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Output Regulation of Linear Systems Subject to Input Constraints*

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Abstract— **This paper studies the output regulation problem for linear systems subject to input constraints. It presents a new controller that uses a conditional servocompensator. Previous work introduced the conditional servocompensator, which acts as a traditional servocompensator only in a neighborhood of the zero-error manifold, while acting as a stable system outside a boundary layer, leading to improvement in the transient response while achieving zero steady-state regulation error. Starting with a low-gain stabilizing state feedback controller, based on the solution of an algebraic Riccati equation, Lyapunov redesign is used to implement the state feedback control in a saturated high-gain feedback form that includes the conditional servocompensator. The design is extended to output feedback via a full-order high-gain observer.**

I. Introduction

In this paper the problem of output regulation for linear systems subject to input saturation is considered. The servomechanism problem deals with the design of a controller to make the output of a plant asymptotically track reference signals and reject disturbance signals, both produced by an autonomous external system called the *exosystem*. Stabilization of linear systems under input constraints has been extensively studied over the past decade, c.f., [4], [5], [9], [11], [13]. When the open-loop eigenvalues are in the closed left-half plane, global or semi-global stabilization can be achieved by low-gain feedback or by a combination of low-gain and high-gain feedback. The techniques are extended to the servomechanism problem in [8]. Although more recent results [6] have also dealt with cases with right-half plane eigenvalues; we consider here the case of left-half-plane eigenvalues. The presence of saturation in the input channel imposes strong limitations to the achievable control objectives such as transient performance. In order to achieve desired control objectives we cast the output regulation problem for linear systems subject to input constraints in the Lyapunov redesign framework as presented in [10]. The key feature of this idea is that the conditional servocompensator acts as a traditional servocompensator only in a neighborhood of the zero-error manifold, while it is a bounded-input-bounded-state system whose state is guaranteed to be of the order of a small design parameter. The use of conditional servocompensators enables us to achieve zero steady-state tracking error without degrading the transient response of the system. The goal of this work is to apply the Lyapunov-redesign-servocompensator approach of [10] to the linear regulation problem under input constraints and compare its performance with the approach presented in [8].

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II. Output Regulation Using Conditional Servocompensators

In this section we briefly review the Lyapunov redesign approach to output regulation problem using conditional servocompensators [10]. Consider the SISO nonlinear system

$$\begin{aligned} \dot{\xi} &= \tilde{f}(\xi, w) + \tilde{g}(\xi, w)u \\ e &= \tilde{h}(\xi, w) \end{aligned} \tag{1}$$

where $\xi \in R^n$ is the state, u is the control input, e is the regulation error and the functions $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ are sufficiently smooth. The plant is subjected to a vector of *exogenous* input variables, which are generated by the known exosystem

$$\dot{w} = S_0 w \tag{2}$$

where S_0 has distinct eigenvalues on the imaginary axis and $w(t)$ belongs to a compact set $\mathcal{W}$. Suppose that for all $w \in \mathcal{W}$, there exist a continuously differentiable mapping $\xi = \pi(w)$, with $\pi(0) = 0$, and a continuous mapping $\chi(w)$, generated by the internal model

$$\frac{\partial \tau(w)}{\partial w} S_0 w = S\tau(w), \qquad \chi(w) = \Gamma \tau(w)$$

where S has distinct eigenvalues on the imaginary axis, such that

$$\begin{aligned} \frac{\partial \pi(w)}{\partial w} S_0 w &= \tilde{f}(\pi, w) + \tilde{g}(\pi, w)\chi(w) \\ 0 &= h(\pi, w) \end{aligned} \tag{3}$$

With the change of variables $x = \xi - \pi$, the system (1) can be represented by

$$\dot{x} = f(x, w) + g(x, w)[u - \chi(w)] \tag{4}$$

The system (4) is in the form where the state feedback regulation problem can be formulated as a state feedback stabilization problem by treating $\chi(w)$ as a matched uncertainty. Suppose there is a locally Lipschitz function $\psi(x, w)$, with $\psi(0, w) = 0$, and a continuously differentiable Lyapunov function $V(x, w)$, possibly unknown, such that

$$\alpha_1(\|x\|) \le V(x, w) \le \alpha_2(\|x\|) \tag{5}$$

$$\frac{\partial V}{\partial w} S_0 w + \frac{\partial V}{\partial x}[f(x, w) + g(x, w)\psi(x, w)] \le -W(x) \tag{6}$$

