Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Static Output Stabilisation of Singular LPV Systems: $\mathcal{LMI}$ Formulation

M. Chadli, M. Darouach and J. Daafouz

Abstract— **This paper addresses the design of static output feedback controller for singular discrete-time LPV systems. Sufficient synthesis conditions are derived in Linear Matrix Inequalities ($\mathcal{LMI}$) formulation. To introduce more of relaxation, polyquadratic Lyapunov functions is proposed instead of quadratic method, the number of $\mathcal{LMI}$ conditions is reduced and extra degrees of freedom are included. Academic example is proposed to illustrate the effectiveness of the derived results.**

Keyword: **Singular systems, LPV systems, static output control, Lyapunov method, $\mathcal{LMI}$.**

I. INTRODUCTION

These last years, many efforts have been carried out for singular systems [1]. Many works dealing with the control of impulsive and switching singular systems and singular LPV systems have been studied recently for the theoretical and practical point of view (see for example [2], [3], [4], [6], [10], [9]). For the switching systems the control techniques based on switching between different controllers have been applied extensively in recent years, due to their advantages in achieving stability [15], [7], [8], [10], [13] whereas gain-scheduling controllers techniques are used for LPV systems [12], [14]. For singular LPV systems, stability and stabilisation are studied using gain-scheduling controllers techniques [19] and recently using multiple state feedback controllers for the class of polytopic LPV [20]. However, if there exist a lot of works on singular linear systems (see for exampe [16], [17] and references therein), to our knowledge there are few studies on singular LPV systems and their corresponding control problems.

In this paper we consider the stabilisation of the class of polytopic singular LPV systems using static output feedback controller. While a single control is often used, this paper proposes to design multiple static output feedback controller obtained by interpolation of linear static output feedback control law. In fact, such controller introduces more of relaxation since a single/common gain fails to solve many control problems.

This paper studies the design of static output controllers for singular LPV systems in polytopic form. $\mathcal{LMI}$ formulation and polyquadratic Lyapunov approach are used to design static output controller obtained by interpolation of multiple linear static output gain. For more of relaxation, a transformation technique of the system is proposed to reduce the number of $\mathcal{LMI}$ constraints and extra degree of freedom is introduced and exploited.

The paper is organized as follows. In section 2, the considered class of a discrete-time singular LPV systems is described. In section 3, a static static output controller is introduced and the stabilisation of the considered class of singular LPV systems is studied. The main result is proposed in section 3-B where relaxations are introduced to design multiple controller gains in $\mathcal{LMI}$ formulation. Example is given in section 4.

Notation. The following notations are used. R^n and $R^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes matrix transposition, the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite) and the symbol $(*)$ denotes the transpose elements in the symmetric positions. I is the identity matrices with compatible dimensions and $I_N = \{1, 2, \cdots, N\}$.

II. PROBLEM STATEMENT

The considered singular LPV system is as follows

$$\begin{array}{l} Ex(t+1) = A(\rho(t))x(t) + B(\rho(t))u(t) \\ y(t) = Cx(t) \end{array} \tag{1}$$

Two important classes of LPV systems can be distinguished; the affine LPV where the state space matrices depend affinely on $\rho(t)$ and the polytopic LPV where the parameter $\rho(t)$ varies in polytope of vertices ρ_i such that $\rho(t) \in Co\{\rho_1, \rho_2, ..., \rho_r\}$ [12]. In the sequel only the second class is used in the following form

$$\begin{array}{l} Ex(t+1) = \sum_{i=1}^{N} \xi_i(\rho(t))(A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{array} \tag{2}$$

where

$$\xi_i(\rho(t)) \geq 0, \sum_{i=1}^{N} \xi_i(\rho(t)) = 1 \tag{3}$$

with $x(t) \in R^n$ is the state vector, $y(t) \in R^p$ is the output vector, $u(t) \in R^m$ is the input vector, $A_i \in R^{n\times n}$, $B_i \in R^{n\times m}$ and $C \in R^{p\times n}$. The matrix E may be singular with $0 \leq \text{rank}(E) = n_E < n$.

