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State reconstruction in complex networks using sliding mode observers

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***Abstract*— This paper focuses on the reconstruction of complete state information in all the individual nodes of a complex network dynamical system, at a supervisory level. Sliding mode observers are designed for this purpose. The proposed network observer is inherently robust, nonlinear and can accommodate time-varying coupling strengths and switching topologies, provided the number of nodes remain fixed. At the supervisory level, decentralised control signals are computed based on the state estimates in order to operate the network of dynamical systems in synchrony. A network of Chua circuits with six nodes is used to demonstrate the novelty of the proposed approach.**

I. INTRODUCTION

It is conspicuous that control applications in distributed and cooperative environments are increasing in number. This has given prominence to the problems associated with the control of network systems as well as systems of systems. A substantial research problem in this area is answering the question of how the collective dynamical systems operating over a network, in most cases having interactions with each other, can achieve synchronised performance. Analysis and control of complex behaviours in networks consisting of a large number of dynamical nodes has attracted the attention of researchers from different fields: see [19] for an overview of work in the area of general complexity problems related to the dynamics of networks; [28] for contributions in synchronisation of complex networks; and [17], [21] for contributions in cooperative control. The key feature of networks is their complexity, including dynamical evolution, topological structure, and time varying coupling strengths. Traditionally, a complex network was realized as a random graph – the so-called E-R model [9]. Later, Watts and Strogatz introduced the concept of a small-world network in [26], [27]. An important discovery, reported in [26], is the observation that a number of complex networks are also scale-free, and scale-free and small-world phenomena contribute to complexity. A general scale-free dynamical network model was discussed in [25], and subsequently conditions for synchronisation were derived [24]. A particular representation of the scale-free dynamical network (in a modified form), that is consistent with the one in [28], will be utilised in this paper.

The motivation for the present work is increasing the level of autonomy in the case of 'systems of systems' i.e. a group of dynamic systems operated over a network to perform synchronously. Systems operated in a distributed and cooperative environment are prevalent in many areas of research. For example mobile robots, cooperating UAV team operations, formation flying of UAV's and satellites, and distributed state estimation applications (for example localisation): for details of these kinds of applications, see [12], [17], [21]. Research in the area of communication, command and control is also very active [12], [20]. It is important that in autonomous operation, the systems are monitored at a supervisory level, and in most cases the situation will be such that the supervisory level node commands and controls the systems during operation; for example see [12], [20].

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In [1] an adaptive sliding mode observer has been designed for synchronisation of coupled nonlinear systems (in a master-slave architecture). Adaptive sliding mode control has also been applied in [30] for synchronisation of general master slave systems. Notice that all these applications follow the master-slave framework. Recently a second order sliding mode observer has been designed specifically for online monitoring and fault detection in satellite formations, in the presence of uncertainty [29]. In [20], an observer-controller combination is discussed for a single unicycle mobile robot. Reference [18] considers a similar problem, providing a sufficient condition for nonlinear observability and an associated Extended Kalman Filter (EKF) scheme for the localization problem. The EKF observer estimates the states of the leader-follower formation from the measurements and control signals computed at the leader level. References [18], [20], [29], [31] give details and examples of the use of observers in complex applications.

In this paper, an observer-controller combination for a class of systems operating over a network is considered. Algebraic graph theoretic tools, based on the connectivity of the graph, are used to represent the dynamical systems operating over the network. The individual node level dynamics are represented as a combination of linear and nonlinear parts. The basic idea and focus of this paper is to estimate the states of a complex network from the measurement signals by a centralised observer at a supervisory level. The reconstructed state signal is then used to generate decentralised control commands sent to the individual subsystems. The focus of this paper is on the reconstruction of the states. Sliding mode observers – well recognized nonlinear robust observers – have been developed over a number of years [2], [4]–[7], [10], [11], [13], [14], [16]. By making use of the particular structure of the problem, appropriate observer gains are designed. The underlying idea is to drive the state estimation error to the sliding plane in finite time and thereby achieve asymptotic observers for the network of subsystems. A systematic state reconstruction and control signal generation approach at a supervisory level is developed. The main contribution of this paper is the extension of existing sliding mode observer ideas to the problem of node state estimation in a network environment in which there are time varying coupling strengths and unknown changes in the topology.

