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Matrix Block Formulation of Closed-Loop Memoryless Stackelberg Strategy for Discrete-Time Games

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Abstract— **Stackelberg strategies with closed-loop information structure are well known to be delicate to design in general. Nevertheless when they are restricted to the case of memoryless ones, that is when the controls are function of the time and the current state, then they are strongly time consistent. Due to this property, it is possible to compute step by step backward in time the value functions associated with the Stackelberg equilibrium. A new method, using a matrix block formulation is provided here to facilitate this numerical computation. An example illustrates this method.**

I. Introduction

The Stackelberg strategies from Game Theory [1–5] for control design were formalized in [6–8]. This kind of strategies, named in honor of the economist Heinrich Stackelberg who introduced this concept [9, 10], assumes that the role of each player is not the same. It involves an hierarchy between the different players. This concept was revisited in [11] for economic and automatic area. Only the case of a two-player nonzero-sum game is studied here. In this framework, there is a leader and there is a follower. The leader is able to enforce his strategy on the follower.

The information structure in the game is the set of all available information for the players to make their decisions. When open-loop information structure is considered, no measurement of the state of the system is available and the players are committed to follow a predetermined strategy based on their knowledge of the initial state, the system's model and the cost functional to be minimized. For the closed-loop information structure case each player has access to state measurements and thus can adapt his strategy in function of the system's evolution. More precisely when only the current state is available, the closed-loop information structure is called memoryless or pure state feedback.

The necessary conditions for obtaining a Stackelberg equilibrium with an open-loop information structure are well known (see [6–8]). Especially in the linear-quadratic case, closed-form solutions are provided by a characteristic matrix, which is symplectic in discrete-time [12] or Hamiltonian in continuous-time [13]. These solutions are related to non-standard Riccati equations [14, 15].

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Obtaining Stackelberg strategies with a closed-loop information structure is a very difficult optimization problem, because the dependency of the leader control with respect to the current state. This problem defined in [7, 8] has been treated in [16–21]. Several attempts have been proposed in the literature to avoid this main difficulty, like Team Approach [22]. This approach is based on a common minimization of the leader's criterion with respect to both leader and follower controls. For the restriction of memoryless closed-loop Stackelberg strategies, the optimization problem is strongly time consistent [1, 23] and allow to determine the solution by a value function approach.

In this paper, Discrete time Stackelberg strategy with memoryless closed-loop information structure is studied by a block matrix approach in the same spirit as in [24]. By using particular properties of a parametrized matrix, we provide an iterative procedure, with respect to discrete time, to compute backward in time the value of the criteria associated with both players. The outline of the paper is as follows: in Section II the optimization problem is formulated. Section III presents the main theorem associated with the proposed matrix block method. An scalar exemple illustrates this new approach in Section IV. Some concluding remarks are gathered in Section V.

Notation : in the whole paper, the indices 1 and 2 are dedicated respectively to the follower and the leader. $i = 1,2$ denotes one player and $j = 1,2$ with $j \neq i$ denotes the other player.

II. Problem Statement

Consider a Discrete time two-player non-zero game associating the linear dynamic

$$x(k+1) = Ax(k) + B_1 u_1(k) + B_2 u_2(k), \quad (1)$$

$$= f(x(k), u_1(k), u_2(k)), \quad x(0) = x_0, \quad (2)$$

where $x(\cdot) \in \mathbb{R}^n$ $u_1(\cdot) \in \mathbb{R}^{r_1}$, $u_2(\cdot) \in \mathbb{R}^{r_2}$ are respectively the state, the control of the follower and of the leader. The criteria of the players are done by :

$$J_i = \frac{1}{2} x^T(N) K_{iN} x(N) + \frac{1}{2} \sum_{k=0}^{N-1} L_i(x(k), u_1(k), u_2(k)). \quad (3)$$

where

$$L_i(x(k), u_1(k), u_2(k)) = \left(x^T(k) Q_i x(k) + u_1^T(k) R_{i1} u_1(k) + u_2^T(k) R_{i2} u_2(k) \right). \quad (4)$$

All weighting matrices in the criteria J_i are constant and symmetric with $Q_i \geq 0$, $K_{if} \geq 0$, $R_{12} \geq 0$, $R_{21} > 0$ and $R_{ii} > 0$. $R_{ii} > 0$ denotes the convexity of J_i with respect to the control u_i and $R_{21} > 0$ allows the leader to enforce the follower's control.