$\forall x \in \mathcal{X} \subset R^n$, $w \in \mathcal{W}$, where $W(x)$ is a continuous positive definite function and α_1 and α_2 are class $\mathcal{K}$ functions. The system (4) can be re-written as

$$\begin{aligned} \dot{x} = {} & f(x, w) + g(x, w)\psi(x, w) \\ & + g(x, w)u - g(x, w)[\chi(w) + \psi(x, w)] \end{aligned} \tag{7}$$

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Let $\Omega = \{V(w,x) \leq c_1\} \subset \mathcal{X}$ be a compact set for some $c_1 > 0$ and $\delta(x)$ be a function such that

$$\|\chi(w) + \psi(x,w)\| \leq \delta(x) \qquad \forall x \in \Omega, \quad \forall w \in \mathcal{W} \quad (8)$$

Suppose $(\partial V/\partial x)g(x,w)$ can be expressed as

$$(\partial V/\partial x)g(x,w) = \upsilon(x)H(x,w) \quad (9)$$

where $\upsilon(x)$ is a known, locally Lipschitz function, with $\upsilon(0) = 0$, and $H(x,w)$ is a, possibly unknown, function such that $0 < \theta \leq |H(x,w)| \leq k$, $\forall x \in \Omega$, $\forall w \in \mathcal{W}$.

A *conditional servocompensator* [10] is introduced via the saturated high-gain feedback controller

$$u = -\alpha(x)\varrho\left(\frac{s}{\mu}\right) \quad (10)$$

where $s = \upsilon(x) + K_1\sigma$, the function ϱ is defined as

$$\varrho(y) = \begin{cases} y & \text{if} \quad |y| \leq 1 \\ -1 & \text{if} \quad y < -1 \\ 1 & \text{if} \quad y > 1 \end{cases} \quad (11)$$

and σ is the output of the conditional servocompensator

$$\dot{\sigma} = (S - JK_1)\sigma + \mu J\varrho\left(\frac{s}{\mu}\right) \quad (12)$$

with $\mu > 0$ being the width of the boundary layer. The pair (S, J) is controllable and K_1 is chosen such that $S - JK_1$ is Hurwitz. The function $\alpha(x)$ is chosen to satisfy

$$\alpha(x) \geq \frac{k}{\theta}\delta(x) + \alpha_0, \quad \alpha_0 > 0 \quad (13)$$

It is shown in [10] that if $\sigma(0)$ is $O(\mu)$, the state $\sigma(t)$ of the conditional servocompensator (12) will always be $O(\mu)$.

The analysis in [10] shows that, for sufficiently small μ, every trajectory of the closed-loop system (2), (4), (10) and (12) asymptotically approaches a disturbance-dependant manifold of the form $\{x = 0, \sigma = \bar{\sigma}\}$, on which the regulation error is zero. The state feedback design is extended to output feedback for a class of minimum-phase, input-output linearizable systems. For this class of systems, the state feedback control can be designed as a partial state feedback law that does not use the states of the internal dynamics. A reduced-order high-gain observer is used to estimate the states of the linearizable part of the system, which are derivatives of the output. The output feedback controller, obtained by replacing the states by their estimates, recovers the transient and asymptotic properties of the state feedback controller. The performance recovery is shown using the separation principle of [1] and [2].

III. Low-gain design for linear systems

In this section we briefly review the approach presented in [8] for the semiglobal output regulation problem of linear systems subject to input saturation. Consider a single-input single-output linear system

$$\begin{aligned} \dot{\zeta} &= A\zeta + B\varrho(u) + Ew \\ e &= C\zeta + Fw \end{aligned} \quad (14)$$

where $\zeta \in R^n$ is the state, u is the control input, e is the regulation error and $w(t)$ is an *exogenous* input that belongs to a compact set $\mathcal{W} \in R^w$, and is generated by the internal model $\dot{w} = Sw$, where S has distinct eigenvalues on the imaginary axis. It is assumed that A has all eigenvalues in the closed left-half plane, (A, B) is stabilizable, and (A, C) is detectable. Moreover, there exist matrices Π, Γ such that

$$\Pi S = A\Pi + B\Gamma + E\ , \quad 0 = C\Pi + F \quad (15)$$

where $|\Gamma w| \leq 1 - \delta$ for all $w \in \mathcal{W}$, for some $0 < \delta < 1$.