II. NOTATION AND PRELIMINARIES

The expression $\mathcal{D}iag(.)$ defines a diagonal matrix and $\mathcal{C}ol(.)$ defines a column vector. For a symmetric positive definite (s.p.d) matrix $P = P^{\mathrm{T}} > 0$, $\lambda_{min}(P)$ and $\lambda_{max}(P)$ are the minimum and maximum eigenvalues. The symbols $\mathcal{N}(\cdot)$ and $\mathcal{R}(\cdot)$ represent the null space and range space of a matrix respectively. For the graph $\mathcal{G}$, the adjacency matrix $\mathcal{A}(\mathcal{G}) = [a_{ij}]$, is defined by setting $a_{ij} = 1$ if i and j are

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adjacent nodes of the graph, and $a_{ij} = 0$ otherwise. This is a symmetric matrix. The symbol $\Delta(\mathcal{G}) = [\delta_{ij}]$ represents the degree matrix, and is an $N \times N$ diagonal matrix, where δ_{ii} is the degree of the vertex i. The Laplacian of $\mathcal{G}$, $\mathcal{L}(\mathcal{G})$, is defined as the difference $\Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G})$. The smallest eigenvalue of $\mathcal{L}(\mathcal{G})$ is exactly zero and the corresponding eigenvector is given by $\mathbf{1}$. The Laplacian $\mathcal{L}(\mathcal{G})$ is always rank deficient and positive semi-definite. Moreover, the rank of $\mathcal{L}(\mathcal{G})$ is $n-1$ if and only if $\mathcal{G}$ is connected.

III. System description

The network considered in this paper consists of N dynamical systems, indexed as $1,2,...,N$, with communication interactions. This is viewed as a graph $\mathcal{G}$ with N labelled vertices, each representing an n-dimensional dynamic system. The nodes are assumed to be coupled linearly and diffusively. As and where there is an interconnection between any two dynamical systems, it constitutes an edge connecting those nodes. In the case of a dynamically varying interaction topology among the fixed number of N systems, the notation $\overline{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_M\}$ will be used, where each $\mathcal{G}_i$ represents an interaction topology for a particular time period. The dynamics of the i^{th} individual node of graph $\mathcal{G}$ are given in equations (1) and (2).

$$\dot{x}_i = Ax_i + Bu_i + Df_i(y_i) - c(t)\sum_{j=1}^{N} \mathcal{L}_{ij}\Gamma x_j \tag{1}$$

$$y_i = Cx_i \tag{2}$$

where $x_i \in \mathbb{R}^n$ represents the n-dimensional state vector of the i^{th} node of the network. The symbol x represents the collective state of the network $x = \mathrm{col}(x_1, x_2, .., x_i, .., x_N)$. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{n\times q}$ and $C \in \mathbb{R}^{p\times n}$ represent the nominal linear part of the system comprising the dynamics of the i^{th} node. Assume that the matrices D and C have full column and row rank respectively. The triplet (A,D,C) is assumed to be a minimal realization of the i^{th} node of the network. The coefficient $c(t) > 0$ is the time varying coupling strength between the i^{th} and j^{th} node. The coupling strength is assumed to be identical for all the connections between the nodes. As described earlier, $\mathcal{L} \in \mathbb{R}^{N\times N}$ denotes the connectivity of the topology of the network being considered.

The matrix $\Gamma = \tau_{ij} \in \mathbb{R}^{n\times n}$ represents the local coupling configuration among the states of the nodes. All the entries of Γ are 1 or 0 and represent the existence or non-existence of coupling in the respective channels in the network. In the paper it is assumed that

$$\Gamma = \mathcal{D}iag\ [\tau_1, \tau_2, .., \tau_i, .., \tau_n]$$

is diagonal, implying the coupling is identical in each node of the network. The signal $y_i \in \mathbb{R}^p$ represents the measured outputs of the i^{th} node respectively. Here it is assumed that $p \geq q$. The functions $f_i(y_i)$, represent the *unknown* nonlinear part of the dynamical system and are assumed to satisfy certain sector bounds which will be precisely defined later in the paper.

Assumption 1 *rank*$(CD) = q$ for each node.

Assumption 2 *The linear part of a decoupled node in the system* (1)-(2)*, represented as* (A,D,C)*, is minimum phase.*

IV. Sliding mode observer design preliminaries

Consider the autonomous system

$$\dot{x} = Ax + Bu + Df(y) \tag{3}$$

$$y = Cx \tag{4}$$

thought of as the decoupled dynamics of a typical node. Note that the structure of (3)-(4) is similar to the one used for absolute stability analysis in classical Lur'e systems.

If Assumptions 1 and 2 hold then, as argued in [8], there exists a nonsingular mapping $x \mapsto T_0 x$ transforming the co-ordinates to ones in which the system triple (A,D,C) has the following structure

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ D_o \end{bmatrix} \quad C = \begin{bmatrix} 0 & C_2 \end{bmatrix} \tag{5}$$

where $A_{11} \in \mathbb{R}^{(n-p)\times(n-p)}, D_o \in \mathbb{R}^{q\times q}$ is non-singular, and $C_2 \in \mathbb{R}^{p\times p}$ is orthogonal. Define A_{211} as the top $p-q$ rows of A_{21}. By construction, the pair (A_{11}, A_{211}) is detectable and the unobservable modes of (A_{11}, A_{211}) are the invariant zeros of (A,D,C) [8]. Also for convenience, define $D_2 \in \mathbb{R}^{p\times q}$ as the bottom p rows of D (and therefore includes D_o).

