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Payoff Suboptimality and Errors in Value Induced by Approximation of the Hamiltonian

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Abstract— **Dynamic programming reduces the solution of optimal control problems to solution of the corresponding Hamilton-Jacobi-Bellman partial differential equations (HJB PDEs). In the case of nonlinear deterministic systems, the HJB PDEs are fully nonlinear, first-order PDEs. Standard, grid-based techniques to the solution of such PDEs are subject to the curse-of-dimensionality, where the computational costs grow exponentially with state-space dimension. Among the recently developed max-plus methods for solution of such PDEs, there is a curse-of-dimensionality-free algorithm. Such an algorithm can be applied in the case where the Hamiltonian takes the form of a pointwise maximum of a finite number of quadratic forms. In order to take advantage of this curse-of-dimensionality-free algorithm for more general HJB PDEs, we need to approximate the general Hamiltonian by a maximum of these quadratic forms. In doing so, one introduces some errors. In this work, we obtain a bound on the difference in solution of two HJB PDEs, as a function of a bound on the difference in the two Hamiltonians. Further, we obtain a bound on the suboptimality of the controller obtained from the solution of the approximate HJB PDE rather than from the original.**

I. Introduction

The use of dynamic programming to solve nonlinear control problems leads to the familiar dynamic programming equation. In the case of problems in continuous space/time governed by finite-dimensional "deterministic" (or max-plus stochastic) dynamics, the dynamic programming equation takes the form of a Hamilton-Jacobi-Bellman partial differential equation (HJB PDE). In the infinite time-horizon case, this is typically a PDE over a region in a space whose dimension is the dimension of the state variable in the control problem. We remark that the solutions are generally nonsmooth, and the theory of viscosity solutions yields the appropriate solution definition (c.f., [4], [6], [10]).

The difficulty lies in computing the solution of the HJB PDE. The most intuitive, and commonly applied, approaches are grid-based (c.f., [4], [5], [7], [10] among many others), and are subject to the curse-of-dimensionality (whereby the computational cost growth is very roughly on the order of $(2D)^n$ where D is the required number of grid points per dimension, and more importantly, n is the space dimension.

A recent development is the discovery of the curse-of-dimensionality-free methods exploiting semiconvex dual operators and max-plus linearity ([13], [14], [15]). Using convex-programming based pruning, a problem over $I\!R^6$ was solved on a desktop machine [12]. This approach has, so far, only been developed for steady-state problems over the entire space, although the class could be enlarged. (For other max-plus-based methods developed for larger classes of problems, see [1], [2], [9], [15], [16].) The curse-of-dimensionality-free approach currently handles HJB PDE problems of form

$$0 = -\widetilde{H}(x, \nabla V) \;\; \forall x \in I\!R^n \setminus \{0\}, \quad V(0) = 0 \quad (1)$$

where

$$\widetilde{H}(x, \nabla V) = \max_{m \in \mathcal{M}} \{H^m(x, \nabla V)\}, \quad (2)$$

$\mathcal{M} = \{1, 2 \ldots M\}$, and the H^m have computationally simpler forms. In particular, the H^m considered to date have been quadratic forms. Also, note that by boundary condition, $V(0) = 0$, we mean that the solution is zero at the origin.

In [14], [15], the method was developed and the curse-of-dimensionality-free nature was made clear. In [13], the convergence rate for the algorithm was obtained. In particular, it was shown that there were two parameters, τ and $T = N\tau$ such that the errors go to zero as $T = N\tau \to \infty$ and $\tau \downarrow 0$. Further, a required relation between the relative T and τ rates was indicated. The errors in the pre-limit solution approximation are bounded in a form $0 \le \widetilde{V} - V^a \le \varepsilon(1 + |x|^2)$ where $\widetilde{V}$ is the true solution and V^a is the computed approximation. Additionally, we had $T = N\tau \propto \varepsilon^{-1}$ and $\tau \propto \varepsilon^2$, and so $N \propto \varepsilon^{-3}$. The computational cost growth with (space dimension) n is only on the order of n^3 (due to some matrix inverses). However, the approach is subject to a curse-of-complexity, where the computational cost can grow like M^N. Attenuation of this curse-of-complexity growth through pruning, using semidefinite programming, is an active area of research [12].

Although the PDEs of (1) are certainly nontrivial nonlinear PDEs, we would like to solve more general HJB PDEs. A function, say $F(y)$, is semiconvex if given any $R < \infty$, there exists $C_R < \infty$ such that $F(y) + \frac{C_R}{2}|y|^2$ is convex over $B_R(0)$. (Note that the space of semiconvex functions certainly contains both C^2 and the space of convex functions as subspaces.) It is well known that one can approximate any semiconvex function as the pointwise maximum of quadratic forms. In fact, this is simply a max-plus basis expansion over the max-plus vector space, or moduloid, of semiconvex functions (c.f., [15]). With this in mind, we see that we could approximate any semiconvex Hamiltonian by a Hamiltonian, $\widetilde{H}$, of the form (2) with quadratic H^m. We could then solve the HJB PDE problem (1) with a curse-of-dimensionality-

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free method, thereby yielding an approximate solution of the HJB PDE with the original semiconvex Hamiltonian. Such a procedure would induce two error sources. The first consists of the errors in the solution of (1) generated by the curse-of-dimensionality-free algorithm. These are briefly discussed in the previous paragraph, and fully discussed in [13]. The second source are those induced by the approximation of the original Hamiltonian by $\widetilde{H}$. This latter error source is under discussion here. Although the analysis to follow is specifically oriented toward approximation by $\widetilde{H}$ of the above form, the general concepts may be more widely applicable. Further, in addition to obtaining bounds on the difference between the solution of the original and approximating HJB PDE problems, we also obtain a lower bound on the suboptimality of the controller obtained by use of the solution of (1) in the controller computation. This latter question is, of course, of significant practical value.