The leader knows how the follower will rationally react, but the follower does not know the leader's rational reaction. Define the rational reaction set of the follower $\mathscr{R}_1(u)$

$$\left\{\tilde{u}_1 \;\middle|\; J_1(\tilde{u}_1, u) \leq J_1(u_1, u), \forall u_1 \in \mathscr{U}_{ad,1}\right\}. \tag{5}$$

$\mathscr{U}_{ad,i}$ denotes the set of admissible controls $u_i(k)$ for the player i.

For a closed-loop information structure, the leader (player 2) is seeking a J_2-minimizing strategy $u_2^*(k)$, as a function of time only, that he announces before the game starts knowing the follower's rational reaction. The follower (player 1) will then minimize his cost functional J_1 with the strategy $u_1^*(k)$, a function of time only. Mathematically, the definition of a Stackelberg equilibrium (u_1^*, u_2^*) is

$$\begin{cases} u_1^* \in \mathscr{R}_1(u_2^*) \\ \text{and} \quad \forall u_2 \in \mathscr{U}_{ad,2} \\ \max\limits_{u_1 \in \mathscr{R}_1(u_2^*)} J_2(u_1, u_2^*) \leq \max\limits_{u_1 \in \mathscr{R}_1(u_2)} J_2(u_1, u_2). \end{cases} \tag{6}$$

In practice, for memoryless closed-loop information structure, the controls (u_1^*, u_2^*) are only functions of the time and the current state. This property helps to determine the solution. One denotes $J_{i,[k;N]}$ the cost associated to J_i, but restricted to the time horizon $[k;N]$, that is

$$J_{i,[k;N]} = \frac{1}{2}x^T(N)K_{iN}x(N) + \frac{1}{2}\sum_{m=k}^{N-1} L_i(x(m), u_1(m), u_2(m)). \tag{7}$$

In addition, the notation $V_i(x(k),k)$ corresponds to the value of the criterion $J_{i,[k;N]}$ at the Stackelberg equilibrium for a game starting at time k and ending at time N. The quadratic class of criteria J_i allow to look for the values $V_i(x(k),k)$ as a quadratic term with respect to $x(k)$ that is

$$V_i(x(k),k) = \frac{1}{2}(x(k))^T P_i(k)x(k). \tag{8}$$

III. Main Theorem

The following theorem is the main result of this paper. It provides a matrix block formulation to compute step by step the both value functions associated with the memoryless closed-loop Stackelberg strategy.

Theorem 1: The optimization problem of memoryless closed-loop Stackelberg strategy has a single solution given by controls

$$u_i^*(k) = -R_{ii}^{-1}B_i^T\lambda_i(k+1), \tag{9}$$

with

$$\begin{pmatrix} x(k+1) \\ \gamma(k+1) \\ \lambda_2(k+1) \\ \lambda_1(k+1) \end{pmatrix} = \mathscr{M}_{k+1}^{-1} \begin{bmatrix} A \\ 0_n \\ 0_n \\ 0_n \end{bmatrix} x(k); \tag{10}$$

where $\mathscr{M}_{k+1}$ is Hamiltonian, invertible and given by

$$\mathscr{M}_{k+1} = \begin{bmatrix} I_n & 0_n & S_2 & S_1 \\ 0_n & I_n & S_1 & -S_{21} \\ P_2(k+1) & P_1(k+1) & -I_n & 0_n \\ P_1(k+1) & 0_n & 0_n & -I_n \end{bmatrix}. \tag{11}$$

