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Dynamic Traveling Repairperson Problem for Dynamic Systems

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Abstract— **In this paper, we study the Dynamic Traveling Repairman Problem (DTRP) for dynamic systems. In the DTRP, customers are arising dynamically and randomly in a bounded region R, and when customers arrive, they wait for the repairperson to visit their location and offer a "service" -that will take a certain random amount of time s-. In our study, the repairperson is modeled as a dynamic system whose output space contains R and our objective is the average time a customer has to wait to be serviced. We present schemes (for low and high traffic intensities) that guarantee that the expected waiting time for a customer scales within a constant factor of the optimum in terms of traffic intensity.**

I. Introduction

The problem we tackle in this paper is a dynamic parallel to our previous work [12] on the (static) Traveling Salesperson Problem for dynamic systems. Both problems have two components; the first is the dynamic system and the second is an optimal routing problem (in the output space of the system) with points to be visited. The idea of introducing the kinematics or dynamics into the study of the path optimization problems is gaining a considerable amount of interest lately. Historically, these path optimization problems (for example TSP and DTRP) have generally been studied as combinatorial optimization problems over a graph. Recently though, the properties the dynamic system that has to tour the points have been considered; and the TSP has been recently studied for the Dubins vehicle [1], [10], the double integrator[2], the Reeds-Shepp car, and the differential drive[7]. The DTRP has been studied for the Dubins vehicle [1].

The main motivation for injecting the dynamics of the system into these problems are real life applications where separating the dynamics from the planning problem leads to worse performance. These applications are usually for autonomous robots and unmanned vehicles, where the robot or vehicle should be able to trace the optimal path designed. That requirement is hard to achieve by solving the path planning problem first and then modifying the resulting optimal path. Thus the natural problem to solve is the one in which the dynamics of the system are part of the problem. This way, the optimal paths designed will be appropriate for the real life vehicles and robots the problem was designed for.

With all the advances in robotics and the growth of interest in Unmanned Aerial Vehicles (UAV's), the applications that need a fusion of dynamics and optimal path planning through a set of points are countless. The possible use of robots and UAV's in search and rescue missions, surveillance and many other applications that require optimized planning of a route make the problem we are tackling important for the near future. In addition to that, studying the TSP and DTRP for dynamic systems might also offer insight to the solution of different path planning problems for dynamic systems. Lastly, injecting the dynamics into the DTRP is a natural step in the evolution of the research on the DTRP and similar problems, where the constraints on the system that will tour the points are added to the optimal path planning problem that was historically solved without those constraints.

In this paper, we use state space models that are affine in control to model the dynamic systems. This class of systems is very general, and can be used to model a wide range of vehicles, robots and other machines. It is therefore an interesting and natural family of models to introduce into the framework of the TSP and DTRP. Systems that are affine in control have been widely studied in the literature, due to their elegance, simplicity and wide scope. Much of the research on such systems targets their reachability and steering properties; those aspects of dynamical systems are very interesting for problems that seek an optimal path through a set of points.

The rest of this paper is organized as follows: In Section 2 we introduce the notation, and define the problem rigorously. Section 3 gives some background for our study by citing some relevant results and expanding on their interpretation and usefulness. Section 4 has the main results: lower bounds on the expected customer waiting time for a DTRP with a dynamic system and algorithms that produce an expected customer waiting time that scales like the lower bounds (in terms of the traffic intensity). Section 5 has the conclusions and the future work.

II. Notation and Problem statement

In this paper, we model the dynamic systems that we are studying with state space models that are affine in control and have an output in $\mathbb{R}^2$:

$$\dot{x} = g_0(x) + \sum_{i=1}^{m} g_i(x)u_i, \tag{1}$$

$$y = h(x),$$

$$x(0) = x_0,$$

$$x \in \mathbb{R}^p,\ y \in \mathbb{R}^2,\ u_i \in \mathbb{U},$$

$$\mathbb{U} = \{u(.) : \mathbb{R} \to [-M, M]\}.$$

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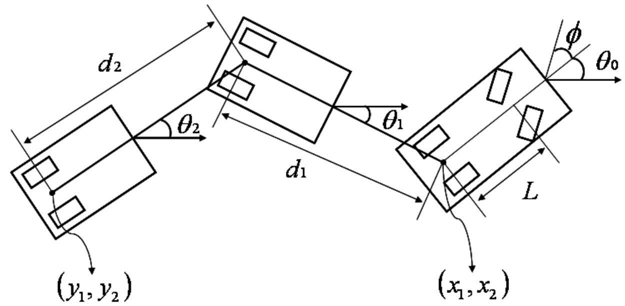

Fig. 1. Parameters for a car pulling k-trailers

We will concentrate on the DTRP policies for different dynamic systems and different traffic intensities. This statement will be made more rigorous in the following section.