II. Problem Statement and Assumptions

We will consider HJB PDE problem

$$0 = -H(x, \nabla V) = -\sup_{w \in I\!R^k} \left[f'(x,w)\nabla V + l(x,w) \right], \qquad V(0) = 0 \tag{3}$$

where $x \in I\!R^n$. More specifically, we are seeking the particular viscosity solution of (3) which is the value function of the following optimal control problem. The dynamics are given by

$$\dot{\xi}_t = f(\xi_t, w_t) \doteq g(\xi_t) + \sigma(\xi_t) w_t, \tag{4}$$

and the running cost is

$$l(\xi_t, w_t) \doteq L(\xi_t) - \frac{\gamma^2}{2}|w_t|^2. \tag{5}$$

The value function we seek, maximizing the payoff over controls $w \in \mathcal{W} \doteq L_2([0,\infty); I\!R^k)$, is

$$\widehat{V}(x) = \sup_{w \in \mathcal{W}} \sup_{T<\infty} \left\{ \int_0^T l(\xi_t, w_t)\, dt \,\middle|\, \xi_0 = x \right\}. \tag{6}$$

We assume, $\exists K, c, d_\sigma, C, \alpha \in (0,\infty)$ such that the following hold. $g(x)$ is globally Lipschitz continuous with constant K, $(x-y)^T(g(x)-g(y)) \le -c|x-y|^2$ for all x, y, and $g(0)=0$. $\sigma(x)$ is Lipschitz continuous with constant K, and its norm is bounded globally by d_σ. $|L(x)-L(y)| \le C(1+|x|+|y|)|x-y|$ for all x, y, and $0 \le L(x) \le \alpha|x|^2$ for all x. Finally, we assume $\gamma^2/d_\sigma^2 > \alpha/c^2$. $(A.V)$

It is worth noting, that with the above forms for f and l,

$$H(x,p) = g(x)'p + L(x) + \tfrac{1}{2\gamma^2} p'\sigma(x)\sigma'(x)p.$$

In [15], it was demonstrated that the above assumptions guarantee the following:

Theorem 2.1: $\widehat{V}$ (given by (6)) is a continuous viscosity solution of (3), and is the unique such solution within the class

$$\mathcal{G}_{\bar\delta} \doteq \{\phi : \phi \text{ is semiconvex, } 0 \le \phi(x) \le c\frac{\gamma^2 - \bar\delta^2}{2d_\sigma^2}|x|^2\} \tag{7}$$

for $\bar\delta > 0$ sufficiently small.

The goal is to approximately compute $\widehat{V}$ by approximating H by an $\widetilde{H}$ taking the form (2) with quadratic H^m, and then to solve (1) with the curse-of-dimensionality-free method [14], [15].

In particular, we assume that H and $\widetilde{H}$ are *close* in following sense. Assume that:

There exists $\theta > 0$ such that, for all $x, p \in I\!R^n$ such that $\widetilde{H}(x,p) \le 0$, one has $(A.c)$

$$\widetilde{H}(x,p) \le H(x,p) \le \widetilde{H}(x,p) + \theta\left[|x|^2 + |p|^2\right].$$

Note that the coefficient θ parameterizes the degree of closeness between H and $\widetilde{H}$. As we are dealing with max-plus vector spaces, $\widetilde{H}$ approximates H from below (c.f. [15]), and so this approximation assumption is one-sided.

Let $D^-V(x)$ denote the subdifferential of V at x, i.e.,

$$D^-V(x) = \left\{ p \in I\!R^n \,\middle|\, \liminf_{y \to x} \frac{V(y) - V(x) - p\cdot(y-x)}{|y-x|} \ge 0 \right\}.$$

Remark 2.2: If $\widetilde{V}$ is a viscosity solution of (1), and $p \in D^-\widetilde{V}(x)$, then by the definition of viscosity solutions, $\widetilde{H}(x,p) \le 0$. Consequently, the inequalities of Assumption $(A.c)$ hold for all x, p such that $p \in D^-\widetilde{V}(x)$.

We will suppose that the H^m are generalized quadratic forms, with parameters meeting certain conditions which guarantee existence and uniqueness within a certain function class. The H^m take the form

$$H^m(x,p) = \tfrac{1}{2}x'D^m x + \tfrac{1}{2}p'\Sigma^m p + (A^m x)'p + (l_1^m)'x + (l_2^m)'p + \alpha^m \tag{8}$$

where each $\Sigma^m = (1/\gamma^2)\sigma^m(\sigma^m)'$ for appropriate matrices σ^m. In regards to H^m, we make following assumptions, which ensure existence of a solution meeting the boundary condition at the origin (c.f. [13]).

