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Explicit Solutions for Stabilization and H_∞ Control of Time-Delayed State-Space Symmetric Systems

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***Abstract*— The paper examines the stabilization and the H_∞ norm performance analysis and control of state-delayed systems with a symmetric state-space realization. Such symmetric realizations appear in many engineering applications. By exploiting the state-space symmetry, we obtain an explicit solution for a bound on the H_∞ norm of such symmetric time-delayed systems. Also, we derive an explicit parametrization of static output feedback gains that solve the stabilization and H_∞ control synthesis problems. Computational examples are used to demonstrate the significance and validity of the proposed methods and results.**

I. INTRODUCTION

Time delays often appear in many control systems either in the state, in the control input, or in the measurements. Time delay commonly exists in various engineering, biological, and economical systems because of the finite speed of information processing and it is a source of performance degradation and instability (see [4], [3] and numerous references therein). Since the control process involves measuring response data, computing control laws, and transmitting data and signals to actuators, a time-delay in processing and applying control inputs to the system cannot be avoided. Application of unsynchronized control inputs due to time-delay may result in a degradation of the control performance and may even render the controlled system to be unstable. Therefore, the stability and performance analysis and the control of systems with time-delays are both theoretically and practically important (see [1] and the references therein). Recent efforts concerning the topic of stability and stabilization of time-delay systems can be divided into two categories, namely, delay-independent and delay-dependent stability criteria (see [4], [3]).

In this paper, we consider the analysis and control problem for state delayed systems with a symmetry property in their state-space realization. Symmetric systems appear in many different engineering fields, such as electrical and power networks, structural systems, and chemical reaction systems. In particular, physical systems with only one type of energy storage capability, such as mechanical systems with only potential energy or only kinematic energy, and electrical systems with only electric energy or only magnetic energy (e.g., RL or RC circuits) provide models of such symmetric systems (see [2], [5]). Moreover, systems with zeros interlacing the poles (ZIP) can be modeled as symmetric systems as shown in [8]. Stability criteria for the state-space symmetric systems have been examined in [10]. The H_∞ control of symmetric systems has been also addressed in [9].

In the present paper, we examine the H_∞ control analysis and the output feedback stabilization and H_∞ control synthesis problems for time-delayed state-space symmetric systems. The objective of the paper is to show that, by exploiting the particular structure of these systems, an *explicit* expression for an upper bound on the H_∞ norm of a symmetric delayed system can be developed, which requires only the computation of the maximum eigenvalue of a matrix containing the state-space data. In addition, an *explicit* solution for the output feedback controllers that guarantee the stability of the closed-loop system and a prescribed level of H_∞ performance is obtained for such systems.

The notation used in this paper is standard. Given a symmetric matrix $X = X^T \in \mathbb{R}^{n\times n}$, $X > 0$ ($X \geq 0$) denotes matrix positive definiteness (semi-definiteness). $(.)^T$ denotes the transpose of a real matrix, $(.)^\dagger$ represents the Moore-Penrose generalized inverse of a matrix. Given a real $n\times m$ matrix Y with rank r, the orthogonal complement $Y^\perp$ is defined as the $(n-r)\times n$ matrix that satisfies $Y^\perp Y = 0$ and $Y^\perp Y^{\perp\top} > 0$. The maximum eigenvalue of a matrix M will be denoted by $\lambda_{max}(M)$. Finally, in a symmetric block matrix, the star $(*)$ is used to denote the sub-matrices lying above the diagonal.

II. PLANT FORMULATION AND PRELIMINARIES

In this section, we define the notion of time delayed systems with symmetry in their state-space realization. Also, some preliminary concepts and results that will be used later in the paper are introduced.

Definition 1: Consider the following state-space system that includes a delay term in its state dynamics

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_h x(t-h) + Bw(t)\\ z(t) &= Cx(t)\end{aligned} \tag{1}$$

We will say that the system is *symmetric* if the state-space matrices satisfy the following conditions

$$A = A^T, \qquad A_h = A_h^T, \qquad B = C^T. \tag{2}$$

The conditions in (2) define the symmetric property of the delayed system under study. The next lemmas and definitions will be useful in the proofs of the main results of the paper.