If ζ and w were available for feedback, a stabilizing state feedback control law can be taken as

$$u = -K(\lambda)\zeta + [K(\lambda)\Pi + \Gamma]w \quad (16)$$

where $K(\lambda) = B^TP(\lambda)$, and $P(\lambda)$ is the positive definite solution of the Riccati equation

$$P(\lambda)A + A^TP(\lambda) - P(\lambda)BB^TP(\lambda) + Q(\lambda) = 0 \quad (17)$$

where $Q(\lambda)$ is a positive definite matrix that satisfies $\lim_{\lambda\to 0}\ Q(\lambda) = 0$. The parameter λ is chosen small enough such that the control does not saturate over the domain of interest. Assuming that

$$\left(\begin{bmatrix} C & F \end{bmatrix},\ \begin{bmatrix} A & E \\ 0 & S \end{bmatrix} \right)$$

is detectable, the state feedback design is extended to an error feedback design by using the observer

$$\begin{aligned} \begin{bmatrix} \dot{\hat{\zeta}} \\ \dot{\hat{w}} \end{bmatrix} &= \begin{bmatrix} A & E \\ 0 & S \end{bmatrix} \begin{bmatrix} \hat{\zeta} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \varrho(u) \\ &\quad + \begin{bmatrix} L_A \\ L_S \end{bmatrix} \left(e - \begin{bmatrix} C & F \end{bmatrix} \begin{bmatrix} \hat{\zeta} \\ \hat{w} \end{bmatrix} \right) \end{aligned} \quad (18)$$

where the matrices L_A and L_S are chosen such that the matrix

$$\bar{A} \triangleq \begin{pmatrix} A - L_AC & E - L_AF \\ -L_SC & S - L_SF \end{pmatrix} \quad (19)$$

is Hurwitz. The error feedback control law is given by

$$u \;=\; -K(\lambda)\hat{\zeta} + [K(\lambda)\Pi + \Gamma]\hat{w} \quad (20)$$

With the change of variables, $\xi = \zeta - \Pi w$, $\tilde{\zeta} = \zeta - \hat{\zeta}$ and $\tilde{w} = w - \hat{w}$, the closed-loop system can be written as

$$\begin{aligned} \dot{\xi} &= A\xi + B\varrho[-K(\lambda)\xi + \Gamma(w - \tilde{w}) + K(\lambda)(\tilde{\zeta} - \Pi\tilde{w})] \\ &\quad + (A\Pi - \Pi S + E)w \\ \dot{\tilde{\zeta}} &= (A - L_AC)\tilde{\zeta} + (E - L_AF)\tilde{w} \\ \dot{\tilde{w}} &= -L_SC\tilde{\zeta} + (S - L_SF)\tilde{w} \end{aligned} \quad (21)$$

The key feature of the approach [8] is designing an observer with much faster dynamics than those of the original system. Since the last two equations of (21) are homogeneous, designing the observer dynamics arbitrarily fast yields rapid error convergence to zero. Hence, $\lim_{t\to\infty}\ e(t) = 0$.

IV. Problem Statement and Control Design

We now cast the output regulation problem for linear systems subject to input constraints in the Lyapunov redesign framework as presented in [10]. Our goal here is to design an output feedback controller for the system (14) to stabilize the system when $w = 0$ and to asymptotically regulate e to zero when $w \neq 0$. With the change of variables $x = \zeta - \pi$, the system (14) can be written as

$$\begin{aligned} \dot{x} &= Ax + B[\varrho(u) - \Gamma w] \\ e &= Cx \end{aligned} \tag{23}$$

The system (23) is in a form where the state feedback regulation problem can be formulated as a state feedback stabilization problem by treating Γw as a matched uncertainty. We design a low-gain feedback control law to achieve stabilization and then introduce a conditional servocompensator through a saturated high-gain feedback [10], [12]. Towards that end, let $K(\lambda) = B^T P(\lambda)$ be the state feedback gain matrix. The derivative of the Lyapunov function $V(x) = x^T P(\lambda) x$ with respect to the nominal system $\dot{x} = [A - BK(\lambda)]x$ is

$$\dot{V}(x) = -x^T[Q(\lambda) + P(\lambda)BB^T P(\lambda)]x \tag{24}$$

For convenience we write $(\partial V/\partial x)B = \upsilon(x)$, where $\upsilon(x) = 2B^T P(\lambda)x$. The system (23) can be written as

$$\dot{x} = (A - BK)x + B\varrho(u) + B[Kx - \Gamma w] \tag{25}$$

We design a saturated high-gain feedback controller for this system to deal with the uncertain term Γw. Let $\Omega = \{V(x) \le c_1\} \subset \mathcal{X}$ be a compact set for some $c_1 > 0$ and

$$|K(\lambda)x - \Gamma w| \le 1 - \delta_0, \quad \delta_0 > 0 \quad \forall x \in \Omega, \forall w \in \mathcal{W} \tag{26}$$