We now introduce two running examples of dynamic systems that we'll use through the paper to clarify some concepts in our study.

The first is a simplified model of a car pulling k trailers (from [13]). The states in that model are the location of the first car and the angles at the axles of the trailers (Fig. 1). The output is the location of the last trailer. The state space model for the car is therefore [13]:

$$\begin{aligned}
\dot{x_1} &= \cos(\theta_0) \\
\dot{x_2} &= \sin(\theta_0) \\
\dot{\theta_0} &= \frac{u}{L} \\
\dot{\theta_1} &= \frac{1}{d_1}\sin(\theta_0-\theta_1) \\
&\vdots \\
\dot{\theta_i} &= \frac{1}{d_i}\left(\prod_{j=1}^{i-1}\cos(\theta_{j-1}-\theta_j)\right)\sin(\theta_{i-1}-\theta_i) \\
&\vdots \\
\dot{\theta_k} &= \frac{1}{d_k}\left(\prod_{j=1}^{k-1}\cos(\theta_{j-1}-\theta_j)\right)\sin(\theta_{k-1}-\theta_k) \\
y &= \begin{bmatrix} x_1-\sum_{i=1}^{k} d_i\cos(\theta_i) \\ x_2-\sum_{i=1}^{k} d_i\sin(\theta_i)\end{bmatrix} \\
\mathbb{U} &= \{u(.)=\tan(\phi(.)), \phi(.):\mathbb{R}\to[-\phi_0,\phi_0]\}.
\end{aligned} \tag{2}$$

The car is assumed to have a constant speed forward, and the control we have on the car is the steering angle ϕ (actually $\tan(\phi)$). The second model is that of a linear time-invariant system (a two-dimensional double integrator.) The state space model of that system is as follows:

$$\begin{aligned}
\dot{x} &= Ax+Bu \\
y &= Cx,
\end{aligned} \tag{3}$$

where

$$A=\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{bmatrix}, B=\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix}, \tag{4}$$

$$C=\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix}, \tag{5}$$

and

$$\mathbb{U}=\begin{bmatrix}u_1\\u_2\end{bmatrix}, u_i(.):\mathbb{R}\to[-1,1]. \tag{6}$$

It is obvious how both of these systems are special cases of our general dynamic system (1); for the car pulling k-trailers, we have:

$$g_0=\begin{bmatrix}\cos(\theta_0)\\ \sin(\theta_0)\\ 0\\ \frac{1}{d_1}\sin(\theta_0-\theta_1)\\ \vdots\\ \frac{1}{d_i}\left(\prod_{j=1}^{i-1}\cos(\theta_{j-1}-\theta_j)\right)\sin(\theta_{i-1}-\theta_i)\\ \vdots\\ \frac{1}{d_k}\left(\prod_{j=1}^{k-1}\cos(\theta_{j-1}-\theta_j)\right)\sin(\theta_{k-1}-\theta_k)\end{bmatrix},$$
$$g_1=\begin{bmatrix}0\\0\\\frac{1}{L}\\0\\\vdots\\0\end{bmatrix}. \tag{7}$$

For the linear system,

$$g_0=Ax=\begin{bmatrix}0\\0\\x_1\\x_2\end{bmatrix}, g_1=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, g_2=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}. \tag{8}$$

We will follow these examples through the paper and generate the asymptotic solutions of the DTRP for them as special cases of our main result and to illustrate some properties.