The *discontinuous* observer structure which will be used here is of the form

$$\dot{z}(t) = Az(t) + Bu(t) - G_l e_y(t) - G_n \nu \tag{6}$$

where the discontinuous vector

$$\nu = -\rho(t,y)\frac{P_o e_y}{\|P_o e_y\|} \qquad \text{if } e_y \neq 0 \tag{7}$$

and z denotes the state estimate. The output estimation error is $e_y(t) := C(x(t) - z(t))$ and $P_o \in \mathbb{R}^{p\times p}$ is a s.p.d. matrix. The scalar modulation function $\rho : \mathbb{R}_+ \times \mathbb{R}^p \times \to \mathbb{R}_+$ will be described formally later but it represents an upper bound on the magnitude of the nonlinear term.

A suitable choice for the matrix G_n in the co-ordinate system in (5) is

$$G_n = \begin{bmatrix} LC_2^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{bmatrix} \tag{8}$$

where $L = \begin{bmatrix} L^o & 0 \end{bmatrix}$ with $L^o \in \mathbb{R}^{(n-p)\times(p-q)}$. The matrices L^o, P_o and G_l are to be determined. In [8], the system associated with the state estimation error $e := x - z$ was analyzed and the following result was proved: Consider a s.p.d. matrix P, with the structure

$$P = \begin{bmatrix} P_1 & P_1 L \\ L^{\mathrm{T}} P_1 & P_2 + L^{\mathrm{T}} P_1 L \end{bmatrix} > 0, \tag{9}$$

where $P_1 \in \mathbb{R}^{(n-p)\times(n-p)}$, $P_2 \in \mathbb{R}^{p\times p}$ such that

$$P(A + G_l C) + (A + G_l C)^{\mathrm{T}} P < 0 \tag{10}$$

If $P_o := C_2 P_2 C_2^{\mathrm{T}}$, the error $e(t)$ which satisfies

$$\dot{e} = (A + G_l C)e + Df(x) + G_n \nu$$

is quadratically stable. Furthermore, sliding occurs in finite time on

$$\mathcal{S} = \{e : e_y = 0\}$$

and the reduced order dynamics are governed by the system matrix $A_{11} - L^o A_{211}$. If Assumptions 1 and 2 hold, a solution to this problem is guaranteed to exist [8] and L^o can be chosen to make $A_{11} - L^o A_{211}$ stable.

In the case when $p=q$, L^o is the empty matrix and $L=0$. In this situation the approach is still valid provided A_{11} is stable. It can be easily verified if $p=q$ then (A,D,C) has exactly $n-q$ invariant zeros which are the eigenvalues of A_{11} and so Assumption 2 is equivalent to A_{11} being stable.

V. A NETWORK OBSERVER

Now consider these ideas extended to the problem of network supervision. Consider the observer dynamical system

$$\dot{z}_i = Az_i + Bu_i - c(t)\sum_{j=1}^{N}\mathcal{L}_{ij}\Gamma z_j - G_{l_i}(e_y) - G_{n_i}\nu_i \quad (11)$$

$$y_i = Cx_i \quad (12)$$

for $i=1,...,N$ where $e_{y_i}=C(x_i-z_i)$, $e_y=\mathcal{C}ol(e_{y_1},\dots e_{y_N})$ and ν_i is a discontinuous injection term which depends only on e_{y_i} for $i=1,...,N$: specifically

$$\nu_i = -\rho(t,y)\frac{e_{y_i}}{\|e_{y_i}\|}, \quad \text{if } e_{y_i}\neq 0 \quad (13)$$

The gain G_{n_i} is assumed to have the structure given in (8). Note this is a 'centralised' observer in the sense that the supervisory node receives information y_i from all other subsystem nodes in the network in order to utilise the output estimation error term e_y. The gain matrix $G_{l_i}(e_y)$ is a function of the output estimation error from the whole network and unlike in (6) will be time varying since it depends on $c(t)$. The error in the state estimate of the i^{th} node is given by

$$\dot{e}_i = Ae_i + Df_i(y_i) - c(t)\sum_{j=1}^{N}\mathcal{L}_{ij}\Gamma e_j + G_{l_i}(e_y) + G_{n_i}\nu_i \quad (14)$$

where $i=1,...,N$.