Lemma 1: [4] Consider the time delay system given by (1). Suppose that there exist positive definite matrices P and Q and a positive scalar γ such that the following linear matrix

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inequality condition holds

$$\begin{bmatrix} A^TP+PA+Q & * & * & * \\ A_h^TP & -Q & * & * \\ B^TP & 0 & -\gamma I & * \\ C & 0 & 0 & -\gamma I \end{bmatrix} \leq 0. \qquad (3)$$

Then the time delay system (1) is asymptotically stable and has induced $\mathcal{L}_2$-gain from $w(t)$ to $z(t)$ less than γ, that is $\|T_{zw}\|_\infty \leq \gamma$.

Lemma 2: [7] Consider the matrices Γ and Λ such that Γ has full column rank, and Λ is symmetric positive definite. Then $\Lambda \geq \Gamma\Gamma^T$ if and only if $\lambda_{max}(\Gamma^T\Lambda^{-1}\Gamma) \leq 1$.

Lemma 3: (Finsler's Lemma [7]) Consider the matrices M and Z such that M has full column rank and $Z = Z^T$. Then the following statements are equivalent:

1) There exists a scalar μ such that

$$\mu MM^T - Z > 0. \qquad (4)$$

2) The following condition holds

$$M^\perp Z M^{\perp T} < 0.$$

If the above statements hold, then all scalars μ satisfying (4) are given by

$$\mu > \lambda_{max}[M^\dagger(Z - ZM^{\perp T}(M^\perp ZM^{\perp T})^{-1}M^\perp Z)M^{\dagger T}]. \qquad (5)$$

Lemma 4: (Generalized Finsler's Lemma [7]) Consider matrices M and Z such that M has full rank column rank and $Z = Z^T$. Then the following statements are equivalent:

1) There exists a symmetric matrix X such that

$$MXM^T - Z > 0. \qquad (6)$$

2) The following condition holds

$$M^\perp Z M^{\perp T} < 0.$$

If the above statements hold, then all matrices X satisfying (6) are given by

$$X > M^\dagger[Z - ZM^{\perp T}(M^\perp ZM^{\perp T})^{-1}M^\perp Z]M^{\dagger T}. \qquad (7)$$

Definition 2: Consider a symmetric matrix N with eigenvalue/eigenvector decomposition

$$N = E\Lambda E^T$$

where E is the orthogonal matrix of eigenvectors and Λ the diagonal matrix of eigenvalues of N. Hence, $\lambda = diag(\lambda_1, ..., \lambda_n)$ where λ_k, $k = 1, ..., n$, are the real eigenvalues of N. We define the *absolute value* of the matrix N as

$$|N| = E|\Lambda|E^T$$

where $|\Lambda| = diag(|\lambda_1|, ..., |\lambda_n|)$.

III. THE H_∞ ANALYSIS OF DELAYED SYMMETRIC SYSTEMS

The next result shows that for a delayed state-space symmetric system represented by (1)-(2), an upper bound on its H_∞ norm can be computed using a simple explicit formula.

Theorem 1: A stable state-space symmetric delayed system described by (1)-(2) has an H_∞ norm that satisfies the following bound

$$\gamma \leq \bar{\gamma} = \lambda_{max}\{-B^T(A + |A_h|)^{-1}B\}. \qquad (8)$$

Proof. Let us consider the Lyapunov matrix in the analysis condition (3) as $P = I$. Taking (2) into account, the matrix inequality (3) can be written as

$$\begin{bmatrix} 2A+Q & * & * & * \\ A_h & -Q & * & * \\ B^T & 0 & -\gamma I & * \\ B^T & 0 & 0 & -\gamma I \end{bmatrix} \leq 0. \qquad (9)$$

Applying Schur complement formula, we obtain the following inequality

$$\frac{2}{\gamma}BB^T \leq -2A - (Q + A_hQ^{-1}A_h). \qquad (10)$$

Using Lemma 2, we obtain that

$$\gamma \geq 2\lambda_{max}\{B^T(-2A - Q - A_hQ^{-1}A_h)^{-1}B\}.$$

Hence, the best bound is obtained as

$$\bar{\gamma} = 2\lambda_{max}\{B^T(-2A - Q - A_hQ^{-1}A_h)^{-1}B\}. \qquad (11)$$

We seek to minimize this bound with respect to the parameter matrix Q. To this end, notice that since the right hand side in (10) is positive

$$\begin{aligned} &\lambda_{max}\{B^T(-2A - Q - A_hQ^{-1}A_h)^{-1}B\} \leq \\ &trace\{B^T(-2A - Q - A_hQ^{-1}A_h)^{-1}B\} \leq \\ &trace(BB^T)trace\{(-2A - Q - A_hQ^{-1}A_h)^{-1}\}. \end{aligned}$$