A. Problem Statement

Given a dynamic system that is modeled as in (1), a closed, and a bounded region R (assumed to be a rectangle with dimensions W_1, W_2, without loss of generality) in the output space of the system (can be thought of as the location of the system), we consider the following problem:

In the DTRP, "customer service requests" are arising according to a Poisson process with rate λ and once a request arrives, it is randomly assigned a position in R uniformly and independently. The repairperson (modeled as in (1)) is required to visit the customers and service their requests. At each customer's location, the repairperson spends a service time s which is a random variable with mean $\overline{s}$ and second moment $\overline{s^2}$. We study the expected waiting time a customer has to wait between the request time and the time of service, and we are mainly interested in how that quantity scales in terms of the traffic intensity $\lambda\overline{s}$ for low traffic ($\lambda\overline{s} \to 0$) and high traffic ($\lambda\overline{s} \to 1$). We also study the stability of the system (whether the number of waiting customers is bounded for all intensities).

B. Notation and Definitions

Next, we introduce some terminology and definitions for systems that are affine in control; most of these definitions are classical in the literature [8] [9]. We start by introducing the most basic quantity we need, the reachable set of a dynamic system.

Definition 1: Reachable set:

Given $T \geq 0$, the reachable set from state x_0 for any dynamic system is the set $R_T(x_0)$ of states x such that $\forall\, x_1 \in R_T$, $\exists\, u_1^*, u_2^*, ..., u_m^* \in \mathbb{U}$ such that:

$$x(0) = x_0,\; x(T) = x_1,$$

$$x(t) \in R_T \;\forall\, t < T.$$

This is the set of states that are reachable in exactly T. We define the set of states reachable in time less than or equal to T by:

$$R_{\leq T}(x_0) = \cup_{0 \leq t \leq T} R_T(x_0).$$

We extend the previous definition to the output space, and so we define the output-reachable set from a state x_0 to be the set $O_T(x_0)$ of points

$$y = h(x), x \in R_T(x_0),$$

and

$$O_{\leq T}(x_0) = \cup_{0 \leq t \leq T} O_T(x_0).$$

We turn to some important properties of some systems that are affine in control.

Definition 2: Vector Fields:

For all the purposes of this work, a vector field $f(x)$ is an infinitely differentiable mapping from $\mathbb{R}^p$ to $\mathbb{R}^p$.

Given a vector field f and a function $w(x) : \mathbb{R}^p \to \mathbb{R}$, we denote the derivative of w along f by :

$$\mathcal{L}_f w(x) = \sum_{i=1}^{p} \frac{\partial w(x)}{\partial x_i} f_i(x).$$

Given a vector field f and $g(x) : \mathbb{R}^p \to \mathbb{R}^q$, we call the derivative of g along f the new $\mathbb{R}^p \to \mathbb{R}^q$ function:

$$\mathcal{L}_f g = \frac{\partial g}{\partial x} f(x).$$

Note that the i^{th} component of $\mathcal{L}_f g$ is the derivative of the function g_i along f. Thus the use of similar notation should not be confusing.

A simple piece of notation that we will use is the function $x_j(x)$ which extracts the j^{th} component of x. Notice that

$$\mathcal{L}_f x_j(x) = f_j(x).$$

We now solve two sub-problems whose solution is essential to our study. One is helpful in finding a lower bounds on the expected customer waiting time (for the DTRP). The other is an important piece of our scheme for the DTRP that achieves an expected customer waiting time that scales as the lower bound in terms of $\lambda\overline{s}$.

III. Small Time Reachable Set and the Level Algorithm

In this section, we will study some results from our previous work in [12] and elaborate on them. These results are important for our study here and they offer insight into the behavior of dynamic systems that are affine in control (and thus into the behavior of linear time-invariant systems.) We start with some properties of the output small time reachable set of systems that are affine in control.

A. Small Time Reachable Sets of Dynamical Systems

We are interested in $A_{\leq T}$, the area of the output small time reachable set $O_{\leq T}$ (definition (1)). In particular, we are interested in how $A_{\leq T}$ scales in terms of T as $T \to 0$. Towards this purpose, let r_1 and r_2 be the two smallest natural numbers such that the f_1 and f_2 defined by:

$$f_1 = \mathcal{L}_{g_{i_0}} ... \mathcal{L}_{g_{i_{r_1-1}}} h(x_0) \neq 0,$$
$$f_2 = \mathcal{L}_{g_{j_0}} ... \mathcal{L}_{g_{j_{r_2-1}}} h(x_0) \neq 0,$$