Assumption 3 *Assume that in the coordinate system of* (5)

$$\Gamma = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} \quad (15)$$

where $\Gamma_2\in\mathbb{R}^{p\times p}$. By changing the coordinates within the null space of C, it can be assumed that

$$\Gamma_1 = \mathcal{D}iag\{\tau_1,\tau_2,\dots\tau_{n-p}\}$$

where the scalars $\tau_i\in\{0,1\}$. In fact it can be assumed without loss of generality

$$\Gamma_1 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad (16)$$

where $r:=rank(\Gamma_1)$.

To obtain an expression for the reduced order sliding mode, consistent with the 4-block partition of the triple (A,D,C), write the node state estimation error as

$$e_i := \begin{bmatrix} e_{1_i} \\ e_{2_i} \end{bmatrix} \quad (17)$$

where $e_{1_i}\in\mathbb{R}^{n-p}$. Suppose a sliding motion can be attained on the surface $\mathcal{S}=\bigcap_{i=1\dots N}\mathcal{S}_i$ where

$$\mathcal{S}_i = \{(e_{1_i},e_{2_i}) \mid e_{2_i}=0\} \quad (18)$$

The sliding motion on $\mathcal{S}$ associated with system (14) is governed by:

$$\dot{e}_{1_i} = A_{11}e_{1_i} - c(t)\sum_{j=1}^{N}\mathcal{L}_{ij}\Gamma_1 e_{1_j} + LC_2^{\mathrm{T}}\nu_{eq,i} \quad (19)$$

$$0 = A_{21}e_{1_i} + D_2 f_i(y_i) + C_2^{\mathrm{T}}\nu_{eq,i} \quad (20)$$

where $\nu_{eq,i}$ is the so-called equivalent injection necessary to maintain sliding and represents the 'average' switched signal [7]. To develop an expression for the reduced order sliding motion write

$$C_2^{\mathrm{T}}\nu_{eq,i} = \begin{bmatrix} \nu_{eq,i}^1 \\ \nu_{eq,i}^2 \end{bmatrix}$$

where $\nu_{eq,i}^1\in\mathbb{R}^{p-q}$. From the structure of D_2 and the definition of A_{211}, it follows from the top $p-q$ rows of equation (20) that

$$0 = A_{211}e_{1_i} + \nu_{eq,i}^1$$

Note since $L=\begin{bmatrix} L^o & 0 \end{bmatrix}$ where $L^o\in\mathbb{R}^{(n-p)\times(p-q)}$ then

$$LC_2^{\mathrm{T}}\nu_{eq,i} = L^o\nu_{eq,i}^1 = -L^oA_{211}e_{1_i} \quad (21)$$

Substituting the expression from (21) into (19), it follows:

$$\dot{e}_{1_i} = (A_{11}-L^oA_{211})e_{1_i} - c(t)\sum_{j=1}^{N}\mathcal{L}_{ij}\Gamma_1 e_{1_j}, \quad i=1\dots N \quad (22)$$

Since the Laplacian matrix $\mathcal{L}$ is symmetric, by applying spectral decomposition [15], it can be written as

$$\mathcal{L} = V^{\mathrm{T}}DV \quad (23)$$

where

$$D = diag\{d_1,d_2,\dots,d_N\}$$

are the eigenvalues, and V is an orthogonal matrix composed of the eigenvectors. By making use of the Kronecker representation, equation (22) at a network level can be written

$$\dot{e}_1 = (I_N\otimes(A_{11}-L^oA_{211}))e_1 - c(t)(\mathcal{L}\otimes\Gamma_1)e_1 \quad (24)$$

where $e_1=\mathcal{C}ol(e_{1_1},e_{1_2},\dots e_{1_N})$. In order to study the stability of the reduced order sliding motion a suitable change of coordinates $e_1\mapsto T_d e_1=\eta$ is introduced, where

$$T_d := (V^{\mathrm{T}}\otimes I_{(n-p)}) \quad (25)$$

From the properties of Kronecker products

$$(V^{\mathrm{T}}\otimes I_{(n-p)})(I_N\otimes(A_{11}-L^oA_{211}))(V\otimes I_{(n-p)}) = I_N\otimes(A_{11}-L^oA_{211}) \quad (26)$$

and

$$(V^{\mathrm{T}}\otimes I_{(n-p)})(\mathcal{L}\otimes\Gamma_1)(V\otimes I_{(n-p)}) = (V^{\mathrm{T}}\mathcal{L}V)\otimes\Gamma_1 = D\otimes\Gamma_1$$

Since $\dot{\eta}=T_d\dot{e}_i$, from applying the transformation in (25) to (24), it follows

$$\dot{\eta} = (I_N\otimes(A_{11}-L^oA_{211}) - c(t)(D\otimes\Gamma_1))\eta \quad (27)$$

Since D is diagonal, (27) can be decoupled to node level dynamics of the form

$$\dot{\eta}_i = (A_{11}-L^oA_{211}-c(t)d_i\Gamma_1)\eta_i, \quad i=1\dots N \quad (28)$$

where $\eta = \mathcal{C}ol(\eta_1, \eta_2 \dots \eta_N)$. Consequently the stability of the reduced order sliding motion depends on the stability of the system matrices

$$A_i(t) =: (A_{11} - L^o A_{211} - c(t) d_i \Gamma_1) \tag{29}$$

for $i = 1 \dots N$. Notice that $c(t)d_i \geq 0$ for $i = 1 \dots N$ since $\mathcal{L}$ is positive semi-definite.