We introduce the *conditional servocompensator* [10], [12] via the saturated high gain feedback controller

$$u = -\left(\frac{s}{\mu}\right) \tag{27}$$

where $s = \upsilon(x) + K_1\sigma$ and σ is the output of the conditional servocompensator

$$\dot{\sigma} = (S - JK_1)\sigma + \mu J \varrho\left(\frac{s}{\mu}\right) \tag{28}$$

where $\mu > 0$ is the width of the boundary layer, (S, J) is controllable and K_1 is chosen such that $S - JK_1$ is Hurwitz. Equation (28) is a perturbation of the exponentially stable system $\dot{\sigma} = (S - JK_1)\sigma$, with the norm of the perturbation bounded by μ. In order to show that σ is always $O(\mu)$, we define the Lyapunov function

$$V_0(\sigma) = \sigma^T P_0 \sigma$$

where the symmetric positive definite matrix P_0 is the solution of $P_0 A_\sigma + A_\sigma^T P_0 = -I$ and $A_\sigma \triangleq S - JK_1$. Consider the compact set $\{\sigma : V_0(\sigma) \le \mu^2 c_2\}$, where c_2 is a positive constant. Let $\sigma(0)$ belong to this set. Using the inequality

$$\dot{V}_0(\sigma) \le -\|\sigma\|^2 + 2\mu \|\sigma\| \|P_0 J\|$$

it is easy to show that $\dot{V}_0(\sigma) \le 0$ on the boundary $V_0(\sigma) = \mu^2 c_2$ for the choice $c_2 = 4\|P_0 J\|^2 \lambda_{max}(P_0)$. Hence, the set $\{\sigma : V_0(\sigma) \le \mu^2 c_2\}$ is positively invariant.

V. Closed-Loop Analysis

In this section we will show that, for sufficiently small μ, every trajectory of the closed-loop system asymptotically approaches an invariant manifold on which the error is zero. The closed-loop system is given by

$$\begin{aligned} \dot{w} &= S_0 w \\ \dot{x} &= [A - BK]x - B\varrho\left(\frac{s}{\mu}\right) + B[Kx - \Gamma w] \\ \dot{\sigma} &= A_\sigma \sigma + \mu J \varrho\left(\frac{s}{\mu}\right) \end{aligned} \tag{29}$$

We start by showing that the set $\Psi = \Omega \times \{V_0(\sigma) \le \mu^2 c_2\}$ is positively invariant and every trajectory in Ψ reaches the positively invariant set $\Psi_\mu = \{V(x) \le \rho(\mu)\} \times \{V_0(\sigma) \le \mu^2 c_2\}$ in finite time, where ρ is a class $\mathcal{K}$ function.

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x}[A - BK]x - \frac{\partial V}{\partial x}B\varrho\left(\frac{s}{\mu}\right) + \frac{\partial V}{\partial x}B[Kx - \Gamma w] \\ &= -x^T[Q(\lambda) + P(\lambda)BB^T P(\lambda)]x - (s - K_1\sigma)\varrho\left(\frac{s}{\mu}\right) \\ &\quad + (s - K_1\sigma)[Kx - \Gamma w] \\ &= -x^T[Q(\lambda) + P(\lambda)BB^T P(\lambda)]x - s\varrho\left(\frac{s}{\mu}\right) \\ &\quad + K_1\sigma\varrho\left(\frac{s}{\mu}\right) + s[Kx - \Gamma w] - K_1\sigma[Kx - \Gamma w] \end{aligned}$$