M. Chadli is with MIS, University of Picardie Jules Verne, 7, Rue du Moulin Neuf, 80000, Amiens, France.Tel.: +33(0)3 82227680, mohammed.chadli@u-picardie.fr

M. Darouach is with CRAN-CNRS (UMR 7039), UHP, Nancy I IUT de Longwy, 186 Rue de Lorraine, 54400 Cosnes et Romain, France. Mohamed.darouach@iut-Longwy.uhp-Nancy.fr

J. Daafouz is with CRAN-CNRS (UMR 7039), INPL, 2 Avenue de la foret de Haye, 54516 Vandoeuvre-Les-Nancy Cedex, France. jamal.daafouz@ensem.inpl-nancy.fr

Before studding the static output feedback stabilisation, let us now recall some basic stability conditions for the unforced singular system (2). Consider the Lyapunov dependant parameter function of the form [20]:

$$V(x(t),\rho(t)) = x(t)^\top \mathscr{P}(\rho(t))x(t) \tag{4}$$

with

$$\mathscr{P}(\rho(t)) = \sum_{i=1}^{N} \xi_i(\rho(t))E^\top P_i E, E^\top P_i E \geq 0, i \in I_N \tag{5}$$

The difference of (4) along the solution of the unforced system of (2) is given by

$$\begin{aligned} \Delta V &= V(x(t+1),\rho(t+1)) - V(x(t),\rho(t)) \\ &= x(t+1)^\top \mathscr{P}(\rho(t+1))x(t+1) - x(t)^\top \mathscr{P}(\rho(t))x(t) \end{aligned} \tag{6}$$

Thus, the unforced singular system of (2) is stable if there exist nonsingular symmetric matrices P_i such that the following hold for all $(i,j) \in I_N^2$:

$$E^\top P_i E \geq 0 \tag{7}$$
$$A_i^\top P_j A_i - E^\top P_i E < 0 \tag{8}$$

To design control laws, introduction of extra degrees of freedom is largely used in literature (see for example [18], [15] and references therein). Theses techniques are very helpful for the development of an $\mathcal{LMI}$-based conditions for controllers design and particulary for static output feedback controller. Thus the unforced singular system of (2) is stable if there exist nonsingular symmetric matrices P_i, matrices F_i and G_i such that the following $\mathcal{LMI}$ hold for all $(i,j) \in I_N^2$ [20]:

$$EP_iE^\top \geq 0 \tag{9}$$
$$\begin{pmatrix} -EP_iE^\top + A_iF_i + (A_iF_i)^\top & -F_i^\top + A_iG_i \\ (*) & P_j - (G_i + G_i^\top) \end{pmatrix} < 0 \tag{10}$$

The stability conditions (10) and (8) are equivalents. However, this last formulation is very helpful for the control problem design.

In this paper our objective is to synthesis static output controller for discrete-time singular systems. Sufficient conditions for controller design will be derived based on the Lyapunov theory and $\mathcal{LMI}$ formulation.

III. Stabilisation of discrete-time LPV singular systems

A. Static output feedback controller design

Consider the following control law obtained by interpolation of linear static output controller:

$$u(t) = \sum_{i=1}^{N} \xi_i(\rho(t))K_i y(t) \tag{11}$$

where $K_i \in R^{m\times p}$. The closed loop system is

$$Ex(t+1) = \sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i(\rho(t))\xi_j(\rho(t))\mathbf{A}_{ij}x(t) \tag{12}$$

with

$$\mathbf{A}_{ij} = A_i + B_iK_jC \tag{13}$$

In the sequel, we assume

Assumption 1: The matrix C is full row rank.

To derive stability conditions of (12) it is possible to substitute A_i by $\mathbf{A}_{ij} = A_i + B_iK_jC$ in conditions (10). However the obtained conditions are Bilinear Matrix Inequalities in K_i, F_i and G_i. $\mathcal{LMI}$ conditions can be obtained by using the transformation used in [21] and choosing $F_i = G_i = G$.

Theorem 1: The singular system (12) is stable if there exist nonsingular matrices G, P_i, M and N_j such that the following $\mathcal{LMI}$ hold for all $(i,j,k) \in I_N^3$:

$$EP_iE^\top \geq 0 \tag{14}$$
$$\begin{pmatrix} -EP_iE^\top + \Phi_{ij} + \Phi_{ij}^\top & -G^\top + \Phi_{ij} \\ (*) & P_k - (G + G^\top) \end{pmatrix} < 0 \tag{15}$$
$$CG = MC \tag{16}$$

with

$$\Phi_{ij} = A_iG + B_iN_jC \tag{17}$$

The controller gains are defined by:

$$K_i = N_iM^{-1} \tag{18}$$

Proof: Multiplying (15) by $\sum_{i=1}^N \sum_{j=1}^N \xi_i(\rho(t))\xi_j(\rho(t))$ and according to (18) and (16), we obtain

$$\sum_{i=1}^N \sum_{j=1}^N \xi_i(\rho(t))\xi_j(\rho(t)). \begin{pmatrix} -EP_iE^\top + \mathbf{A}_{ij}G + (\mathbf{A}_{ij}G)^\top & -G^\top + \mathbf{A}_{ij}G \\ (*) & P_k - (G + G^\top) \end{pmatrix} < 0 \tag{19}$$