The notations $S_{ij} = B_jR_{jj}^{-1}R_{ij}R_{jj}^{-1}B_j^T$ and $S_i = B_iR_{ii}^{-1}B_i^T$ are used. The matrices $P_i(k)$ are symmetric and semi-definite positive. They verify

$$P_i(k) = \begin{bmatrix} A \\ 0_n \\ 0_n \\ 0_n \end{bmatrix}^T \mathscr{M}_{k+1}^{-T}\mathscr{P}_i(k+1)\mathscr{M}_{k+1}^{-1} \begin{bmatrix} A \\ 0_n \\ 0_n \\ 0_n \end{bmatrix} + Q_i, \tag{12}$$

where

$$\mathscr{P}_1(k+1) = \mathrm{diag}(P_1(k+1); 0_n; S_{12}; S_1) \tag{13}$$

and

$$\mathscr{P}_2(k+1) = \mathrm{diag}(P_2(k+1); 0_n; S_2; S_{21}). \tag{14}$$

The final conditions are

$$P_i(N) = K_{iN}. \tag{15}$$

Proof: First we proove that $\mathscr{M}_{k+1}$ is Hamiltonian and invertible, if $P_i(k+1)$ are symmetric. By defining

$$\mathscr{J} = \begin{bmatrix} 0_{2n} & I_{2n} \\ -I_{2n} & 0_{2n} \end{bmatrix}; \tag{16}$$

and by noticing

$$\mathscr{J}^T = -\mathscr{J} = \mathscr{J}^{-1}; \tag{17}$$

the following relation ensures that $\mathscr{M}_{k+1}$ is Hamiltonian [25]:

$$\mathscr{J}^T\mathscr{M}_{k+1}\mathscr{J} = -\mathscr{M}_{k+1}^T. \tag{18}$$

$\mathscr{M}_{k+1}$ is invertible if its kernel contains only the null vector. Assume that

$$\mathscr{M}_{k+1}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = 0_{4n\times 1}. \tag{19}$$

We will show that $y_1 = y_2 = y_3 = y_4 = 0_{n\times 1}$. By developing equation (19), we obtain:

$$0 = y_1 + S_2y_3 + S_1y_4, \tag{20}$$

$$0 = y_2 + S_1y_3 - S_{21}y_4, \tag{21}$$

$$0 = P_2(k+1)y_1 + P_1(k+1)y_2 - y_3, \tag{22}$$

$$0 = P_1y_1 - y_4. \tag{23}$$

By multiplying equation (20) (respectively (21), (22) and (23)) at left by y_3^T (respectively by y_4^T, y_1^T and y_2^T), we obtain:

$$\begin{aligned} 0 &= y_3^T y_1 + y_3^T S_2 y_3 + y_3^T S_1 y_4, && (24)\\ y_4^T S_{21} y_4 &= y_4^T y_2 + y_4^T S_1 y_3, && (25)\\ y_1^T y_3 &= y_1^T P_2(k+1) y_1 + y_1^T P_1(k+1) y_2, && (26)\\ y_2^T P_1 y_1 &= y_2^T y_4. && (27)\end{aligned}$$

Combining these equations lead to

$$y_1^T P_2(k+1) y_1 + y_3^T S_2 y_3 + y_4^T S_{21} y_4 = 0. \quad (28)$$

The semi-definite positivness of $P_2(k+1)$ and the definite positivness of R_{22} and R_{21} imply

$$P_2(k+1) y_1 = 0; \quad B_2^T y_3 = B_1^T y_4 = 0. \quad (29)$$

These last equations and equation (20) ensure that $y_1 = 0$, and then from (23) that $y_4 = 0$. Equations (21) and (22) become

$$\begin{aligned} 0 &= y_2 + S_1 y_3, && (30)\\ 0 &= P_1(k+1) y_2 - y_3, && (31)\\ & && (32)\end{aligned}$$

Multiplying equation (30) at left by y_3^T and equation (31) at left by y_2^T, we obtain

$$y_3^T S_1 y_3 + y_2^T P_1(k+1) y_2 = 0, \quad (33)$$

which implies, by the definite positivness of R_{11} and the semi-definite positivness of $P_1(k+1)$

$$B_1^T y_3 = 0; \quad P_1(k+1) y_2 = 0. \quad (34)$$

By injecting these last relations in equations (30) and (31), we conclude that $y_2 = 0$ and $y_3 = 0$.

The memoryless closed-loop Stackelberg strategy is well known to be strongly time consistent [1, Corollary 7.2]. The main idea, due to the fact that the controls depend only on the current state, is that if a memoryless Stackelberg controls $(u^*_{1,[k,N]}, u^*_{2,[k,N]})$ is associated to a game starting at time k and ending at time N, then the restriction of these controls $(u^*_{1,[k+1,N]}, u^*_{2,[k+1,N]})$ is a memoryless Stackelberg controls associated to the restricted game starting at time $k+1$ and ending at time N. In accordance with this crucial property, the Stackelberg controls could be find step by step from the terminal game and going backward in time.