Inside Ψ, $\|\sigma\| \le \mu\sqrt{c_2/\lambda_{min}(P_0)}$. Using this along with (11) and (26), it can be shown that when $|s| \ge \mu$ we have

$$\dot{V} \le -x^T[Q(\lambda) + P(\lambda)BB^T P(\lambda)]x + \mu\gamma_1 \tag{30}$$

where $\gamma_1 = 2\|K_1\| \sqrt{c_2/\lambda_{min}(P_0)}$. Similarly, when $|s| \le \mu$

$$\dot{V} \le -x^T[Q(\lambda) + P(\lambda)BB^T P(\lambda)]x + \mu\gamma_2 \tag{31}$$

where $\gamma_2 = \gamma_1 + (1/4)$. From (30) and (31), $\dot{V} \le -x^T[Q(\lambda) + P(\lambda)BB^T P(\lambda)]x + \mu\gamma_2$, $\forall (x, \sigma) \in \Psi$. Hence, from [7, Theorem 4.18], for sufficiently small μ, Ψ is positively invariant and all trajectories starting in Ψ enter a positively invariant set $\Psi_\mu = \{V(x) \le \rho(\mu)\} \times \{V_0(\sigma) \le \mu^2 c_2\}$ in finite time.

Next, we use $V_1 = \frac{1}{2}s^2$ to show that the trajectories reach the boundary layer $\{|s| \le \mu\}$ in finite time. Since $P(\lambda)$ is positive definite, $B^T P(\lambda) B > 0$. Let $k_p = 2B^T P(\lambda)B$. For $(x, \sigma) \in \Psi_\mu$, we have

$$\begin{aligned} s\dot{s} &= -2sB^T PB\varrho\left(\frac{s}{\mu}\right) + 2sB^T P[A - BK]x \\ &\quad + 2sB^T PB[Kx - \Gamma w] + sK_1 A_\sigma + \mu s K_1 J \varrho\left(\frac{s}{\mu}\right) \end{aligned}$$

Outside the boundary layer, i.e. when $|s| \ge \mu$, we have

$$\begin{aligned} s\dot{s} \le& -k_p|s| + \|2B^T P[A - BK]x\||s| + k_p\big|Kx - \Gamma w\big||s| \\ &+ (\|\sigma\| \|K_1\| \|A_\sigma\| + \mu \|K_1\| \|J\|)|s| \end{aligned} \tag{32}$$

Inside Ψ_μ, $\|\sigma\| \le \mu\sqrt{c_2/\lambda_{min}(P_0)}$. Also, the function $[A - BK(\lambda)]x$ is continuous and vanishes at $x = 0$. Therefore, the norm $\|2B^T P(\lambda)[A - BK(\lambda)]x\|$ together with the norms

$\|\sigma\| \, \|K_1\| \, \|A_\sigma\|$ and $\mu \, \|K_1\| \, \|J\|$ can be bounded by a class $\mathcal{K}$ function $\rho_1(\mu)$. Hence,

$$\begin{aligned} s\dot{s} &\leq -k_p|s| + k_p(1-\delta_0)|s| + \rho_1(\mu)|s| \\ \Longrightarrow \qquad \dot{V}_1 &\leq -k_p\left[\delta_0 - \frac{\rho_1(\mu)}{k_p}\right]|s| \end{aligned}$$

Thus, for sufficiently small μ, all trajectories inside Ψ_μ would reach the boundary layer $\{|s| \leq \mu\}$ in finite time.

Finally, we show that inside the boundary layer the trajectories of the closed-loop system asymptotically approach an invariant manifold on which the error is zero. Inside the boundary layer, the closed-loop system (29) is given by

$$\begin{aligned} \dot{w} &= S_0 w \\ \dot{x} &= [A - BK]x - B\left(\frac{s}{\mu}\right) + B[Kx - \Gamma w] \qquad (33) \\ \dot{\sigma} &= S\sigma + J\upsilon(x) \end{aligned}$$

From [12], there exists a unique matrix Λ such that

$$S\Lambda = \Lambda S \qquad \text{and} \qquad -K_1\Lambda = \Gamma$$

We define $\mathcal{N}_\mu = \{x = 0, \sigma = \bar{\sigma}\}$, where $\bar{\sigma} = \mu\Lambda w$. It is easy to verify by direct substitution that $\mathcal{N}_\mu$ is an invariant manifold of (33) for all $w \in \mathcal{W}$. Defining $\tilde{\sigma} = \sigma - \bar{\sigma}$ and $\tilde{s} = \upsilon + K_1\tilde{\sigma}$, the closed-loop system inside the boundary layer can be written as

$$\begin{aligned} \dot{w} &= S_0 w \\ \dot{x} &= [A - BK]x - B\left(\frac{\tilde{s}}{\mu}\right) + BKx \qquad (34) \\ \dot{\tilde{\sigma}} &= A_\sigma\tilde{\sigma} + J\tilde{s} \;=\; S\tilde{\sigma} + J\upsilon \end{aligned}$$