Which is only stability conditions (10) by substituting A_i by $A_i + B_iK_jC$ and $F_i = G_i$ by G.

In the following subsection, we introduce multiple matrices G_i instead of single matrix G which leads to more relaxation.

B. Main result

To give more relaxation, we propose to derive $\mathcal{LMI}$ conditions, to introduce different matrices G_i and to reduce the number of $\mathcal{LMI}$ constraints. For this, we propose to write the model (2) and the controller (11) as follows

$$\mathbf{E}\mathbf{x}(t+1) = \sum_{i=1}^{N} \xi_i(\rho(t))\mathbf{A}_i\mathbf{x}(t) \quad (20)$$

Where

$$\mathbf{x}(t) = \left(x(t)^\top, u(t)^\top\right), \mathbf{A}_i = \begin{pmatrix} A_i & B_i \\ K_iC & -I \end{pmatrix} \quad (21)$$

$$\mathbf{E} = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \quad (22)$$

Thus, to design static output controller it suffices to substitute A_i by $\mathbf{A}_i$ in conditions (10). However the obtained conditions are BMI in K_i, F_i and G_i even if we choose $F_i = G_i$.

Without lost of generality, we will further assume that C has a specific structure $C = (I, 0)$ where $I \in R^{p\times p}$ is the identity matrix and $0 \in R^{p\times(n-p)}$ is the null matrix. Note that this is not a restricting condition. Indeed, if C is a full rank matrix (assumption 1) then there always exist a transformation matrix that transforms C into the above form.

In the following we propose $\mathcal{LMI}$ conditions obtained by considering the output matrix structure $C = (I, 0)$ and also specific structure of matrices $F_i = G_i$.

Theorem 2: The singular system (20) is stable if there exist nonsingular symmetric matrices P_{1i}, P_{3i} and matrices P_{2i}, G_{2i}, G_{3i} and G_{1i} of the form

$$G_{1i} = \begin{pmatrix} G_{11i} & 0 \\ G_{12i} & G_{13i} \end{pmatrix} \quad (23)$$

such that the following $\mathcal{LMI}$ hold for all $(i,j) \in I_N^2$.

$$EP_{1i}E^\top \geq 0 \quad (24)$$

$$\begin{pmatrix} \Sigma_i & \begin{pmatrix} B_iG_{3i}+ \\ \bar{N}_i^\top - \\ G_{2i}^\top \end{pmatrix} & \begin{pmatrix} -G_{1i}^\top \\ +A_iG_{1i} \\ +B_iG_{2i} \end{pmatrix} & \begin{pmatrix} -G_{2i}^\top \\ +B_iG_{3i} \end{pmatrix} \\ (*) & \begin{pmatrix} -G_{3i} \\ -G_{3i}^\top \end{pmatrix} & \begin{pmatrix} \bar{N}_i - \\ G_{2i} \end{pmatrix} & \begin{pmatrix} -G_{3i}^\top \\ -G_{3i} \end{pmatrix} \\ (*) & (*) & \begin{pmatrix} P_{1j}- \\ G_{1i} \\ -G_{1i}^\top \end{pmatrix} & \begin{pmatrix} P_{2j}- \\ G_{2i}^\top \end{pmatrix} \\ (*) & (*) & (*) & \begin{pmatrix} P_{3j}- \\ G_{3i} \\ -G_{3i}^\top \end{pmatrix} \end{pmatrix} < 0 \quad (25)$$

with

$$\Sigma_i = -EP_{1i}E^\top + A_iG_{1i} + B_iG_{2i} + (A_iG_{1i} + B_iG_{2i})^\top \quad (26)$$

$$\bar{N}_i = (N_i, 0) \quad (27)$$

The controller gains are defined by:

$$K_i = N_iG_{11i}^{-1} \quad (28)$$

Proof: Substituting A_i by $\mathbf{A}_i$ in conditions (10) with $F_i = G_i$ we get

$$\mathbf{E}P_i\mathbf{E}^\top \geq 0 \quad (29)$$

$$\begin{pmatrix} -\mathbf{E}P_i\mathbf{E}^\top + \mathbf{A}_iG_i + (\mathbf{A}_iG_i)^\top & -G_i^\top + \mathbf{A}_iG_i \\ (*) & P_j - (G_i + G_i^\top) \end{pmatrix} < 0 \quad (30)$$