Assume there exists $c_A \in (0,\infty)$ such that $x'A^m x \le -c_A|x|^2$ for all $x \in I\!R^n$ and all $m \in \mathcal{M}$.
Assume $H^1(x,p)$ has coefficients satisfying the following: $l_1^1 = l_2^1 = 0$; $\alpha^1 = 0$; there exists $c_{A,1} \in (0,\infty)$ such that $x'A^1 x \le -c_{A,1}|x|^2$ $\forall x \in I\!R^n$; D^1 is positive definite, symmetric; $\Sigma^1 > 0$; and $\gamma^2/c_\sigma^2 > c_D/c_{A,1}^2$, where c_D is such that $x'D^1x \le c_D|x|^2$ $\forall x \in I\!R^n$ and $c_\sigma \doteq |\sigma^1|$. $(A.m)$
Assume that system $\dot{\xi}^{\mu_t} = A^{\mu_t}\xi^{\mu_t} + l_2^{\mu_t} + \sigma^{\mu_t}w$ is controllable in the sense that given $x, y \in I\!R^n$ and $T > 0$, there exist processes $w \in \mathcal{W}$ and μ measurable with range in $\mathcal{M}$, such that $\xi_T = y$ when $\xi_0 = x$ and one applies controls w, μ.

The first assumption in $(A.m)$ is not restrictive, as without this nominal stability, sensible problems with positive definite running cost would have unbounded value. The second of the assumptions assures that at least one of the Hamiltonians has a purely quadratic structure, and this one typically "looks like" the H near the origin. The controllability assumption is (currently) needed for technical reasons.

We let

$$\widetilde{V}(x) \doteq \sup_{T<\infty} \sup_{\mu\in\mathcal{D}_\infty} \sup_{w\in\mathcal{W}} \int_0^T L^{\mu_t}(\xi_t) - \frac{\gamma^2}{2}|w_t|^2\,dt$$

where

$$L^m(x) = \tfrac{1}{2}x' D^m x + (l_1^m)' x + \alpha^m,$$
$$\dot{\xi} = A^{\mu_t}\xi_t + l_2^{\mu_t} + \sigma^{\mu_t} w_t,$$

and

$$\mathcal{D}_\infty = \{\mu : [0,\infty) \to \mathcal{M} \mid \text{ measurable } \}.$$

In [15], [14], it was shown that:

Theorem 2.3: $\widetilde{V}$ is the unique viscosity solution of (1) in the class of continuous functions satisfying $V(x) \in [0, \widehat{V}(x)]$ for all $x \in I\!R^n$.

III. Preliminaries

The following lemmas will be useful further below. Let $T \in (0,\infty)$, and let W be the finite horizon value function given by

$$W(x,T) = \sup_{w\in\mathcal{W}} \int_0^T l(\xi_t, w_t)dt, \qquad \xi_0 = x, \qquad (9)$$

where ξ satisfies (4). Noting that $\widehat{V} \geq 0$, we see that

$$W(x,T) \leq \sup_{w\in\mathcal{W}} \int_0^T l(\xi_t, w_t)dt + \widehat{V}(\xi_T), \qquad \xi_0 = x.$$

With this definition, and [15], Ch. 2, we immediately obtain the following two lemmas.

Lemma 3.1: Let w_t^ε be ε–optimal (with $\varepsilon \in (0,1]$) for problem (9). Then,

$$\tfrac{1}{2}\|w^\varepsilon\|^2_{L_2[0,T]} \leq \frac{\varepsilon}{\bar{\delta}} + \frac{1}{\bar{\delta}}\left[\frac{c\gamma^2}{2d_\sigma^2}e^{-cT} + \frac{\alpha}{c}\right]|x|^2. \qquad (10)$$

Lemma 3.2: Let w_t^ε be ε–optimal (with $\varepsilon \in (0,1]$) for problem (9), and let ξ_t^ε be the corresponding state process. Then,

$$\int_0^T |\xi_t^\varepsilon|^2 dt \leq \frac{2\varepsilon}{\bar{\delta}}\frac{d_\sigma^2}{c} + \frac{d_\sigma^2}{\bar{\delta}c}\left[\left(\frac{2\alpha}{c^2} + \frac{\gamma^2}{d_\sigma}\right) + \frac{1}{c}\right]|x|^2. \quad (11)$$

IV. Error in the Value Function

As noted in Theorem 2.3, $0 \leq \widetilde{V}(x) \leq \widehat{V}(x)$ for all $x \in I\!R^n$. Now we obtain an upper bound on $\widehat{V} - \widetilde{V}$. The main result and core of the proof are Theorem 4.5 below and its corresponding proof. Prior to this we obtain some technical results.