The stability of $(A_{11} - L^o A_{211} - c(t)d_i\Gamma_1)$ may be examined by synthesizing a Lyapunov function of the form $V_i = \eta_i^{\mathrm{T}} P_1 \eta_i$.

$$\begin{aligned} \dot{V}_i \ = \ & \eta_i \left(P_1 (A_{11} - L^o A_{211}) + (A_{11} - L^o A_{211})^{\mathrm{T}} P_1 \right. \\ & \left. - c(t) d_i \left(P_1 \Gamma_1 + \Gamma_1^{\mathrm{T}} P_1 \right) \right) \eta_i \end{aligned} \tag{30}$$

A sufficient condition to ensure negative definiteness of $\dot{V}_i$ is to synthesize a s.p.d matrix P_1 such that

$$-P_1\Gamma_1 - \Gamma_1^{\mathrm{T}} P_1 \ \leq \ 0 \tag{31}$$

$$P_1(A_{11} - L^o A_{211}) + (A_{11} - L^o A_{211})^{\mathrm{T}} P_1 \ < \ 0 \tag{32}$$

Because of the specific structure of Γ_1 in assumption 3, the matrix inequality (31) can only hold if

$$P_1 = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} \tag{33}$$

where $P_{11} \in \mathbb{R}^{r \times r}$. This follows because for a general s.p.d matrix P_1 partitioned as

$$P_1 = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix}$$

it follows by direct computation that

$$P_1\Gamma_1 + \Gamma_1 P_1 = \begin{bmatrix} 2P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & 0 \end{bmatrix}$$

and hence $P_1\Gamma_1 + \Gamma_1 P_1 \geq 0$ if and only if $P_{12} \equiv 0$.

From the structure of P_1 in (33), the problem can be posed as an LMI in the decision variables P_1 and M of the form

$$P_1 A_{11} + A_{11} P_1 - M A_{211} - A_{211}^{\mathrm{T}} M^{\mathrm{T}} < 0 \tag{34}$$

where the variable change $M := P_1 L^o$ has been employed. Efficient routines exist to solve these problems [3].

In the situation in which an s.p.d. matrix P_1 can be found to satisfy (31)-(32)*, then $A_i(t)$ from (29) is stable for all* $d_i \geq 0$. This means that the stability of $A_i(t)$ is independent of the Laplacian $\mathcal{L}$ since, whilst a change in $\mathcal{L}$ alters D and V in (23) and hence d_i for $i = 1 \dots N$, the stability of $A_i(t)$ from (29) is unaffected.

Consider the change of coordinates $e_i \mapsto T_L e_i$ where

$$T_L := \begin{bmatrix} I_{n-p} & -L \\ 0 & I_p \end{bmatrix}$$

Partition the transformed error state vector as

$$\begin{bmatrix} \tilde{e}_{1_i} \\ e_{2_i} \end{bmatrix} = T_L e_i \tag{35}$$

where $\tilde{e}_{1i} = e_{1i} - L e_{2i}$. Write the injection gain as

$$\begin{bmatrix} G_{1_i}(e_y) \\ G_{2_i}(e_y) \end{bmatrix} = G_{l_i}(e_y) \tag{36}$$

It follows from (14) that

$$\begin{aligned} \dot{\tilde{e}}_{1_i} \ = \ & (A_{11} - L^o A_{211})\tilde{e}_{1_i} + \tilde{A}_{12} e_{2_i} \\ & - c(t) \sum_{j=1}^{N} \mathcal{L}_{ij} (\Gamma_1 \tilde{e}_{1_j} + \Gamma_{12} e_{2_j}) + G_{1_i}(e_y) - L G_{2_i}(e_y) \end{aligned} \tag{37}$$

$$\begin{aligned} \dot{e}_{2_i} \ = \ & A_{21}\tilde{e}_{1_i} + \tilde{A}_{22} e_{2_i} + D_2 f_i(y_i) \\ & - c(t) \sum_{j=1}^{N} \mathcal{L}_{ij} \Gamma_2 e_{2_j} + G_{2_i}(e_y) + C_2^{\mathrm{T}} \nu_{eq,i} \end{aligned} \tag{38}$$

where $\tilde{A}_{12} \in \mathbb{R}^{(n-p)\times p}$ and $\tilde{A}_{22} \in \mathbb{R}^{p \times p}$ come from the partition

$$\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ A_{21} & \tilde{A}_{22} \end{bmatrix} = TAT^{-1}$$

and $\tilde{A}_{11} = (A_{11} - L^o A_{211})$. The matrix $\Gamma_{12} := \Gamma_1 L - L\Gamma_2$ in (37) is obtained from the partition

$$\begin{bmatrix} \Gamma_1 & \Gamma_{12} \\ 0 & \Gamma_2 \end{bmatrix} = T\Gamma T^{-1}$$