The obtained conditions (30) are BMI in K_i and G_i. To derive $\mathcal{LMI}$ conditions we propose to chose matrix G_i with the following structure

$$G_i = \begin{pmatrix} G_{1i} & 0 \\ G_{2i} & G_{3i} \end{pmatrix} \quad (31)$$

and G_{1i} has the structure (23). Thus

$$\mathbf{A}_iG_i = \begin{pmatrix} A_iG_{1i} + B_iG_{2i} & B_iG_{3i} \\ K_iCG_{1i} - G_{2i} & -G_{3i} \end{pmatrix} \quad (32)$$

Tacking account the structure of the matrices G_{1i} (23) and C and condition (28), we get

$$\mathbf{A}_iG_i = \begin{pmatrix} A_iG_{1i} + B_iG_{2i} & B_iG_{3i} \\ \bar{N}_i - G_{2i} & -G_{3i} \end{pmatrix} \quad (33)$$

Writing the symmetric matrices P_i as follows

$$P_i = \begin{pmatrix} P_{1i} & P_{2i} \\ P_{2i}^\top & P_{3i} \end{pmatrix} \quad (34)$$

and according to (33) we obtain (25) from (30). Conditions (24) are obtained directly from (29).

The result of theorem 2 leads to more relaxation compared with the result of the theorem 1 by reducing the number of constraint to $(N^2 + N)$ $\mathcal{LMI}$ instead of $(N^3 + N)$ and by introducing different matrices G_i.

IV. Numerical Example

Consider a numerical example with the following data:

$$A_1 = \begin{pmatrix} 2.018 & 1.326 & -0.08691 & -0.1644 \\ 2.277 & 2.018 & -0.1359 & -0.2908 \\ 0.4259 & 0.1666 & 0.5714 & 0.1732 \\ 0.6659 & 0.4259 & -0.4822 & -0.152 \end{pmatrix} \quad (35)$$

$$A_2 = \begin{pmatrix} 1 & 1 & -0.06988 & -0.1351 \\ 0 & 1 & -0.07577 & -0.1835 \\ 0 & 0 & 0.5807 & 0.1894 \\ 0 & 0 & -0.4546 & -0.1013 \end{pmatrix} \quad (36)$$

$$B_1 = \begin{pmatrix} 0.6047 \\ 1.644 \\ -1.709 \\ -1.732 \end{pmatrix}, B_2 = \begin{pmatrix} 0.5422 \\ 1.351 \\ -1.747 \\ -1.894 \end{pmatrix} \quad (37)$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (38)$$

The singular matrix, E, are given by:

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (39)$$

Solving $\mathcal{LMI}$ (24)-(25), we get feasible problem:

$$G_{11} = \begin{pmatrix} 0.0058 & -0.0042 & 0 & 0 \\ -0.0043 & 0.0056 & 0 & 0 \\ -0.0027 & -0.0018 & 0.0786 & 0.0020 \\ 0.0012 & -0.0026 & 0.0490 & 0.1963 \end{pmatrix} \quad (40)$$

$$G_{12} = \begin{pmatrix} 0.0071 & -0.0019 & 0 & 0 \\ -0.0029 & 0.0053 & 0 & 0 \\ 0.0027 & -0.0028 & 0.0790 & 0.0030 \\ 0.0188 & -0.0046 & 0.0470 & 0.1924 \end{pmatrix} \quad (41)$$

which give the controller gains

$$K_1 = \begin{pmatrix} -2.3661 & -2.4425 \end{pmatrix} \quad (42)$$
$$K_2 = \begin{pmatrix} -0.3311 & -0.8934 \end{pmatrix} \quad (43)$$

We note that the synthesis conditions of theorem 1 fail to prove the stabilisation of the given example.