Therefore, to minimize the above bound with respect to Q we set

$$\frac{\partial}{\partial Q}trace(-2A - Q - A_hQ^{-1}A_h)^{-1} = 0$$

or

$$\begin{aligned}(-2A - Q - A_hQ^{-1}A_h)^{-1}\{\frac{\partial}{\partial Q}trace(-2A - Q-\\ A_hQ^{-1}A_h)\}(-2A - Q - A_hQ^{-1}A_h)^{-1} = 0\end{aligned}$$

where [6]

$$\frac{\partial}{\partial Q}trace(-2A - Q - A_hQ^{-1}A_h) = -I + Q^{-1}A_h^2Q^{-1}.$$

Hence, the minimization occurs for

$$Q^2 = A_h^2.$$

The unique positive definite solution for Q is obtained as

$$Q = |A_h|$$

where $|A_h|$ is the absolute value of A_h. Since

$$-|A_h| - A_h|A_h|^{-1}A_h = -2|A_h|$$

for the above value of Q the bound (11) results in expression (8) and this concludes the proof. □

We now generalize the above results to symmetric systems that include a feedthrough matrix D as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_h x(t-h) + Bw(t) \\ z(t) &= Cx(t) + Dw(t) \end{aligned} \tag{12}$$

where $D = D^T$, and all the conditions given in (2) hold.

Theorem 2: Consider the symmetric delayed system described by (12). The H_∞ norm of this system satisfies the following bound

$$\gamma \leq \bar{\gamma} = \max(\lambda_{max}(D), \lambda_{max}\{D - B^T(A + |A_h|)^{-1}B\}). \tag{13}$$

Proof. The proof follows similar lines as in the proof of Theorem 1. Considering the Lyapunov matrix as $P = I$, the associated inequality condition (3) becomes

$$\begin{bmatrix} 2A + Q & * & * & * \\ A_h & -Q & * & * \\ B^T & 0 & -\gamma I & * \\ B^T & 0 & D & -\gamma I \end{bmatrix} \leq 0. \tag{14}$$

Applying the Schur complement, we obtain

$$2B(\gamma I - D)^{-1}B^T \leq -2A - Q - A_h Q^{-1} A_h. \tag{15}$$

Solving a similar minimization problem to the one in the proof of Theorem 1 results in the same solution as $Q = |A_h|$. Substituting Q back into the LMI (15) leads to

$$B(\gamma I - D)^{-1}B^T \leq -(A + |A_h|). \tag{16}$$

This inequality can be alternatively rewritten as

$$\begin{bmatrix} A + |A_h| & B \\ B^T & -(\gamma I - D) \end{bmatrix} \leq 0 \tag{17}$$

if and only if

$$\lambda_{max}(D) \leq \gamma. \tag{18}$$

Taking advantage of the Finsler's Lemma, (17) can be written as

$$\gamma M M^T - Z \geq 0$$

where

$$M = \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad Z = \begin{bmatrix} A + |A_h| & B \\ B^T & D \end{bmatrix}.$$

Note that

$$M^\perp = \begin{bmatrix} I & 0 \end{bmatrix}.$$

Hence, if the solvability condition in Lemma 3, *i.e.*, $A + |A_h| \leq 0$, is satisfied, the H_∞ norm of the symmetric delayed system is given by (5). After simplifying the expression and taking (18) into account, (13) is obtained. □

Remark 1: It can be shown, (*e.g.* see the algebraic steps in the proofs of the Theorems in [9]), that for the case of symmetric systems examined here the results in Theorems 1 and 2 provide the best achievable bound obtained from the BRL-type LMI condition (3). That is, for a symmetric time delay realization if the LMI condition (3) has a positive definite solution P, then $P = I$ is a solution of (3).

Remark 2: It is easy to observe that the H_∞ norm bound condition given in (8) is recovered from the bound given in (13) by setting $D = 0$.

Remark 3: It is noted that for a time invariant delay-free symmetric system, *i.e.,* for $A_h = 0$, Theorem 2 provides the exact H_∞ norm given in [9].