Define

$$G_{2_i}(e_y) := -\tilde{A}_{22} C_2^{\mathrm{T}} e_{y_i} + \Phi C_2^{\mathrm{T}} e_{y_i} + c(t) \sum_{j=1}^{N} \mathcal{L}_{ij} \Gamma_2 C_2^{\mathrm{T}} e_{y_j} \tag{39}$$

where $\Phi \in \mathbb{R}^{p \times p}$ is a stable matrix. Then define

$$G_{1_i}(e_y) := L G_{2_i}(e_y) - \tilde{A}_{12} C_2^{\mathrm{T}} e_{y_i} + c(t) \sum_{j=1}^{N} \mathcal{L}_{ij} \Gamma_{12} C_2^{\mathrm{T}} e_{y_j} \tag{40}$$

Substituting the expressions for $G_{1_i}(e_y)$ and $G_{2_i}(e_y)$ into equations (37)-(38)

$$\dot{\tilde{e}}_{1_i} \ = \ (A_{11} - L^o A_{211})\tilde{e}_{1_i} - c(t) \sum_{j=1}^{N} \mathcal{L}_{ij} \Gamma_1 \tilde{e}_{1_j} \tag{41}$$

$$\dot{e}_{2_i} \ = \ A_{21}\tilde{e}_{1_i} + \Phi e_{2_i} + D_2 f_i(y_i) - C_2^{\mathrm{T}} \nu_i \tag{42}$$

for $i = 1 \dots N$.

Write $\tilde{e}_1 = \mathcal{C}ol(\tilde{e}_{1_1}, \dots \tilde{e}_{1_N})$ and $e_2 = \mathcal{C}ol(e_{2_1}, \dots e_{2_N})$ then (41)-(42) can be written as

$$\dot{\tilde{e}}_1 \ = \ (I_N \otimes (A_{11} - L^o A_{211}) - (\mathcal{L}_{ij} \otimes \Gamma_1))\tilde{e}_1 \tag{43}$$

$$\begin{aligned} \dot{e}_2 \ = \ & (I_N \otimes A_{21})\tilde{e}_1 + (I_N \otimes \Phi) e_2 \\ & + (I_N \otimes D_2) f(y) - (I_N \otimes C_2^{\mathrm{T}})\nu \end{aligned} \tag{44}$$

where $\nu = \mathcal{C}ol(\nu_1, \dots \nu_N)$. It was demonstrated earlier that by construction $(I_N \otimes (A_{11} - L^o A_{211}) - (\mathcal{L}_{ij} \otimes \Gamma_1))$ is stable. Let $V_1 = \tilde{e}_1^{\mathrm{T}} \tilde{P}_1 \tilde{e}_1$ be a quadratic Lyapunov equation for the sub-system (43) such that

$$\dot{V}_1|_{(43)} \leq -q \tilde{e}_1^{\mathrm{T}} \tilde{e}_1$$

Note that

$$\|(I_N \otimes D_2) f(y)\| \leq \|D_2\| \|f(y)\|$$

and $e_{y_i} = C_2 e_{2_i}$ which implies $\|e_y\| = \|e_2\|$. Consider

$$V := V_1 + k e_2^{\mathrm{T}} e_2 \tag{45}$$

where k is a positive scalar to be decided upon. For simplicity assume $\Phi = -\phi I_p$ for some positive scalar ϕ. It can be shown using the quadratic form in (45) that (43)-(44) is quadratically stable: taking time derivatives along the trajectories

$$\begin{aligned}\dot{V} &\leq -q\tilde{e}_1^{\mathrm{T}}\tilde{e}_1 + 2ke_2^{\mathrm{T}}\dot{e}_2\\ &\leq -q\|\tilde{e}_1\|^2 + 2k\|e_2\|\|(I_N\otimes A_{21})\|\|\tilde{e}_1\| - 2\phi k\|e_2\|^2\\ &\quad +2k\|e_2\|\|D_2\|\|f\| - 2ke_2^{\mathrm{T}}(I_N\otimes C_2^{\mathrm{T}})\nu\end{aligned}$$