V. CONCLUSION

This paper deals with the output stabilisation of class of singular LPV systems. Thus, static output stabilisation controller is studied for polytopoic singular LPV systems. Using polyquadratic Lyapunov functions, sufficient conditions to design multiple static output controllers are developed in $\mathcal{LMI}$ terms. For more of relaxation, a technique to reduce the number of $\mathcal{LMI}$ constraints is proposed and different extra degrees of freedom are introduced. Numerical example is given to illustrate the benefit of derived results.

REFERENCES

[1] L. Dai. Singular Control Systems, Springer, New York, 1989.
[2] J.K. Mills, A.A. Goldenberg. Force and position control of manipulators during constrained motion tasks. IEEE Trans. Robot. Automat. 38, pp 30 - 46, 1989.
[3] R.W. Newcomb, B. Dziurla. Some circuits and systems applications of semistate theory Circuits Systems Signal Process 8, pp. 235 - 260, 1989.
[4] B.L. Stevens, F.L. Lewis. Aircraft Modeling, Dynamics and Control, Wiley, New York, 1991.
[5] S. Boyd, El Ghaoui, L., Feron, E., and V. Balakrishnan. 'Linear matrix inequalities in systems and control theory'. SIAM, Philadelphia, PA, 1994.
[6] A.K. Gelig, A.N. Churilov. Stability and Oscillations of Nonlinear Pulse-Modulated Systems, Birkhuser, Boston, 1998.
[7] S.S. Ge, Z. Sun, T.H. Lee. On reachability and stabilization of switched linear discrete-time systems, IEEE Trans. Automat. Control 46 (9), pp. 1437 - 1441, 2001.
[8] J.L. Mancilla-Aguilar. A condition for the stability of switched nonlinear systems. IEEE Trans. Automat. Control 45 (11), pp. 2077 - 2079, 2003.
[9] I. Masubuchi, J. Kato., M. Saeki. and A. Ohara. Gain-Scheduled Controller Design Based on Descriptor Representation of LPV Systems: Application to Flight Vehicle Control. 43rd IEEE Conference on Decision and Control, Atlantis, Bahamas, Dec. 2004.
[10] J. Yao, Z-H. Guan, G. Chenb, D. Hoc. Stability, robust stabilization and Hinf control of singular-impulsive systems via switching control. Systems and Control Letters 55(11). 2006.
[11] J. Daafouz and J. Bernussou. Parameter dependent lyapunov functions for discrete time systems with time varying parametric uncertainties. Systems and Control Letters, vol. 43, pp. 355 - 359, 2001.
[12] P. Apkarian, P. Gahinet, A convex Characterization of Parameter-Dependent H_∞ controllers. IEEE Transactions on Automatic Control 40(5), 1995.
[13] D. Liberzon, Switching in Systems and Control, Birkhuser, Boston, 2003.
[14] P. Apkarian, P. Pellanda, H.D. Tuan, Mixed H_2/H_∞ multi-channel linear parameter-varying control in discrete time. Systems and Control Letters, vol. 41, pp. 333-346, 2000.
[15] J. Daafouz, Riedinger, P.; C. Iung. Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach. IEEE Trans. Automat. Control 47(11), pp. 2093 - 2094, 2002.
[16] M. Darouach, M. Boutayeb. Design of observers for descriptor systems. IEEE Transactions on Automatic Control 40(7), pp. 1323 - 1327, 1995.
[17] M. Darouach. Solution to Sylvester equation associated to linear descriptor systems. Systems and Control Letters, 55 (10), pp. 835 - 838, 2006.
[18] D. Peaucelle, D. Arzelier, O. Bachelier and J. Bernussou. A new robust stability condition for real convex polytopic uncertainty. Systems and Control Letters 40(1), 2000.
[19] I. Masubuchi, T. Akiyama, M. Saeki. Synthesis of Output Feedback Gain-Scheduling Controllers Based on Descriptor LPV System Representation. Conference On Decision and Control Maui, Hawaii USA, Dec. 2003.
[20] M. Chadli, J. Daafouz, M. Darouach. Stabilisation of Singular LPV Systems. 17th IFAC World congress, Seoul, Korea, July, 6-11, 2008.
[21] J. Daafouz, P. Riedinger, C. Iung. Static output feedback control for switched systems. 40th IEEE Conference on Decision and Contro Vol. 3 pp. 2093 - 2094,4-7 Dec. 2001.