Remark 4: The determined explicit expressions given in Theorems 1 and 2 are based on the *delay-independent* analysis condition represented by LMI (3). *Delay-independent* analysis conditions are known to provide conservative results compared to the *delay-dependent* analysis conditions due to ignoring the size of the delay (see [4], [3]). Similar algebraic tools as he ones presented in this paper can be used to address the delay-dependent analysis problem for symmetric systems.

IV. The Output Feedback Stabilization Problem

For the output feedback control synthesis problem we consider the following state-space representation of a delayed symmetric system.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_h x(t-h) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{19}$$

where $u(t)$ is the vector of control inputs, and $y(t)$ is the vector of measured outputs. We call this system state-space symmetric if conditions (2) hold.

The static symmetric output feedback stabilization problem is to design a symmetric static feedback gain $G = G^T$ such that the control law

$$u = -Gy \tag{20}$$

stabilizes the closed-loop system formed by the interconnection of (19) and (20). The following result provides a solution to the symmetric time-delayed output feedback stabilization problem.

Theorem 3: Consider the delayed symmetric system (19). There exists a symmetric output feedback control law (20) to asymptotically stabilize the closed-loop system if

$$B^\perp \mathcal{N} B^{\perp T} \leq 0 \tag{21}$$

where $\mathcal{N}$ is given by

$$\mathcal{N} = A + |A_h|. \tag{22}$$

If the above condition is satisfied, then all stabilizing symmetric output feedback gains G satisfy

- If B is square and invertible, then G can be selected as any matrix such that

$$G \geq B^{-1}\mathcal{N}B^{-T}. \tag{23}$$

- If BB^T is singular, then G can be selected as any matrix such that

$$G \geq B^{\dagger}\mathcal{N}B^{\dagger T} - B^{\dagger}\mathcal{N}B^{\perp T}(B^{\perp}\mathcal{N}B^{\perp T})^{-1}B^{\perp}\mathcal{N}B^{\dagger T} \tag{24}$$

Proof. The equation of the closed-loop state-space representation of the system reads

$$\dot{x}(t) = (A - BGB^T)x(t) + A_h x(t-h). \tag{25}$$

The stability condition for this system is the existence of symmetric positive definite matrices P and Q such that

$$\begin{bmatrix} PA_{cl} + A_{cl}^T P + Q & PA_h \\ A_h^T P & -Q \end{bmatrix} \leq 0 \tag{26}$$

where $A_{cl} = A - BGB^T$. Following the same approach as in the proof of Theorem 1, we obtain

$$BGB^T \geq \mathcal{N}. \tag{27}$$

If matrix B is square and nonsingular, the solution G to (27) is determined by inequality (23). Otherwise, Lemma 4 may be used to find a matrix bound on G, which results in the inequality (24), and this concludes the proof. □

In the next section a similar approach is applied to derive the symmetric output feedback control laws that guarantee the closed-loop H_∞ specifications.

V. The H_∞ Control Synthesis Problem

Now consider the following state-space system representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_h x(t-h) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) \\ y(t) &= C_2 x(t) \end{aligned} \tag{28}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^{m_1}$ is the vector of exogenous inputs, $u(t) \in \mathbb{R}^{m_2}$ is the vector of control inputs, $z(t) \in \mathbb{R}^{p_1}$ is the vector of controlled outputs, and $y(t) \in \mathbb{R}^{p_2}$ is the vector of measured outputs. We call this system state-space symmetric if the system state-space data satisfy the following symmetry conditions

$$A = A^T, \quad A_h = A_h^T, \quad B_1 = C_1^T, \quad B_2 = C_2^T. \tag{29}$$

The static symmetric output feedback H_∞ control synthesis problem is to design a symmetric static output feedback gain G such that the control law

$$u(t) = -Gy(t) \tag{30}$$

renders the system stable and guarantees a prescribed level of the H_∞ performance for the closed-loop system.

The closed-loop system of the open-loop system (28) and the controller (30) becomes

$$\begin{aligned} \dot{x}(t) &= (A - B_2 G B_2^T)x(t) + B_1 w(t) \\ z(t) &= C_1 x(t). \end{aligned} \tag{31}$$

Note that the closed-loop system (31) is also symmetric. The following result provides an explicit expression for an upper bound on the H_∞ norm of the closed-loop system and an explicit parametrization of controller gains that guarantee this performance bound for the closed-loop system.