Lemma 4.1: There exists $K_g < \infty$ such that for any $x \in I\!R^n$,

$$|p| \leq K_g|x| \qquad \forall\, p \in D^-\widetilde{V}(x).$$

Proof: By Theorem 2.3, Remark 2.2, and (2), for all $p \in D^-\widetilde{V}(x)$, one has

$$H^1(x,p) \leq \widetilde{H}(x,p) \leq 0.$$

Using (8) and Assumption $(A.m)$, this implies

$$x^T D^1 x + p^T \Sigma^1 p + (A^1 x)^T p \leq 0 \qquad \forall\, p \in D^-\widetilde{V}(x).$$

Rearranging this, and dropping superscripts for convenience, yields

$$\left(p + \frac{\Sigma^{-1}Ax}{2}\right)^T \Sigma \left(p + \frac{\Sigma^{-1}Ax}{2}\right) \leq x^T(A\Sigma^{-1}A - D)x.$$

Thus

$$\left|p + \frac{\Sigma^{-1}Ax}{2}\right|^2 \lambda_{\min}[\Sigma] \leq |x|^2 \lambda_{\max}[A\Sigma^{-1}A - D],$$

where, by Assumption $(A.m)$, $\lambda_{\min}[\Sigma] = \lambda_{\min}[\Sigma^1] > 0$. With a little calculation, this implies the desired result. ∎

Fix $R < \infty$, and let $x \in B_R$. Let $\varepsilon \in (0,1]$, and let w^ε be an ε–optimal controller for (9). Also, let ξ^ε denote the corresponding state process.

Lemma 4.2: For any $T \in [0,\infty)$, ξ_t^ε is absolutely continuous on $[0,T]$.

Proof: Fix any $\delta > 0$. Consider any finite set of disjoint subintervals of $[0,T]$, say $\{[s_i,t_i] \mid i \in]i,N[\}$, such that $t_i < s_{i+1}$ for all $i \in]1, N-1[$ and such that $\sum_{i\leq N}|t_i - s_i| = \delta$. We have

$$\sum_{i=1}^N |\xi_{t_i}^\varepsilon - \xi_{s_i}^\varepsilon| = \sum_{i=1}^N \left|\int_{s_i}^{t_i} g(\xi_t^\varepsilon) + \sigma(\xi_t^\varepsilon)w_t^\varepsilon\,dt\right|,$$

which by Assumption $(A.V)$

$$\leq \sum_{i=1}^N \int_{s_i}^{t_i} K|\xi_t^\varepsilon| + d_\sigma|w_t^\varepsilon|\,dt \qquad (12)$$

for proper choice of K_1.

From inequality (3.17) from [15] (which follows easily from Assumptions $(A.V)$), there exists $C_4 < \infty$, independent of T, such that

$$|\xi_t^\varepsilon|^2 \;\leq C_4\left(1 + |x|^2 + \int_0^t |w_r^\varepsilon|^2\,dr\right) \qquad \forall\, t \in [0,T]$$

which by Lemma 3.1,

$$\leq C_5(1 + |x|^2) \qquad \forall\, t \in [0,T], \qquad (13)$$

for proper choice of $C_5 < \infty$ (independent of $T \in [0,\infty)$ and $\varepsilon \in (0,1]$). Combining (12) and (13), one finds for proper choice of $C_6 < \infty$ (independent of $T \in [0,\infty)$ and $\varepsilon \in (0,1]$),

$$\sum_{i=1}^N |\xi_{t_i}^\varepsilon - \xi_{s_i}^\varepsilon| \;\leq \sum_{i=1}^N \left\{\int_{s_i}^{t_i} C_6(1+|x|)\,dt + \int_{s_i}^{t_i} d_\sigma|w_t^\varepsilon|\,dt\right\}$$
$$= C_6(1+|x|)\delta + \sum_{i=1}^N \int_{s_i}^{t_i} d_\sigma|w_t^\varepsilon|\,dt. \qquad (14)$$

Define

$$\phi(t) = \begin{cases} 1 & \text{if } t \in [s_i,t_i] \text{ for some } i \\ 0 & \text{otherwise.}\end{cases}$$

With this definition, (14) becomes

$$\sum_{i=1}^N |\xi_{t_i}^\varepsilon - \xi_{s_i}^\varepsilon| \;\leq C_6(1+|x|)\delta + d_\sigma \int_0^T \phi(t)|w_t^\varepsilon|\,dt.$$

which by the Cauchy-Schwarz inequality

$$\le C_6(1+|x|)\delta + d_\sigma \left[\int_0^T (\phi(t))^2\,dt\right]^{1/2} \|w_t^\varepsilon\|$$
$$\le C_6(1+|x|)\delta + d_\sigma \delta^{1/2}\|w_t^\varepsilon\|$$
which, by Lemma 3.1 again
$$\le C_6(1+|x|)\delta + C_7(1+|x|)\delta^{1/2}$$
for proper choice of $C_7 < \infty$. This last inequality implies absolute continuity. ■

Lemma 4.3: For any $T \in [0,\infty)$, $\widetilde{V}(\xi_t^\varepsilon)$ is absolutely continuous on $[0,T]$, and
$$\widetilde{V}(\xi_T^\varepsilon) - \widetilde{V}(x) = \int_0^T \frac{d}{dt}\widetilde{V}(\xi_t^\varepsilon)\,dt,$$
where the time-derivative exists almost everywhere.