It is easy to see that

$$e_2^{\mathrm{T}}(I_N\otimes C_2^{\mathrm{T}}) = e_2^{\mathrm{T}}(I_N\otimes C_2)^{\mathrm{T}} = ((I_N\otimes C_2)e_2)^{\mathrm{T}} = e_y^{\mathrm{T}} \quad (46)$$

Applying (46) and the definition of ν in (13), it follows:

$$e_2^{\mathrm{T}}(I_N\otimes C_2^{\mathrm{T}})\nu = -\rho\sum_{i=1}^{N}\|e_{yi}\| \leq -\rho\|e_y\|$$

Hence,

$$\begin{aligned}\dot{V} &\leq -q\|\tilde{e}_1\|^2 + 2k\|e_2\|\|(I_N\otimes A_{21})\|\|\tilde{e}_1\| - 2\phi k\|e_2\|^2\\ &\quad +2k\|e_2\|\|D_2\|\|f\| - 2k\rho(t,y)\|e_y\|\\ &\leq -q\|\tilde{e}_1\|^2 + 2k\|e_2\|\|(I_N\otimes A_{21})\|\|\tilde{e}_1\| - 2\phi k\|e_2\|^2\end{aligned}$$

if $\rho(t,y)\geq\|D_2\|\|f(y)\|$ since $\|e_2\| = \|e_y\|$. Furthermore since $\|I\otimes A_{21}\| = \|A_{21}\|$,

$$\dot{V} \leq \begin{bmatrix}\|\tilde{e}_1\|\\ \|e_2\|\end{bmatrix}^{\mathrm{T}} \underbrace{\begin{bmatrix}-q & k\|A_{21}\|\\ k\|A_{21}^{\mathrm{T}}\| & -2k\phi\end{bmatrix}}_{Q} \begin{bmatrix}\|\tilde{e}_1\|\\ \|e_2\|\end{bmatrix}$$

It can be verified that $Q<0$ if and only if

$$k\|A_{21}\|^2 \leq 2\phi q \quad (47)$$

Since k is a free parameter, it can always be chosen to ensure (47) is satisfied.

A. Static state error feedback control

The control strategy adopted in the supervisory level node is the static state error feedback controller discussed in [23]. Synchronisation in [23] to a pre-defined master system behaviour is obtained with full static-state error feedback to the individual nodes using the control signal $u_i \in \mathbb{R}^m$ with feedback matrix $F \in \mathbb{R}^{m\times n}$. An identical procedure to the one discussed in [23] is employed to synthesise the feedback matrix for control purposes. However, estimated state information for the individual nodes has been used in place of the true node level state information. Here, an $F \in \mathbb{R}^{m\times n}$ has been constructed with the intention of obtaining 'double-scroll' attractor behaviour and to force all the individual nodes to reach consensus to this trajectory.

VI. Numerical Example

To demonstrate the theory developed in this paper, a network of Chua oscillators is considered where

$$A = \begin{bmatrix}-am_1 & a & 0\\ 1 & -1 & 1\\ 0 & -b & 0\end{bmatrix},\ D = \begin{bmatrix}-a(m_0-m_1)\\ 0\\ 0\end{bmatrix} \quad (48)$$

$$C = \begin{bmatrix}1 & 0 & 0\end{bmatrix}, \qquad \Gamma = \mathcal{D}iag\{1,0,0\} \quad (49)$$

and $B = I_3$. The nonlinearity is $f_i(y_i) = \frac{1}{2}(|x_{i1}+c| - |x_{i1}-c|)$, which has a sector bound $[0,1]$. The chosen values of the parameters are $a = 9, b = 14.286, c = 1, m_0 = -1/7, m_1 = 2/7$ in order to obtain the double scroll attractor [23]. A decentralised control signal u_i is generated following the procedure described in subsection V-A, which is essentially the one reported in [23]. Consistent with [23], the control input matrix B is taken as the identity. Here, one assumption is that each node communicates its output information to the supervisory monitoring node. Hence, this could be viewed as a network monitoring framework.

The objective is to demonstrate robust state estimation even in the presence of time varying coupling strengths and varying network topologies at different time intervals. A 100 second simulation time window is considered. A chirp signal with an input frequency of 0.5Hz and with an amplitude bias of 1.25, is considered as the 'random' time varying coupling strength $c(t) > 0$.