Theorem 4: Consider the symmetric system represented by (28)-(29). There exists a symmetric output feedback control law (30) to stabilize the system and satisfies the suboptimal H_∞ performance if

$$B_2^{\perp}\mathcal{N}B_2^{\perp T} < 0 \tag{32}$$

where $\mathcal{N}$ is given by (22). If the above condition is satisfied, an upper bound on the achievable level of the H_∞ performance can be computed from

$$\gamma_{bound} = \lambda_{max}\{B_1^T B_2^{\perp T}(B_2^{\perp}\mathcal{N}B_2^{\perp T})^{-1}B_2^{\perp}B_1\} \tag{33}$$

For any $\gamma \geq \gamma_{bound}$, a static symmetric output feedback H_∞ control gain which makes the closed-loop system stable with H_∞ norm less than γ can be selected as any matrix G such that

$$G \geq B_2^{\dagger}[\Sigma - \Sigma B_2^{\perp T}(B_2^{\perp}\Sigma B_2^{\perp T})^{-1}B_2^{\perp}\Sigma]B_2^{\dagger T} \tag{34}$$

where Σ is defined as

$$\Sigma = A + |A_h| + \frac{1}{\gamma}B_1 B_1^T. \tag{35}$$

Proof. Substituting the closed-loop system (31) into LMI (3), and following the same lines as in proof of Theorem 3, we obtain the results of Theorem 4. □

VI. Numerical Examples

In this section, we validate the proposed analysis and design results of the previous sections using numerical examples.

In the first example we calculate the H_∞ bound for a scalar symmetric system with state-delay and compare our proposed bound with the H_∞ norm computed via linear analysis when a Pade approximation for the delay is used. Then, two additional numerical examples are presented to validate the analysis and synthesis results of this paper.

Example 1: Consider the scalar state-delay system represented as follows

$$\begin{aligned} \dot{x}(t) &= ax(t) + a_h x(t-h) + bw(t) \\ z(t) &= bx(t). \end{aligned} \tag{36}$$

For this particular case, employing Theorem 1 results in the following explicit H_∞ norm bound

$$\gamma \leq \bar{\gamma} = -b^2(a + |a_h|)^{-1}. \tag{37}$$

Let us assume that in the above system $a = -2$, $a_h = -0.5$, $b = 0.5$. Then, the bound $\bar{\gamma} = 0.167$ is obtained. Note that the same exact value of $\bar{\gamma}$ is obtained from the solution of the Bounded Real Lemma (BRL)-type analysis LMI in (3).

Next, we use the first- and the second-order Pade approximation to approximate the delay term $x(t-h)$ as follows

$$e^{-hs} \approx \frac{1-hs/2}{1+hs/2}, \qquad e^{-hs} \approx \frac{1-hs/2+(hs)^2/12}{1+hs/2+(hs)^2/12}. \tag{38}$$

Hence, we obtain LTI systems that approximate the delay system (36) and the state-space data are functions of the delay h. The exact H_∞ norm of these two systems is obtained from the standard BRL condition [7] and it is plotted in Figure 2. Figure 2 also shows the analytical bound $\bar{\gamma}$ obtained from (37). It is observed that as the delay size h increases, the H_∞ norms of the LTI systems obtained from the Pade approximations converge to the explicit H_∞ bound computed using the delay-independent analysis condition. Hence, these results demonstrate that for the above scalar system, the proposed explicit bound on the H_∞ norm is tight and provides the exact H_∞ norm for the delay-independent case.

Example 2: We consider the symmetric system (12) with the following randomly generated state-space data:

$$A = A^T = \begin{bmatrix} -1.9092 & -0.4588 & -0.0902 & -0.6758 \\ -0.4588 & -1.9149 & -0.7137 & -1.4493 \\ -0.0902 & -0.7137 & -0.8758 & -0.7530 \\ -0.6758 & -1.4493 & -0.7530 & -1.7289 \end{bmatrix}$$

$$A_h = A_h^T = \begin{bmatrix} -0.1\beta & 0 & -0.1 & 0 \\ 0 & 0.2 & 0 & 0 \\ -0.1 & 0 & 0.09\beta & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$$

$$B = C^T = \begin{bmatrix} 0.9516 & 0.4010 & 0.0431 & 0.4776 \\ 0.2603 & 0.4866 & 0.3709 & 0.1291 \\ 0.5147 & 0.7505 & 0.6933 & 0.4838 \\ 0.6363 & 0.1262 & 0.9358 & 0.9456 \end{bmatrix}$$

$$D = D^T = \begin{bmatrix} 0.8437 & 0.9280 & 0.7998 & 0.4248 \\ 0.9280 & 0.4022 & 0.1510 & 0.6874 \\ 0.7998 & 0.1510 & 0.0430 & 0.7157 \\ 0.4248 & 0.6874 & 0.7157 & 0.4577 \end{bmatrix}$$

Notice that the delay matrix A_h contains a scalar parameter β that we will vary to examine its effect on the H_∞ norm bound of the system.