The necessary conditions for both players are now pointed out. The necessary conditions for the follower are provided in [8] and [16]. The control $u_1(k)$ of the follower at time k is looking for minimizing, for all $x(k+1)$, $J_{1,[k,N]}$ when $u_1^*(m)$ ($\forall m \in [k+1,N]$) is applied and $u_2^*(m)$ ($\forall m \in [k,N]$), under the constraint (2), which is associated to the costate vector $\lambda_1(k+1)$ that is minimizing

$$V_1(x(k+1),k+1) + L_1(x(k),u_1(k),u_2^*(k)) + \gamma^T(k+1)\left[f(x(k),u_1(k),u_2^*(k)) - x(k+1)\right]. \quad (35)$$

We obtain

$$\frac{\partial V_1}{\partial x(k+1)}(x(k+1);k+1) - \lambda_1(k+1) = 0, \quad (36)$$

and

$$\frac{\partial L_1}{\partial u_1}(x(k),u_1(k),u_2^*(k)) + \frac{\partial f}{\partial u_1}(x(k),u_1(k),u_2^*(k))\lambda_1(k+1) = 0. \quad (37)$$

This last equation leads to:

$$B_1^T \lambda_1(k+1) + R_{11} u_1^*(k) = 0, \quad (38)$$

or

$$u_1^*(k) = -R_{11}^{-1} B_1^T \lambda_1(k+1). \quad (39)$$

with

$$\begin{aligned} \lambda_1(k+1) &= \frac{\partial V_1}{\partial x(k+1)}(x(k+1);k+1)\\ &= P_1(k+1)x(k+1). && (40)\end{aligned}$$

These relations (39) and (40) characterize the rational reaction set of the follower defined by (5). The initial step is given by the following relation

$$V_1(x(N),N) = (x(N))^T K_{1N} x(N). \quad (41)$$

It implies the final condition

$$P_1(N) = K_{1N}. \quad (42)$$

The leader knowing the rational reaction set of the follower, should design the control $u_2(k) = u_2^*(k)$ minimizing, for all $x(k+1)$, $J_{2,[k,N]}$ when $u_2^*(m)$ ($\forall m \in [k+1,N]$) is applied and $u_1^*(m)$ ($\forall m \in [k,N]$), under the constraint (39). That is

$$L_2(x(k),u_1^*(k),u_2(k)) + V_2(x(k+1),k+1) + \gamma(k+1)^T\left[\frac{\partial V_1}{\partial x(k+1)}(x(k+1);k+1) - \lambda_1(k+1)\right] + \lambda_2(k+1)^T\left[f(x(k),u_1^*(k),u_2(k)) - x(k+1)\right] \quad (43)$$

should be minimized by $u_2(k)$ and $\lambda_1(k+1)$. These conditions imply

$$\frac{\partial V_2}{\partial x(k+1)}(x(k+1);k+1) - \lambda_2(k+1) - \frac{\partial^2 V_1}{\partial x(k+1)^2}(x(k+1);k+1)\gamma(k+1) = 0, \quad (44)$$

$$\frac{\partial u_1^*}{\partial \lambda_1(k+1)}\frac{\partial L_2}{\partial u_1}(x(k),u_1^*(k),u_2(k)) + \gamma(k+1) + \frac{\partial u_1^*}{\partial \lambda_1(k+1)}\frac{\partial f}{\partial u_1}(x(k),u_1^*(k),u_2(k)) = 0, \quad (45)$$

and

$$\frac{\partial f}{\partial u_2}(x(k),u_1^*(k),u_2(k))\lambda_2(k+1) + \frac{\partial L_2}{\partial u_2}(x(k),u_1^*(k),u_2(k)) = 0. \quad (46)$$