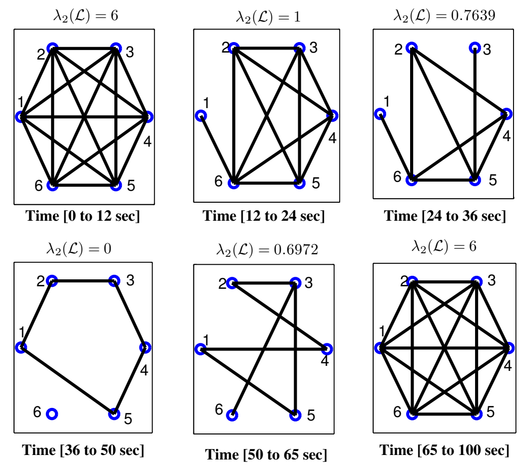

Fig. 1. Schematic representation of varying Graph topology

Note that by assumption there is no change in the number of nodes/dynamic systems comprising the network over the simulation time. However, the interaction topologies can vary over time as shown in figure 1. Initially at time $t = 0$, the Chua oscillator network is assumed to be fully connected, i.e. $\mathcal{G}(6,15)$, see the first subfigure in figure 1. At time $t = 12$ seconds, the network configuration is changed to the one shown in the second upper subfigure in figure 1, i.e. $\mathcal{G}(6,11)$, and remains this way until time $t = 24$ seconds. There is a change in λ_2, (representing the algebraic connectivity [22]), from 6 to 1 (the λ_2 values of each graph are given above in the subfigures in figure 1). During the time intervals $[24,36), [36,50), [50,65)$ and $[65,100]$, the network topologies are $\mathcal{G}(6,8), \mathcal{G}(6,5), \mathcal{G}(6,6)$ and $\mathcal{G}(6,15)$ respectively. It is important to notice that during the period - 36 seconds to 50 seconds - the algebraic connectivity is zero, with the 6^{th} node being completely isolated while the remainder form a cyclic structure.

Figure 2 shows the convergence of the error in the difference between the true state and the estimated state at the individual node level. To illustrate the convergence of the estimation error, only a 10 second time window is provided in the six subfigures in Fig. 2. The two bottom subfigures ($e_5(t)$ and $e_6(t)$ vs. Time) in Fig. 2 are drawn over a period of 100 seconds. Note that at 12, 24, 36, 50 and 65 seconds the network topology varies. *However, it is important to note that at the supervisory level there is no information about the variation of the network topologies.* Furthermore, once sliding is achieved, provided the nonlinear gain $\rho(t)$ is sufficiently large to maintain sliding, the linear

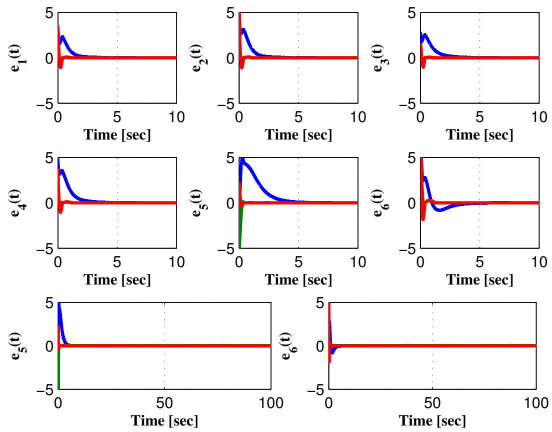

Fig. 2. Error in state estimation at each individual node

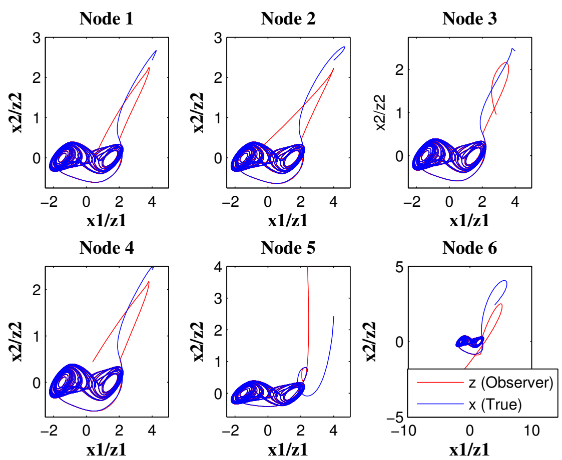

Fig. 3. Phase portrait of true state x_1 vs x_2 with estimated state z_1 vs z_2

gain term $G_{li}(e_y) \equiv 0$ and so the observer is independent of the coupling strength $c(t)$ and $\mathcal{L}$. At the supervisory level, the control signal is generated as a static error feedback based on the estimated state and the supervisory level master node state at that instant of time. This establishes the effectiveness of sliding mode observers in estimating the state of node level systems in the network robustly, in the presence of the time varying coupling strength and the varying unknown network topologies given in figure 1.

VII. Conclusions

The primary objective of this paper is to reconstruct complete state information in a complex network dynamical system at a supervisory node level. Sliding mode observers are designed for this purpose. The proposed network observer is inherently robust and can accommodate time varying coupling strengths and switching topologies. At the supervisory level, decentralised control signals are computed based on the state estimates in order to operate the network in a synchronous fashion. A network of Chua circuits with six nodes is used to demonstrate the proposed approach.

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