Proof: The semiconvexity of $\widetilde{V}$ (given in Theorem 2.3) implies local Lipschitz behavior (c.f., [8]). Further, by the continuity given in Lemma 4.2 and finiteness of T, ξ_t^ε remains in a bounded set. Combining the absolute continuity of ξ^ε obtained in Lemma 4.2 with the Lipschitz property of $\widetilde{V}$ over the bounded set immediately implies the absolute continuity of $\widetilde{V}(\xi_t^\varepsilon)$. The remaining assertion is a direct result of the absolute continuity. ■

Lemma 4.4: For any $T \in [0,\infty)$,
$$\widetilde{V}(\xi_T^\varepsilon) - \widetilde{V}(x) = \int_0^T \max_{p\in D^-\widetilde{V}(\xi_t^\varepsilon)} p\cdot f(\xi_t^\varepsilon, w_t^\varepsilon)\,dt.$$

Proof: By the semiconvexity of $\widetilde{V}$, the directional derivative, $\widetilde{V}_u(x)$, exists for all $x \in I\!R^n$ and all $|u| = 1$ in $I\!R^n$ (c.f., [4], Th. II.4.7). Now,
$$\begin{aligned}\frac{d}{dt}\widetilde{V}(\xi_t^\varepsilon) &= \lim_{\delta\to 0}\frac{1}{\delta}[\widetilde{V}(\xi_{t+\delta}^\varepsilon) - \widetilde{V}(\xi_t^\varepsilon)]\\ &= \lim_{\delta\to 0}\frac{1}{\delta}[\widetilde{V}(\xi_t^\varepsilon + \delta f(\xi_t^\varepsilon, w_t^\varepsilon) + \mathcal{O}(\delta^2)) - \widetilde{V}(\xi_t^\varepsilon)]\\ &= |f(\xi_t^\varepsilon, w_t^\varepsilon)|\widetilde{V}_{u_t}(\xi_t^\varepsilon)\end{aligned}$$
with $u_t = f(\xi_t^\varepsilon, w_t^\varepsilon)/|f(\xi_t^\varepsilon, w_t^\varepsilon)|$ when $f(\xi_t^\varepsilon, w_t^\varepsilon) \neq 0$, and u_t an arbitrary unit vector otherwise. Again applying [4], Th. II.4.7, this yields
$$\begin{aligned}\frac{d}{dt}\widetilde{V}(\xi_t^\varepsilon) &= |f(\xi_t^\varepsilon, w_t^\varepsilon)| \max_{p\in D^-\widetilde{V}(\xi_t^\varepsilon)} p\cdot u_t\\ &= \max_{p\in D^-\widetilde{V}(\xi_t^\varepsilon)} p\cdot f(\xi_t^\varepsilon, w_t^\varepsilon).\end{aligned}$$
■

We now proceed to obtain the main result of the section. For any $t \in [0,T]$, let
$$v_t^\varepsilon \doteq \max_{p\in D^-\widetilde{V}(\xi_t^\varepsilon)} p\cdot f(\xi_t^\varepsilon, w_t^\varepsilon).$$
By the ε-optimality of w^ε, one has
$$W(x,T) \le \int_0^T [l(\xi_t^\varepsilon, w_t^\varepsilon) + v_t^\varepsilon]\,dt - \int_0^T v_t^\varepsilon\,dt + \varepsilon$$
(where existence of the integrals follows from Lemma 4.4). Then, by Lemma 4.4,
$$W(x,T) \le \widetilde{V}(x) - \widetilde{V}(\xi_T^\varepsilon) + \int_0^T [l(\xi_t^\varepsilon, w_t^\varepsilon) + v_t^\varepsilon]\,dt + \varepsilon. \quad (15)$$

For any $t \in [0,T]$, let
$$p_t^\varepsilon \in \operatorname*{argmax}_{p\in D^-\widetilde{V}(\xi_t^\varepsilon)} p\cdot f(\xi_t^\varepsilon, w_t^\varepsilon).$$
Then,
$$l(\xi_t^\varepsilon, w_t^\varepsilon) + v_t^\varepsilon = l(\xi_t^\varepsilon, w_t^\varepsilon) + p_t^\varepsilon\cdot f(\xi_t^\varepsilon, w_t^\varepsilon)$$
which by Assumption $(A.c)$,
$$\le \widetilde{H}(\xi_t^\varepsilon, p_t^\varepsilon) + \theta(|\xi_t^\varepsilon|^2 + |p_t^\varepsilon|^2). \quad (16)$$
However, by the definition of a viscosity solution, and the fact that $p_t^\varepsilon \in D^-\widetilde{V}(\xi_t^\varepsilon)$, $\widetilde{H}(\xi_t^\varepsilon, p_t^\varepsilon) \le 0$, and so, (16) yields
$$l(\xi_t^\varepsilon, w_t^\varepsilon) + v_t^\varepsilon \le \theta(|\xi_t^\varepsilon|^2 + |p_t^\varepsilon|^2)$$
which by Lemma 4.1,
$$\le \theta(1+K_g^2)|\xi_t^\varepsilon|^2. \quad (17)$$