Define the Lyapunov function

$$V_2 = V(x) + \frac{b}{\mu}\tilde{\sigma}^T P_0 \tilde{\sigma} + \frac{c}{2}\tilde{s}^2 \qquad (35)$$

where b and c are positive constants to be chosen. Calculating $\dot{V}_2$ along the trajectories of the system (34), we obtain

$$\dot{V}_2 = \dot{V} + \frac{b}{\mu}\left[\tilde{\sigma}^T P_0 \dot{\tilde{\sigma}} + \dot{\tilde{\sigma}}^T P_0 \tilde{\sigma}\right] + c\tilde{s}\dot{\tilde{s}} \qquad (36)$$

Calculating $\dot{V}$ along the trajectories of (34), we have

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x}[A - BK(\lambda)]x - \frac{\partial V}{\partial x}B\left(\frac{\tilde{s}}{\mu}\right) + \frac{\partial V}{\partial x}BK(\lambda)x \\ &\leq -\lambda_{min}(Q)\|x\|^2 - \frac{|\upsilon|^2}{\mu} + \frac{k_a}{\mu}|\upsilon| \, \|\tilde{\sigma}\| + k_0|\upsilon|\|x\| \qquad (37) \end{aligned}$$

where k_a and k_0 are the upper bounds on $\|K_1\|$ and $\|K\|$, respectively. The second term of $\dot{V}_2$ satisfies the inequality

$$\begin{aligned} \frac{b}{\mu}\left[\tilde{\sigma}^T P_0 \dot{\tilde{\sigma}} + \dot{\tilde{\sigma}}^T P_0 \tilde{\sigma}\right] \quad &\leq \quad -\frac{b}{\mu}\|\tilde{\sigma}\|^2 \\ &\quad + \frac{2bk_1}{\mu}\|\tilde{\sigma}\| \, |\tilde{s}| \lambda_{max}(P_0) \qquad (38) \end{aligned}$$

where k_1 is the upper bound on $\|J\|$. Next, we have

$$\begin{aligned} \dot{\tilde{s}} &= \frac{\partial \upsilon}{\partial x}[A - BK(\lambda)]x - \frac{\partial \upsilon}{\partial x}B\left(\frac{\tilde{s}}{\mu}\right) + \frac{\partial \upsilon}{\partial x}BK(\lambda)x \\ &\quad + K_1(S\tilde{\sigma} + J\upsilon) \\ c\tilde{s}\dot{\tilde{s}} &\leq -c(k_p/\mu)|\tilde{s}|^2 + ck_2|\tilde{s}|\|x\| + ck_3|\tilde{s}| \, \|\tilde{\sigma}\| \\ &\quad + ck_4|\tilde{s}||\upsilon| \qquad (39) \end{aligned}$$

where k_2, k_3 and k_4 are some positive constants. From (37), (38) and (39), we have

$$\begin{aligned} \dot{V}_2 \quad \leq \quad & -\lambda_{min}(Q)\|x\|^2 - \frac{1}{\mu}|\upsilon|^2 - \frac{b}{\mu}\|\tilde{\sigma}\|^2 - c(k_p/\mu)|\tilde{s}|^2 \\ & + k_0|\upsilon|\|x\| + \frac{k_a}{\mu}|\upsilon| \, \|\tilde{\sigma}\| + ck_4|\tilde{s}||\upsilon| + ck_2|\tilde{s}|\|x\| \\ & + [(2bk_1/\mu)\lambda_{max}(P_0) + ck_3] \, \|\tilde{\sigma}\| \, |\tilde{s}| \qquad (40) \end{aligned}$$

The right-hand side of (40) can be arranged in the following quadratic form of $\Pi = [\; \|x\| \; |\upsilon| \; \|\tilde{\sigma}\| \; |\tilde{s}| \;]^T$:

$$\dot{V}_2 \;\leq\; -\Pi^T \Delta \Pi \qquad (41)$$

where the symmetric matrix Δ is given by

$$\Delta = \begin{pmatrix} \lambda_{min}(Q) & -\frac{k_0}{2} & 0 & \frac{-c\;k_2}{2} \\ & \frac{1}{\mu} & -\frac{k_a}{2\mu} & \frac{-c\;k_4}{2} \\ & & \frac{b}{\mu} & -\frac{bk_1\lambda_{max}(P_0)}{\mu} - \frac{ck_3}{2} \\ & & & \frac{ck_p}{\mu} \end{pmatrix}$$

Similar to [10], the leading principal minors of Δ can be made positive by first choosing b large enough, and then, choosing c large. Finally, by choosing μ sufficiently small, $\dot{V}_2$ will be negative definite. Therefore, inside the boundary layer, the trajectories of the closed-loop system will asymptotically approach $\mathcal{N}_\mu$ as $t \to \infty$. Our conclusions can be summarized in the following theorem.