are linearly independent, where

$$i_0, ..., i_{r_1-1}, j_0, ..., j_{r_2-1} \in \{0, ..., m\}.$$

For the car pulling k-trailers, let

$$P_i = \prod_{j=1}^{i} \cos(\theta_{j-1} - \theta_j),$$

then

$$\begin{aligned} f_1 &= \mathcal{L}_{g_0} h(x_0) \\ &= \begin{bmatrix} \cos(\theta_0) + \sum_{i=1}^{k} d_i \sin(\theta_i)\sin(\theta_{i-1}-\theta_i)P_i \\ \sin(\theta_0) - \sum_{i=1}^{k} d_i \cos(\theta_i)\sin(\theta_{i-1}-\theta_i)P_i \end{bmatrix}, \\ f_2 &= \mathcal{L}_{g_1}\mathcal{L}_{g_0} h(x_0) = \begin{bmatrix} -\frac{\sin(\theta_0)}{L} \\ \frac{\cos(\theta_0)}{L} \end{bmatrix}, \\ r_1 &= 1, r_2 = 2. \end{aligned} \tag{9}$$

For the linear system,

$$\begin{aligned} f_1 &= \mathcal{L}_{g_0} h(x_0) = CAx = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ f_2 &= \mathcal{L}_{g_2}\mathcal{L}_{g_0} h(x_0) = CAB_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ r_1 &= 1, r_2 = 2, \end{aligned} \tag{10}$$

for every point such that $x_1 \neq 0$.

It is obvious in our LTI example that f_1 and f_2 are first two linearly independent column vectors in

$$\left[C[Ax|B]|CAB|CA^2B|\ldots|CA^kB\right],$$

where $|$ stands for matrix concatenation.

In general, given an LTI system and the minimum k such that

$$C\left[[Ax|B]|AB|A^2B|\ldots|A^kB\right]$$

has rank 2, then the larger between r_1 and r_2 is equal to $k+1$. It is obvious that the matrix on the right is the concatenation of Ax and the controllability matrix of the LTI system. This means that the r_i's (and our index k here) are indicators of "output controllability". The inputs don't have to be able to steer the state arbitrarily, they just have to move it in directions that will affect the output. Of course, if the controllability matrix is full rank, and C is full rank, then the system is "output controllable." This means that the r_i's are upper bounded by the controllability index of the system.

Of course, the r_i's for nonlinear systems carry the same interpretation, and thus they can represent a similar index (like the k here). Thus a local version of the controllability index can be defined for nonlinear systems that are affine in control, and its relation to the r_i's is the same as in the LTI case.

Thus relating r_1 and r_2 to $A_{\leq T}$ relates k (and the controllability index) to the small time reachable area, and emphasizes its role in the output curves of the system (at least those for the TSP, DTRP and similar problems). The relation between the r_i's and $A_{\leq T}$ is given by the following Theorem [12]:

Theorem 1: Given a system described in (1) and r_1 and r_2 as above, $\exists\, C_U, C_L > 0$ such that:

$$C_L \leq \lim_{T\to 0}\frac{A_{\leq T}}{T^{r_1+r_2}} \leq C_U.$$

This theorem says that the r_i's are measures of the "difficulty" of moving in different directions in the output space. Thus the higher the r_i's are, the smaller the area of the set reachable in time t is, and so the system will need a longer time on average to travel between any two points. Thus both systems we are using as examples have a direction in the output space in which they can move proportionally to T and a direction where they can more proportionally to T^2 (note that since T is small, $T^2 <<< T$.) Therefore, the area of the set reachable in time T is proportional to T^3.

B. Level Algorithm

The Level Algorithm was introduced in [12] as an algorithm for the TSP for dynamic systems. The TSP for dynamic systems is defined as follows: Given a set P of n points that are distributed in a closed and bounded region (assumed here to be the same rectangle R defined in the DTRP problem statement) in the output space of system (1), find the fastest output curve of system (1) that passes through all of the points of P.