Substituting (17) into (15), one obtains
$$W(x,T) \le \widetilde{V}(x) - \widetilde{V}(\xi_T^\varepsilon) + \theta(1+K_g^2)\int_0^T |\xi_t^\varepsilon|^2\,dt + \varepsilon,$$
and noting $\widetilde{V} \ge 0$,
$$\le \widetilde{V}(x) + \theta(1+K_g^2)\int_0^T |\xi_t^\varepsilon|^2\,dt + \varepsilon,$$
which, by Lemma 3.2,
$$\le \widetilde{V}(x) + \theta(1+K_g^2)[C_1 + C_2|x|^2] + \varepsilon, \quad (18)$$
where
$$C_1 = 2d_\sigma^2/(\bar{\delta}c),$$
$$C_2 = \frac{d_\sigma^2}{\bar{\delta}c}\left[\left(\frac{2\alpha}{c^2} + \frac{\gamma^2}{d_\sigma}\right) + \frac{1}{c}\right].$$
Since this is true for all $\varepsilon \in (0,1]$, we have
$$W(x,T) \le \widetilde{V}(x) + \theta(1+K_g^2)[C_1 + C_2|x|^2]. \quad (19)$$
Then, noting (c.f., [15]) that $W(x,T) \to \widehat{V}(x)$ as $T \to \infty$, (19) yields the value approximation result:

Theorem 4.5: There exists $C_3 < \infty$ such that
$$\widehat{V}(x) - \theta C_3(1+|x|^2) \le \widetilde{V}(x) \le \widehat{V}(x) \qquad \forall\, x \in I\!R^n, \quad (20)$$
where θ is as given in Assumption $(A.c)$.

Thus, we see that $\widetilde{V}$ approximates $\widehat{V}$ arbitrarily well if $\widetilde{H}$ is sufficiently close to H, this closeness being parameterized by θ.

V. Degree of Suboptimality of the Controller

In the previous section, it was shown that if the approximating Hamiltonian is close to the Hamiltonian of the originating problem in a certain sense, then the corresponding viscosity solutions will be close in an appropriate sense. However, recall that we are specifically concerned with a case where we can efficiently solve the HJB PDE with the approximating Hamiltonian, and would like to use this solution to generate a controller for the originating problem. Consequently, we would like to know whether an (approximate) optimal control generated from the solution of the approximate HJB PDE, will perform well when applied to the true system, which is described by the originating Hamiltonian. We begin with some preparatory results, which are minor variations of well-known properties of viscosity solutions and semiconvexity. Between Lemma 5.4 and Lemma 5.5, the optimal control approximation will be introduced. The main development will begin with Theorem 5.7. In the interest of page-length, some proofs are omitted.

Lemma 5.1: Suppose V is a semiconvex viscosity solution of $0 = \widehat{H}(x, \nabla V)$, where $\widehat{H}$ is continuous. Let $D^- V(x)$ denote the subdifferential of V at x. For any $x, q \in I\!R^n$, there exists $\bar{p} \in D^- V(x)$ such that

$$\bar{p} \cdot q = \max_{p \in D^- V(x)} p \cdot q \tag{21}$$

and

$$\widehat{H}(x, \bar{p}) = 0. \tag{22}$$

It will be helpful to make the following definitions. Let

$$\begin{aligned} &\mathcal{P}(x) = \mathcal{P}(x; \widetilde{V}) \\ &\doteq \operatorname{argmax}\big\{ f(x,w) \cdot p + l(x,w) \,|\, (w,p) \in I\!R^k \times D^- \widetilde{V}(x) \big\}. \end{aligned}$$

Also, let

$$\mathcal{W}^0(x) = \operatorname*{argmax}_{w \in I\!R^k} \max_{p \in D^- \widetilde{V}(x)} \left[f(x,w) \cdot p + l(x,w) \right],$$

and

$$\mathcal{P}^0(x) = \operatorname*{argmax}_{p \in D^- \widetilde{V}(x)} \max_{w \in I\!R^k} \left[f(x,w) \cdot p + l(x,w) \right].$$

Lemma 5.2: If $\hat{w} \in \mathcal{W}^0(x)$, then there exists $\hat{p} \in D^- \widetilde{V}(x)$ such that $(\hat{w}, \hat{p}) \in \mathcal{P}(x)$. On the other hand, $(\hat{w}, \hat{p}) \in \mathcal{P}(x)$ implies that $\hat{w} \in \mathcal{W}^0(x)$.

Lemma 5.3: If $\hat{p} \in \mathcal{P}^0(x)$, then there exists $\hat{w} \in I\!R^k$ such that $(\hat{w}, \hat{p}) \in \mathcal{P}(x)$. On the other hand, $(\hat{w}, \hat{p}) \in \mathcal{P}(x)$ implies that $\hat{p} \in \mathcal{P}^0(x)$.

We now get a simple representation for $\hat{w}$, which will be useful in bounding the control effort.