Theorem 1: Under stated assumptions, there exists $\mu^* > 0$ such that $\forall \mu \in (0, \mu^*]$, the state variables of the closed-loop system comprising of the system (23), the servocompensator (28) and the state feedback control (27) are bounded and $\lim_{t\to\infty} \; e(t) = 0$.

VI. Output Feedback Design

In this section we extend the state feedback controller of the previous section to output feedback by using a full-order high-gain observer. This is different from [10] where a reduced-order high-gain observer was used because the state feedback control was a partial one. In the current problem, because of the constraint on the control, the mechanism of solving the stabilization problem through ARE necessitates the use of full-state feedback. We use the singular perturbation approach to the observer design described in [3].

The state feedback control (27) is implemented as an observer-based controller

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B\varrho(u) + L(e - C\hat{x}) \\ u &= -\left(\frac{\hat{s}}{\mu}\right) \qquad (42) \end{aligned}$$

where L is the vector of observer gains to be designed and $\hat{s} = 2B^T P(\lambda)\hat{x} + K_1\sigma$. The estimation error, $\eta = x - \hat{x}$, satisfies the equation

$$\dot{\eta} = (A - LC)\eta - B\Gamma w \qquad (43)$$

The observer design starts by transforming the system into the normal form. There is a nonsingular matrix T such that

$x = T \begin{pmatrix} x_a \\ x_f \end{pmatrix}$ transforms the system (23) into the form

$$\begin{aligned} \dot{x}_a &= A_{aa}x_a + A_{af}y & (44)\\ \dot{x}_f &= A_f x_f + B_f[E_a x_a + E_f x_f + \varrho(u)] & (45)\\ y &= C_f x_f & (46) \end{aligned}$$

where (A_f, B_f, C_f) represents a chain of integrators. The eigenvalues of A_{aa} are the invariant zeros of the triplet (A, B, C). The observer gain L is designed as [3]

$$L(\varepsilon) = T \begin{pmatrix} A_{af} \\ M(\varepsilon)L_f \end{pmatrix} \tag{47}$$

where L_f assigns the eigenvalues of $(A_f - L_f C_f)$ in the open left-half plane, $M(\varepsilon) = blkdiag[\frac{1}{\varepsilon}, \frac{1}{\varepsilon^2}, \cdots, \frac{1}{\varepsilon^q}]$, $q = dim(x_f)$, and ε is a small positive constant. The observer gain $L(\varepsilon)$ assigns the observer eigenvalues into two groups: $(n-q)$ eigenvalues are assigned at the open-loop invariant zeros and q eigenvalues are assigned at $O(1/\varepsilon)$ locations, approaching the eigenvalues of $(A_f - L_f C_f)/\varepsilon$ as $\varepsilon \to 0$. The state component x_a is estimated by the observer

$$\dot{\hat{x}}_a = A_{aa}\hat{x}_a + A_{af}y \tag{48}$$

and the state x_f is estimated using a high-gain observer

$$\dot{\hat{x}}_f = A_f\hat{x}_f + M(\varepsilon)L_f(y - C_f\hat{x}_f) + B_f\varrho(u) \tag{49}$$

The change of variables $\eta = \begin{pmatrix} T_1 & T_2 \end{pmatrix} \begin{pmatrix} \eta_s \\ \bar{\eta}_f \end{pmatrix}$ transforms the error equation (43) into

$$\begin{aligned} \dot{\eta}_s &= A_{aa}\eta_s & (50)\\ \dot{\bar{\eta}}_f &= [A_f - M(\varepsilon)L_f C_f]\bar{\eta}_f + B_f[E_a\eta_s + E_f\bar{\eta}_f - \Gamma w] & (51) \end{aligned}$$