The Level algorithm is a simple iterative scanning of the the area where the points are distributed such that every iteration is done at a different "resolution" of the space. In every iteration (em level), R is divided into a number of rectangles that are sized depending on the dynamics of system (1). The level Algorithm produces a tour LA such that the expected time system (1) needs to trace LA, (T_{LA}), scales as the optimal in terms of n if the points in P are chose randomly and uniformly. This is given by the following theorem [12][1]:

Theorem 2: $T_{LA} = C_{LA}n^{1-\frac{1}{r_1+r_2}} + o(n^{1-\frac{1}{r_1+r_2}})$.

We will use the Level algorithm as a black box in our scheme for the DTRP of dynamic systems under high traffic intensity conditions. The Level Algorithm's performance guarantee will give the performance guarantee that we want from our DTRP scheme.

We are now ready to produce our main results. We will study the DTRP for the cases of low traffic intensity ($\lambda\overline{s}\to 0$) and high traffic intensity. In both cases, we will produce a lower bound on the expected waiting time of a customer and a scheme that will guarantee an expected customer waiting time that scales as the lower bound in terms of $\lambda\overline{s}$.

IV. Main Results

Here we study the minimum customer waiting time for the DTRP T_{DTRP}, and whether there are schemes that will guarantee that the number of waiting customers is always bounded (as long as the $\lambda\overline{s}$. More specifically, we will study how T_{DTRP} scales in terms of the traffic intensity.

For simplicity of presentation, we add a few assumptions here; we assume that r_1 and r_2 are constant over R, and that that f_1 is parallel to H and that $\exists\, T_T > 0$ such that between any two points y_1 and y_2 in R, the system can be steered from y_1 to y_2 in time less than T_T (T_T doesn't have to be small.)

A. Low Traffic Intensity

We will start with the results for low traffic intensity. This means that $\lambda\overline{s}\to 0$ and so almost all of the time can be used to move the system's output from one customer to another. Let x^* be a "time median" of R under the system's dynamical constraints (doesn't have to be unique). So x^* is the point in R that minimizes

$$E[T_v(x, x^*)],$$

where $T_v(x^*, x)$ is the time that the system needs to travel from x^* to x. Note that T_v doesn't have to be small. Let

$$T_1 = E[T_v(x^*, x)],$$

and

$$T_2 = E[T_v^2(x^*, x)].$$

We have the following theorem:

[1]We say a function $f(n)$ is $O(g(n))$ if $\exists c, N > 0$ such that $f(n) \leq cg(n)\ \forall\, n > N$, we say $f(n)$ is $\Omega(g(n))$ if $g(n)$ is $O(f(n))$ and we say $f(n)$ is $\Theta(g(n))$ $f(n)$ is $O(g(n))$ and $\Omega(g(n))$. We say $f(l)$ is $o(g(l))$ if $\lim_{l\to 0}\frac{f(l)}{g(l)} = 0$ (for functions) or $\lim_{n\to\infty}\frac{f(n)}{g(n)} = 0$ (for sequences).

Theorem 3: The expected customer waiting time in the DTRP (T_{DTRP})for a dynamic system is equal to $E[T_v(x, x^*)] + \bar{s}$ as $\lambda \to 0$.

Proving that $T_{DTRP} \geq E[T_v(x, x^*)] + \bar{s}$ is direct. When a customer service request arises, the system's output has to at least move from where it is already to the location of the new customer and service it. Thus the expected time a customer has to wait is at least $E[T_v(x, x_{sys})] + \bar{s}$, where x_{sys} is a random variable determining the location of the system's output when the customer service request arrived. From the definition of x^*, $E[T_v(x, x_{sys})] \geq E[T_v(x, x^*)]$, and thus

$$T_{DTRP} \geq E[T_v(x^*, x)] + \bar{s} = T_1 + \bar{s} = T^*.$$

To get the matching upper bound, the following policy can be followed:
Service customers in a First Come First Serve fashion, while waiting at x^* when there are no customers.