This last equation leads to:

$$B_2^T\lambda_2(k+1)+R_{22}u_2^*(k)=0, \quad (47)$$

or

$$u_2^*(k)=-R_{22}^{-1}B_2^T\lambda_2(k+1), \quad (48)$$

with

$$\begin{aligned}\lambda_2(k+1) &= \frac{\partial V_2}{\partial x(k+1)}(x(k+1);k+1)\\ &= P_2(k+1)x(k+1), \quad (49)\end{aligned}$$

and

$$S_{21}\lambda_1(k+1)-\gamma(k+1)-S_1\lambda_2(k+1)=0. \quad (50)$$

The initial step is given by the following relation

$$V_2(x(N),N)=(x(N))^TK_{2N}x(N). \quad (51)$$

It implies the final condition

$$P_2(N)=K_{2N}. \quad (52)$$

By applying the controls u_1^* and u_2^* defined by (39) and (48), the dynamic (2) becomes

$$x(k+1)=Ax(k)-S_1\lambda_1(k+1)-S_2\lambda_2(k+1). \quad (53)$$

By rearranging equations (40), (49), (50) and (53), we obtain

$$\mathscr{M}_{k+1}\begin{pmatrix}x(k+1)\\ \gamma(k+1)\\ \lambda_2(k+1)\\ \lambda_1(k+1)\end{pmatrix}=\begin{bmatrix}A\\ 0_n\\ 0_n\\ 0_n\end{bmatrix}x(k). \quad (54)$$

We use now the definition of the value function $V_i(x(k),k)$: for all $x(k)$,

$$\begin{aligned}2V_i(x(k),k) &= x^T(k)P_i(k)x(k)\\ &= 2L_i(x(k),u_1^*(k),u_2^*(k))\\ &\quad +2V_i(x(k+1),k+1)\\ &= x^T(k)Q_ix(k)+\lambda_i^T(k+1)S_i\lambda_i(k+1)\\ &\quad +\lambda_j^T(k+1)S_{ij}\lambda_j(k+1)\\ &\quad +x^T(k+1)P_i(k+1)x(k+1) \quad (55)\end{aligned}$$

This relation can be reformulated into

$$x^T(k)P_ix(k)=x^T(k)Q_ix(k) + \begin{pmatrix}x(k+1)\\ \gamma(k+1)\\ \lambda_2(k+1)\\ \lambda_1(k+1)\end{pmatrix}^T\mathscr{P}_i(k+1)\begin{pmatrix}x(k+1)\\ \gamma(k+1)\\ \lambda_2(k+1)\\ \lambda_1(k+1)\end{pmatrix}, \quad (56)$$

with $\mathscr{P}_i(k+1)$ defined by the relations (13) and (14). In addition, the relation (54) allow to obtain relations (12), because $\mathscr{M}_{k+1}$ is invertible.

To conclude the proof, we should still prove that $P_i(k)$ is symmetric and semi-definite positive. First the matrices $\mathscr{P}_i(N)$ are semi-definite positive and symmetric, because R_{ii} and R_{ij} are at least semi-definite positive and $P_i(N)=K_{iN}\geq 0$. The property is proved by recurrence. Assuming that $P_i(k+1)$ is semi-definite positive and symmetric, the matrices $\mathscr{P}_i(k+1)$ are symmetric and semi-definite positive. By construction of the iterative relation (12) and due to the fact that matrices Q_i are semi-definite positive and symmetric, we can conclude that $P_i(k)$ is semi-definite positive and symmetric for all $k\in[0,N]$. ■

The global criteria for memoryless closed-loop Stackelberg strategy are given by

$$J_i^*=V_i(x_0,0)=\frac{1}{2}x_0^TP_i(0)x_0. \quad (57)$$

The Hamiltonian structure of the matrix $\mathscr{M}_{k+1}$ can be compared to the Hamiltonian structure of the characteristic matrix for Stackelberg strategy with open-loop information structure in continuous-time [13, 14, 26, 27]. Nevertheless here the matrix $\mathscr{M}_{k+1}$ depends on time and is not function of the weighting matrix Q_i.

The main improvment of this matrix block formulation for memoryless closed-loop Stackelberg strategy with respect existing methods [1] is its simplicity of calculation and handling. A block computation of the inverse of $\mathscr{M}_{k+1}$ allow to refind the existing methods. For a sake of clarity, this calculation is not presented here.

This technic is inspired from the matrix block formulation of standard linear-quadratic regulator problem proposed in [24]. It should be note that the extension to Stackelberg strategy is possible because $\mathscr{M}_{k+1}$ is Hamiltonian. The Hamiltonian structure ensures a symmetry in the distribution of eigenvalues and allows to prove the invertibility of this matrix. For exemple for Nash strategy, this property does not hold, and it is not possible to prove the invertibility of the obtained matrix.