Lemma 5.4: Suppose $\hat{p} \in \mathcal{P}^0(x)$, and let $\hat{w} = \hat{w}(x, \hat{p}) = \frac{1}{\gamma^2} \sigma'(x) \hat{p}$. Then, $(\hat{w}, \hat{p}) \in \mathcal{P}(x)$, and $\hat{w} \in \mathcal{W}^0(x)$.

Assume there exists a measurable selection $\bar{p}(\cdot) : I\!R^n \to I\!R^n$ from set-valued $\mathcal{P}^0(\cdot)$ such that, with $\bar{w}(x) \doteq \frac{1}{\gamma^2} \sigma'(x) \bar{p}(x)$, there exists a Lipschitz continuous solution to the feedback controlled system

$$\dot{\xi} = f(\xi, \bar{w}(\xi)), \qquad \xi_0 = x. \tag{$A.s$}$$

for any initial $x \in I\!R^n$ (i.e., a solution to $\xi_t = x + \int_0^t f(\xi_r, \bar{w}(\xi_r))\, dr$) over $[0, \infty)$, which we denote by $\bar{\xi}$.

Lemma 5.5: For any $T \in [0, \infty)$,

$$\widetilde{V}(\bar{\xi}_T) - \widetilde{V}(x) = \int_0^T \frac{d}{dt} \widetilde{V}(\bar{\xi}_t)\, dt$$

where $\frac{d}{dt} \widetilde{V}(\bar{\xi}_t)$ exists a.e.

The proof of the following lemma is essentially identical to the proof of Lemma 4.4, and so we do not repeat it.

Lemma 5.6: For any $T \in [0, \infty)$,

$$\widetilde{V}(\bar{\xi}_T) - \widetilde{V}(x) = \int_0^T \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} p \cdot f(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\, dt.$$

It will be necessary to show that solutions driven by our feedback control are well-behaved, i.e., staying bounded and eventually decaying to the origin. This step is comprised of Theorem 5.7 to Lemma 5.10.

Theorem 5.7: For any $T \in [0, \infty)$,

$$\int_0^T l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\, dt \geq \widetilde{V}(x) - \widetilde{V}(\bar{\xi}_T).$$

Proof:

$$\begin{aligned} &\int_0^T l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\, dt \\ &= \int_0^T \left[l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t)) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}(\bar{\xi}_t)) \cdot p \right] dt \\ &\quad - \int_0^T \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}(\bar{\xi}_t)) \cdot p\, dt, \end{aligned} \tag{23}$$

where the integrability follows from Lemma 5.6.

Define

$$\mathcal{H}_0(x; \widetilde{H}) \doteq \{ p \in I\!R^n \,|\, \widetilde{H}(x,p) = 0 \}.$$

Then, note that

$$\begin{aligned} &l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t)) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}(\bar{\xi}_t)) \cdot p \\ &= \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} \left[l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t)) + f(\bar{\xi}_t, \bar{w}(\bar{\xi}_t)) \cdot p \right], \end{aligned}$$

which by the definition of $\bar{w}$, Lemma 5.4 and the definition of $\mathcal{W}^0$,

$$\begin{aligned} &= \max_{w \in I\!R^k} \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} \left[l(\bar{\xi}_t, w) + f(\bar{\xi}_t, w) \cdot p \right], \\ &= \max_{w \in I\!R^k} \left\{ l(\bar{\xi}_t, w) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} \left[f(\bar{\xi}_t, w) \cdot p \right] \right\}, \end{aligned}$$

which by Lemma 5.1,

$$\begin{aligned} &= \max_{w \in I\!R^k} \left\{ l(\bar{\xi}_t, w) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t) \cap \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})} \left[f(\bar{\xi}_t, w) \cdot p \right] \right\}, \\ &= \max_{p \in D^- \widetilde{V}(\bar{\xi}_t) \cap \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})} \max_{w \in I\!R^k} \left[l(\bar{\xi}_t, w) + f(\bar{\xi}_t, w) \cdot p \right], \\ &= \max_{p \in D^- \widetilde{V}(\bar{\xi}_t) \cap \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})} H(\bar{\xi}_t, p), \end{aligned}$$

which by Assumption $(A.c)$,

$$\geq \max_{p \in D^- \widetilde{V}(\bar{\xi}_t) \cap \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})} \widetilde{H}(\bar{\xi}_t, p),$$

which since $p \in \mathcal{H}_0(\bar{\xi}_t; \widetilde{H})$,

$$= 0.$$

Integrating this over time, we see that,

$$\int_0^T \left[l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t)) + \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}(\bar{\xi}_t)) \cdot p \right] dt \geq 0. \tag{24}$$

Substituting (24) into (23), one finds

$$\int_0^T l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\, dt \geq -\int_0^T \max_{p \in D^- \widetilde{V}(\bar{\xi}_t)} f(\bar{\xi}_t, \bar{w}(\bar{\xi}_t)) \cdot p\, dt,$$

which by Lemma 5.6,

$$= \widetilde{V}(x) - \widetilde{V}(\bar{\xi}_T).$$

■

Corollary 5.8: For any $x \in I\!R^n$, and any $T \in [0, \infty)$,

$$\int_0^T l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\, dt + \widehat{V}(\bar{\xi}_T) \geq \widehat{V}(x) - K_x$$

where $K_x \doteq \widehat{V}(x) - \widetilde{V}(x)$.