Lemma 1: The expected time of the previous policy T_{FCFS} satisfies the following relation:

$$\frac{T_{FCFS}}{T^*} \to 1 \; as \; \lambda \to 0.$$

Proof: The proof is similar to the one in [5].
We therefore have a single-server SQM system behaving like an $M/G/1$ queue with first moment $2T_1 + \bar{s}$ and second moment $4T_2 + 4\bar{s}T_1 + \overline{s^2}$. Thus the expected customer waiting time can be bounded by:

$$T_F CFS = \frac{\lambda(4T_2 + 4\bar{s}T_1 + \overline{s^2})}{2(1 - 2\lambda T_1 - \lambda\bar{s})} + T_1 + \bar{s}.$$

Therefore:

$$\lim_{\lambda \to 0} \frac{T_F CFS}{T^*} = \lim_{\lambda \to 0} \frac{\frac{\lambda(4T_2 + 4\bar{s}T_1 + \overline{s^2})}{2(1 - 2\lambda T_1 - \lambda\bar{s})} + T_1 + \bar{s}}{T_1 + \bar{s}} = 1. \quad (11)$$

■

Theorem (3) follows directly.

B. High Traffic Intensity

We now turn to the case where the traffic intensity is high. Heavy traffic intensity is when $\lambda\bar{s} \to 1$. This means that there is little time for travel ($1 - \lambda\bar{s}$ per customer on average.) Thus the system will need to follow a more complicated scheme to allow the number of waiting customers to be bounded.

We first produce a lower bound on the expected customer waiting time. This result will depend on the area of the small time reachable set discussed in Section III-A, and is given by the following lemma:

Lemma 2: T_{DTRP} is $\Omega\left((1 - \lambda\bar{s})^{-(r_1+r_2)}\right)$.

Proof:
The lower bound proof is three steps:

1) *Bound the Expected time travelled per customer:*
Let n be the number of customers waiting in R to be serviced and given that the output of system (1) is at a certain point, let the minimum time needed to travel to a customer be t^*. Then

$$\mathbf{E}[t^*] \geq \int_0^\infty \mathbf{P}[t^* > \tau] d\tau \geq \int_0^\infty \max\{0, 1 - c\tau^{r_1+r_2}\} d\tau,$$

where $c = n\frac{C_U}{W_1 W_2}$, and C_U is from Theorem 1.

$$\begin{aligned}\mathbf{E}[t^*] &\geq \int_0^{c^{-\frac{1}{r_1+r_2}}} 1 - c\tau^{r_1+r_2} d\tau \\ &= c^{-\frac{1}{r_1+r_2}} - c\frac{1}{1 + r_1 + r_2} c^{-\frac{1+r_1+r_2}{r_1+r_2}} \\ &= \frac{r_1 + r_2}{1 + r_1 + r_2}\left(\frac{C_U}{W_1 W_2}\right)^{-\frac{1}{r_1+r_2}} n^{-\frac{1}{r_1+r_2}} \\ &= c_2 n^{-\frac{1}{r_1+r_2}}.\end{aligned}$$

2) *Upper bound the rate of arrival:*
Recall that $\bar{s}$ is the average service time per customer needed, W is the average waiting time, and $T = W + \bar{s}$ is the system waiting time.
The stability condition is that the average time spent travelling on the road plus the average service time is not greater than the average time for a customer to arrive:

$$\lambda(\bar{s} + \mathbf{E}[t^*]) \leq 1,$$

where λ is the rate of arrival.
Therefore,

$$\bar{s} + c_2 n^{-\frac{1}{r_1+r_2}} \leq \frac{1}{\lambda}.$$

3) *Lower Bound the Customer Waiting Time:*
From the previous bound, and using Little's formula: $n = \lambda W$ and $T^* = \bar{s} + W$ is the minimum system waiting time, we get:

$$n \geq \frac{(\lambda c_2)^{r_1+r_2}}{(1 - \lambda\bar{s})^{r_1+r_2}},$$

and

$$T_{DTRP} \geq \bar{s} + \frac{\lambda^{r_1+r_2-1} c_2^{r_1+r_2}}{(1 - \lambda\bar{s})^{r_1+r_2}}.$$

Therefore as $\lambda\bar{s} \to 1$, T_{DTRP} is $\Omega(1 - \lambda\bar{s})^{-(r_1+r_2)}$.

■

To achieve the upper bound corresponding to the high traffic intensity lower bound, we use the TSP policy. Under this policy, the system waits until there are n customers, and then services them with a TSP tour. This means that it first waits for customers number 1 to n, service them using a TSP tour, then waits till the $2n^{th}$ customer arrives, and services customers $n+1$ to $2n$,... Denote the k^{th} set of n customers by $\mathbb{S}_k$ and the system waiting time under this policy by T_{TSP}, we have the following theorem:

Theorem 4: As $1-\lambda\bar{s} \to 0$, $\frac{T_{TSP}}{T_{DTRP}} \leq c_3$.