IV. ILLUSTRATION

In this section, an example is presented to illustrate the new approach in this paper. One consider a game on a finite time horizon, with $N=5$.

$$A=1;\quad B_1=1;\quad B_2=1;$$

$$Q_1=1;\quad R_{11}=2;\quad R_{12}=1;$$

$$Q_2=1;\quad R_{21}=1;\quad R_{22}=3;$$

$$K_{1,N}=3;\quad K_{2,N}=2;\quad x_0=1.$$

The matrices $P_1(k)$ and $P_2(k)$ are computed backward in time by the iterative relation (12) in the main theorem and are drawn on Fig. 1. Then it is possible to compute the controls forwardly in time and the implied states all over the finite time horizon. The value functions $V_1(x(k),k)$ and $V_2(x(k),k)$ are represented on Fig. 2.

The global criteria for memoryless closed-loop Stackelberg strategy are here

$$V_1(x_0,0)=\frac{1}{2}x_0^TP_1(0)x_0=1.6348. \quad (58)$$

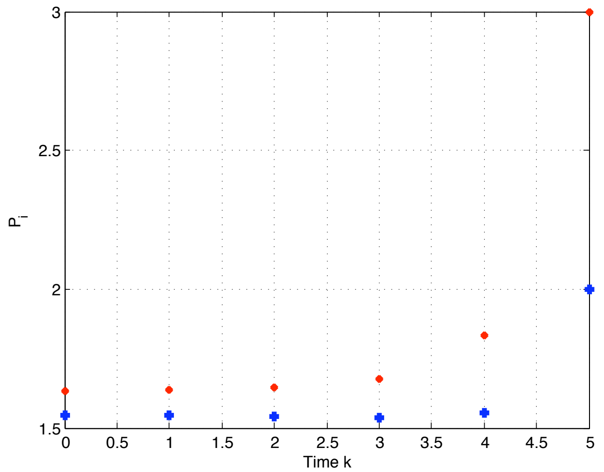

Fig. 1. Solutions $P_1(k)$ and $P_2(k)$ computed backward in time from $P_1(N)$ and $P_2(N)$. $P_1(k)$ is in red and $P_2(k)$ is in blue.

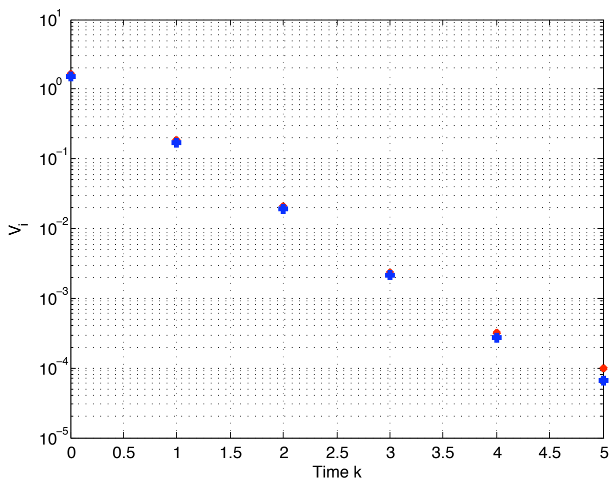

Fig. 2. Value functions $V_1(k)$ in red and $V_2(k)$ in blue.

and

$$V_2(x_0,0) = \frac{1}{2}x_0^T P_2(0)x_0 = 1.5478. \quad (59)$$

As a verification, one can notice that the value functions $V_1(x(k),k)$ and $V_2(x(k),k)$ are decreasing functions of the discrete time k. However it is not necessary the case of the associated matrices $P_1(k)$ and $P_2(k)$.

V. Conclusion

In this paper, the study of memoryless closed-loop Stackelberg strategy for discrete-time games is tackled. This kind of closed-loop Stackelberg strategy is strongly time consistent, instead ones with memory. A new matrix block formulation is proposed to improve and facilitate the computation of the value functions associated with the equilibria. This formulation is feasible, because it is shown that the main block matrix exhibits an Hamiltonian structure and is invertible, with only assumption about convexity of each criteria with respect to the state and the controls. An example points out the applicability of the provided method.

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