Solving the stability LMI condition in (26), it is easy to verify that for $\beta \in [0, 4.5]$, the open-loop system is asymptotically stable. While the parameter β changes in $\beta \in [0, 4.5]$, the H_∞ norm bound of this system is calculated using 1) the LMI condition as given in (3), and 2) the explicit formula of this paper as given by Theorem 2. Figure 2 illustrates this comparison. As observed from this figure, the results provide consistent estimation of the H_∞ norm bound.

Example 3: As a third example, we consider the following state-space matrices to validate our explicit expression for the feedback control gain to stabilize the unstable time delay symmetric systems.

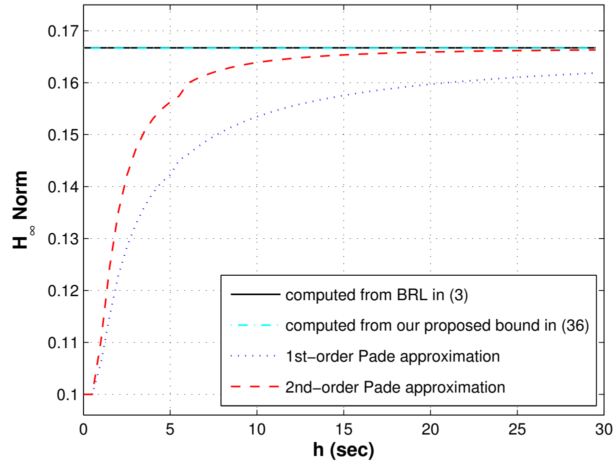

Fig. 1. H_∞ norm of the symmetric delayed system determined using different methods

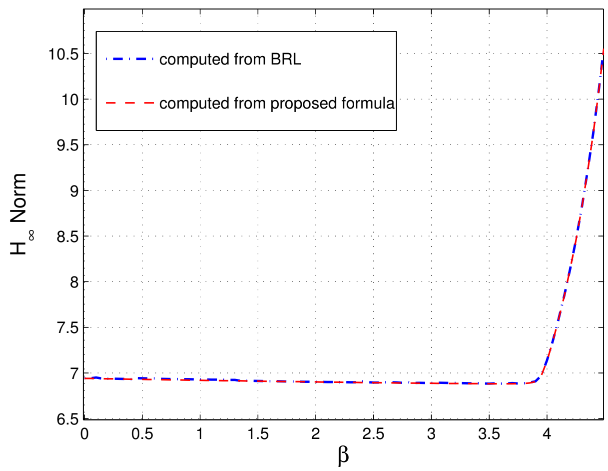

Fig. 2. H_∞ norm comparison

$$A = A^T = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

$$A_h = A_h^T = \begin{bmatrix} -0.1 & 0 & 0 \\ 0 & -0.2 & 0 \\ 0 & 0 & -0.1 \end{bmatrix},$$

$$B = C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad D = 0.$$

It is not difficult to verify that the open-loop system is unstable; hence, we seek to design an output feedback controller to asymptotically stabilize the closed-loop system. To this purpose, we use the result presented in Theorem 3. The stabilizing output feedback controller for this example is obtained from the explicit expression (24) (using the equality