Corollary 5.9: For any $R < \infty$, there exists $M_R < \infty$ such that for all $|x| \le R$ and all $T \in [0,\infty)$,

$$\int_0^T |\bar{\xi}_t|^2\,dt \le M_R.$$

Lemma 5.10: Given $\varepsilon \in (0,1]$, $x \in I\!R^n$ and $\overline{T} < \infty$, there exists $T > \overline{T}$ such that

$$0 \le \widetilde{V}(\bar{\xi}_T) \le \widehat{V}(\bar{\xi}_T) < \varepsilon.$$

Proof: As the other inequalities are already proven, we prove only the rightmost. Using Corollary 5.9, it is easy to show that given $\bar{\varepsilon} > 0$ and $\overline{T} < \infty$, there exists $T \in [\overline{T},\infty)$ such that

$$|\bar{\xi}_T|^2 < \bar{\varepsilon}. \tag{25}$$

From [15], Theorems 3.19 and 3.20, there exists $C_V < \infty$ such that $\widehat{V}(x) \le C_V|x|^2$, and consequently,

$$\widehat{V}(\bar{\xi}_T) \le C_V|\bar{\xi}_T|^2. \tag{26}$$

Combining (25) and (26) yields the result. ∎

We now begin the development leading to the main result of the section. By Corollary 5.8 and Lemma 5.10, we see that given $\varepsilon \in (0,1]$ and $\overline{T} < \infty$, there exists $T \in [\overline{T},\infty)$ such that

$$\int_0^T l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\,dt \ge \widehat{V}(x) - K_x - \varepsilon. \tag{27}$$

Recalling the specific form of l given in (5) and the growth on L given by Assumption $(A.V)$, we see that (27) implies

$$\frac{\gamma^2}{2}\|\bar{w}(\bar{\xi}_\cdot)\|^2_{L_2(0,T)} \le \alpha\|\bar{\xi}\|^2_{L_2(0,T)} + K_x + \varepsilon - \widehat{V}(x)$$

which, by Corollary 5.9

$$\le \alpha M_{|x|} + K_x + \varepsilon - \widehat{V}(x).$$

Since this is true for any $\overline{T} < \infty$,

$$\|\bar{w}(\bar{\xi}_\cdot)\|^2_{L_2(0,\infty)} \le \overline{M}_x \doteq \frac{2}{\gamma^2}\left[\alpha M_{|x|} + K_x + 1 - \widehat{V}(x)\right]. \tag{28}$$

Combining (28) and Corollary 5.9, we see that given $\hat{\varepsilon} > 0$, there exists $\widehat{T} < \infty$ such that

$$\|\bar{\xi}\|^2_{L_2(\widehat{T},\infty)}, \|\bar{w}(\bar{\xi}_\cdot)\|^2_{L_2(\widehat{T},\infty)} < \hat{\varepsilon},$$

which implies that given $\hat{\varepsilon} > 0$, there exists $\widetilde{T} < \infty$ such that

$$\int_{\widehat{T}}^\infty \left|l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\right|dt < \hat{\varepsilon}, \tag{29}$$

which implies that $\lim_{T\to\infty}\int_0^T l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\,dt$ exists. In particular, given $\hat{\varepsilon} > 0$,

$$\left|\int_0^T l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\,dt - \lim_{T\to\infty}\int_0^T l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\,dt\right| < \hat{\varepsilon} \tag{30}$$

for all $T \ge \widetilde{T}$. By (30) and Theorem 5.7, given $\hat{\varepsilon} > 0$,

$$\lim_{T\to\infty}\int_0^T l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\,dt \ge \widetilde{V}(x) - \hat{\varepsilon} - \widetilde{V}(\bar{\xi}_T) \qquad \forall T \ge \widetilde{T}. \tag{31}$$

Combining (31) and Lemma 5.10 (with $\overline{T}$ replacing $\widetilde{T}$), one sees that given $\hat{\varepsilon} > 0$,

$$\lim_{T\to\infty}\int_0^T l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\,dt \ge \widetilde{V}(x) - 2\hat{\varepsilon}.$$

Lastly, since this is true for all $\hat{\varepsilon} > 0$, we obtain:

Theorem 5.11:

$$\lim_{T\to\infty}\int_0^T l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\,dt \ge \widetilde{V}(x).$$

Combining Theorem 5.11 and Theorem 4.5, we have:

Theorem 5.12: For any $x \in I\!R^n$,

$$\lim_{T\to\infty}\int_0^T l(\bar{\xi}_t, \bar{w}(\bar{\xi}_t))\,dt \ge \widehat{V}(x) - \theta C_3(1+|x|^2).$$

In other words, the payoff obtained with a feedback control, $\bar{w}(\cdot)$, based on solution of the approximating problem, will be arbitrarily close to the optimal payoff, $\widehat{V}(x)$. Further, the bound on the difference, $\theta C_3(1+|x|^2)$, goes to zero as $\theta \to 0$, where θ parameterizes the closeness of $\widetilde{H}$ to the originating Hamiltonian, H.

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