\ \Pi(v) & \|\lambda_1+\lambda_2\| < \sqrt{2}d_{\min}\end{cases} \tag{29}$$

and $\Pi(v) : \mathbb{S}^1 \to \mathbb{S}^1$ is such that $\forall u^0 \in \mathbb{S}^1$, the set $\{u \in \mathbb{S}^1 : u = \Pi^m(u^0),\ m \in \mathbb{Z}_{>0}\}$ is dense in $\mathbb{S}^1$. One could design Π to rotate the vector by a rational angle (in radians). The $\bmod$ operator denotes the modulus operation. In (27) , the $\bmod$ operation simply increments k when $k \in \{0, 1\}$ and sets k back to zero when $k = 2$.

We then define g_z as the composite function,

$$\overline{g}_z(z) := \begin{cases}g_1(z) & z \in D_1 \setminus D_2\\ \{g_1(z), g_2(z)\} & z \in D_1 \cap D_2\\ g_2(z) & z \in D_2 \setminus D_1\end{cases} \tag{30}$$

Finally, we define the function g_r, which calculates the rotation needed for the next vehicle orientation. Letting $\overline{g}_z^v$ denote the v component of $\overline{g}_z$, we define the next search direction, $v^n = \overline{g}_z^v(\overline{z})$. The function g_r will calculate the value of $r = [\ r_1 \quad r_2\ ]^T$ which satisfies

$$\begin{aligned} v^n &= v \otimes r\\ \begin{bmatrix}v_1^n\\ v_2^n\end{bmatrix} &= \begin{bmatrix}v_1\\ v_2\end{bmatrix} \otimes \begin{bmatrix}r_1\\ r_2\end{bmatrix} = \begin{bmatrix}v_1 r_1 - v_2 r_2\\ v_2 r_1 + v_1 r_2\end{bmatrix}\\ \Rightarrow \overline{g}_r(z) := \begin{bmatrix}r_1\\ r_2\end{bmatrix} &= \begin{bmatrix}v_1 & v_2\\ -v_2 & v_1\end{bmatrix}\begin{bmatrix}v_1^n\\ v_2^n\end{bmatrix}\end{aligned} \tag{31}$$

With the algorithm fully specified, we now state that this algorithm satisfies assumptions **(A1)–(A4)**. Due to space constraints, the proof is omitted; however, it is easy to see that this is the case, since $\overline{g}_z(z)$ generates linearly independent search directions.

Theorem 6.1: The hybrid source-seeking controller, $(\overline{f}_z, \overline{f}_\omega, \overline{g}_z, \overline{g}_r, \Upsilon)$ has the continual search property.

Figure 1 shows a comparison between a simulation of the proposed algorithm and experimental results from the setup described in [13]. Open-loop control laws used in Figure 1 are those that generate the optimal Dubins paths [8].

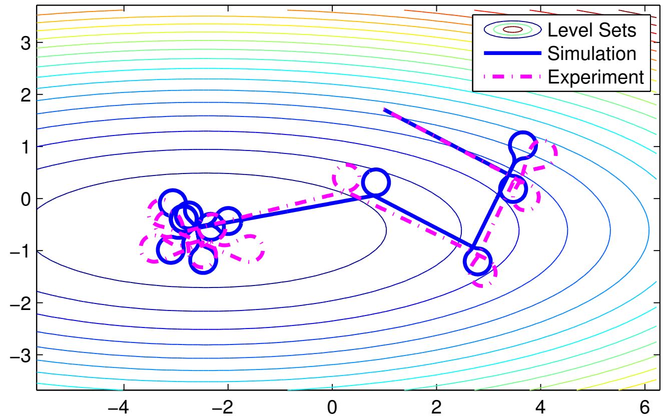

Fig. 1. A comparison between simulation and experiment of the proposed source-seeking algorithm.

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