Proof: We now consider $\mathbb{S}_k$ to be the k^{th} customer in a queue. Since the interarrival and service times are i.i.d, we have a $GI/G/1$ queue with an Erlang distribution of order n. The mean of the sets is $\frac{n}{\lambda}$ and the variance is $\frac{n}{\lambda^2}$.

Therefore, the expected value of the service time of a set is $E[L_{LA}(n)]+n\bar{s}$ and the variance is $var(L_{LA}(n))+n\sigma^2$.

Therefore, we can bound the average waiting time of the sets by:

$$W_S \leq \frac{\frac{\lambda}{n}(\frac{n}{\lambda^2}+var(L_{LA}(n))+n\sigma^2)}{2(1-\frac{\lambda}{n}(E[L_{LA}(n)]+n\bar{s}))}$$

$$= \frac{\lambda(\frac{1}{\lambda^2}+\sigma^2)}{2(1-\lambda(\bar{(s)}+C_{LA}n^{-\frac{1}{r_1+r_2}}))},$$

where C_{LA} is from theorem 2. For stability, we have:

$$1-\lambda(\bar{s}+C_{LA}n^{-\frac{1}{r_1+r_2}})>0.$$

Therefore

$$\frac{(1-\lambda\bar{s})^{r_1+r_2}}{C_{LA}^{r_1+r_2}} > \frac{1}{n},$$

and

$$n > \frac{(\lambda C_{LA})^{r_1+r_2}}{(1-\lambda\bar{s})^{r_1+r_2}}.$$

This means that for high traffic ($1-\lambda\bar{s}\to 0$), n has to be large for the system to be stable. Our assumption for the Level Algorithm performance guarantee (that n is large) is thus satisfied.

The expected waiting time for a certain customer is the sum of the expected time it waits for its set to form, the waiting time for the set to get serviced, and the expected time it needs to wait to get serviced after the service of its set started.

Therefore,

$$T_{TSP} \leq \frac{\lambda(\frac{1}{\lambda^2}+\sigma^2)}{2(1-\lambda(\bar{s}+C_{LA}n^{-\frac{1}{r_1+r_2}}))} + n\frac{1+\lambda\bar{s}}{2\lambda}+C_{LA}n^{1-\frac{1}{r_1+r_2}}. \quad (12)$$

It can be shown that as $1-\lambda\bar{s}\to 0$, the optimal n approaches $\frac{(\lambda C_{LA})^{r_1+r_2}}{(1-\lambda\bar{s})^{r_1+r_2}}$ (which is the stability bound).

Substituting the optimal value of n in 12 gives:

$$T_{TSP} \leq \frac{\lambda^{r_1+r_2-1}C_{LA}^{r_1+r_2}}{(1-\lambda\bar{s})^{r_1+r_2}},$$

and using this with lemma (2) gives the result:

$$\frac{T_{TSP}}{T_{DTRP}} \leq c_3.$$

and thus proves that T_{DTRP} is $\Theta\left((1-\lambda\bar{s})^{r_1+r_2}\right)$. ■

Thus the DTRP is stabilizable for any dynamic system that has basic reachability properties, that is, the expected waiting time for a customer can be guaranteed to be bounded as long as $\lambda\bar{s}<1$. For the examples we are using, the average customer waiting time scales as $(1-\lambda\bar{s})^3$, which is worse than the Euclidean case. This deterioration in behavior is due to the fact that there is a direction in which motions of the systems' output is slow (the direction with $r_i = 2$.)

V. Conclusions and Future Work

In this paper, we tackled the Dynamic Traveling repairperson problem for dynamic systems that are affine in control. We studied the area of the small time reachable set and related it to notions of controllability and observability. We proved that under some weak assumptions on the dynamic system, the number of customers waiting to be serviced will always be bounded for any traffic intensity satisfying the necessary condition $\lambda\bar{s}<1$. We finally studied the expected waiting time for a customer and provided a scheme that allows it to scale optimally in terms of $\lambda\bar{s}$.

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