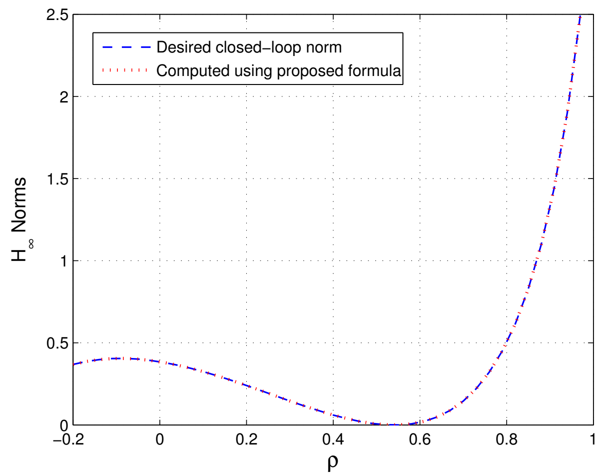

Fig. 3. Closed-loop H_∞ norm comparison

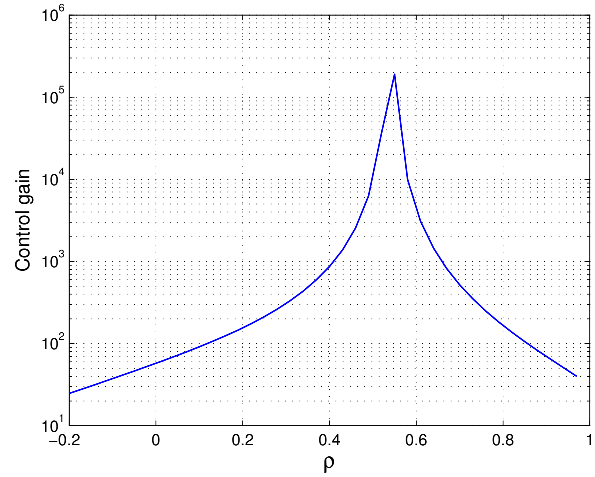

Fig. 4. Controller gain vs. ρ

sign) as

$$u(t) = -0.5516y(t).$$

Example 4: Consider the symmetric time delay system represented by

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -2\rho & 1.1+2\rho \\ 1.1+2\rho & -3.3+\rho \end{bmatrix} x(t) \\ &\quad + \begin{bmatrix} -0.5 & 0 \\ 0 & -1.2 \end{bmatrix} x(t-h) \\ &\quad + \begin{bmatrix} 2\rho \\ 0.1+\rho \end{bmatrix} w(t) + \begin{bmatrix} 1+\rho \\ 0.1+\rho \end{bmatrix} u(t) \\ z(t) &= \begin{bmatrix} 2\rho & 0.1+\rho \end{bmatrix} x(t) \\ y(t) &= \begin{bmatrix} 1+\rho & 0.1+\rho \end{bmatrix} x(t) \end{aligned}$$

We vary the parameter ρ in the interval $(-0.2, 1)$ and compute the best achievable level of the H_∞ performance bound of the closed-loop system γ_{bound} using the explicit expression (33). Then, we calculate a symmetric static output feedback control using the explicit expression (34) (with the equality sign) which guarantees a desired closed-loop system H_∞ performance for any $\gamma > \gamma_{bound}$. For each $\rho \in (-0.2, 1)$, we assume the desired level of the closed-loop system H_∞ norm to be $\gamma = 1.01\gamma_{bound}$. Hence, the control gain calculated from (34) will guarantee a closed-loop H_∞ norm less than γ. Comparison between the H_∞ norm bound of the closed-loop system computed from the BRL condition in (3) and the explicitly computed bound γ is illustrated in Figure 3. It is observed that for any ρ the closed-loop system H_∞ norm bound is matching the explicitly computed bound. Shown in Figure 4 is the control gain required to guarantee the closed-loop stability and desired H_∞ performance. It is noted that for $\rho = 0.55$, γ_{bound} reaches its minimum at $\gamma_{bound} = 0.0003$. This results in a large peak in the profile of the control output $u(t)$ as observed in Figure 4.

VII. Conclusion

We have examined the stabilization and the H_∞ norm analysis and control for state-space symmetric systems with a state delay term. For these systems, we have obtained analytical explicit solutions for an upper bound H_∞ norm calculation, as well as, explicit solutions of the output feedback stabilization and H_∞ performance synthesis problems. For the stabilization problem, we have developed an explicit parametrization of asymptotically stabilizing output feedback gains. For the H_∞ control problem we have obtained an explicit parametrization of static output feedback gains that guarantee a desired closed-loop H_∞ norm bound. The results of the paper are expressed in terms of the state-space data with no need for iterative calculations. The main results follow from a particular solution to the LMI formulation of the above problems and the use of basic matrix algebraic tools. These results represent analytical solutions of the delay-independent analysis and control problems for the class of symmetric systems.

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