where $\eta_s = x_a - \hat{x}_a$ and $\bar{\eta}_f = x_f - \hat{x}_f$. To bring the system (50)-(51) into the standard singularly perturbed form we need to scale $\bar{\eta}_f$ as

$$\eta_f = N^{-1}(\varepsilon)\bar{\eta}_f \tag{52}$$

where $N(\varepsilon) = blkdiag[\varepsilon^{q-1}, \cdots, \varepsilon, 1]$. With the special structure of the matrices $N(\varepsilon)$, A_f, B_f, C_f, $M(\varepsilon)$ and L_f, it is shown in [3] that

$$\begin{aligned} N^{-1}(\varepsilon)B_f &= B_f \\ N^{-1}(\varepsilon)[A_f - M(\varepsilon)L_f C_f]N(\varepsilon) &= \frac{1}{\varepsilon}[A_f - L_f C_f] \end{aligned}$$

where $[A_f - L_f C_f]$ is Hurwitz. The scaling (52) transforms (50)-(51) into the standard singularly perturbed system

$$\begin{aligned} \dot{\eta}_s &= A_{aa}\eta_s \\ \varepsilon\dot{\eta}_f &= [A_f - L_f C_f]\eta_f + \varepsilon B_f[E_a\eta_s + E_f N(\varepsilon)\eta_f - \Gamma w] \end{aligned}$$

We notice that the perturbation term Γw is multiplied by ε so that its effect diminishes asymptotically as $\varepsilon \to 0$.

We now analyze the closed-loop system composed of (23), (28) and (42). Using $(x, \sigma, \eta_s, \eta_f)$ as the state vector, the closed-loop system is given by

$$\begin{aligned} \dot{x} &= Ax + B\varrho\Big[-\frac{1}{\mu}\Big(2B^T P(\lambda)x + K_1\sigma \\ &\quad - 2B^T P(\lambda)\begin{bmatrix} T_1 & T_2 N(\varepsilon)\end{bmatrix}\begin{bmatrix}\eta_s \\ \eta_f\end{bmatrix}\Big)\Big] - B\Gamma w \\ \dot{\sigma} &= A_\sigma\sigma + \mu J\varrho\Big[\frac{1}{\mu}\Big(2B^T P(\lambda)x + K_1\sigma \\ &\quad - 2B^T P(\lambda)\begin{bmatrix} T_1 & T_2 N(\varepsilon)\end{bmatrix}\begin{bmatrix}\eta_s \\ \eta_f\end{bmatrix}\Big)\Big] \qquad (53) \\ \dot{\eta}_s &= A_{aa}\eta_s \\ \varepsilon\dot{\eta}_f &= [A_f - L_f C_f]\eta_f + \varepsilon B_f[E_a\eta_s + E_f N(\varepsilon)\eta_f - \Gamma w] \end{aligned}$$

The system (53) is a standard singularly perturbed system with (x, σ, η_s) as the slow variable and η_f as the fast variable. The slow model of (53) is obtained by setting $\varepsilon = 0$ in the last equation of (53). Since $[A_f - L_f C_f]$ is Hurwitz, hence non-singular, we obtain the unique root $\eta_f = 0$. Substitution of $\eta_f = 0$ in (53) results in the slow model

$$\begin{aligned} \dot{x} &= Ax + B\varrho\Big[-\frac{1}{\mu}\Big(2B^T P(\lambda)x + K_1\sigma \\ &\quad - 2B^T P(\lambda)T_1\eta_s\Big)\Big] - B\Gamma w & (54)\\ \dot{\sigma} &= A_\sigma\sigma + \mu J\varrho\Big[\frac{1}{\mu}\Big(2B^T P(\lambda)x + K_1\sigma \\ &\quad - 2B^T P(\lambda)T_1\eta_s\Big)\Big] & (55)\\ \dot{\eta}_s &= A_{aa}\eta_s & (56) \end{aligned}$$

which appears as the cascade connection of (56) and the closed-loop system under the state feedback (29). Let P_s be the positive definite solution of the Lyapunov equation

$$P_s A_{aa} + A_{aa}^T P_s = -I$$

The function $V_s = \eta_s^T P_s \eta_s$ satisfies $\dot{V}_s \leq -\|\eta_s\|^2$. Using the fact that A_{aa} is Hurwitz and the origin of (54)-(55) is exponentially stable when $\eta_s = 0$, it can be shown that (54)-(56) is exponentially stable with a region of attraction

$$\{(x^T, \sigma^T, \eta_s^T)^T : V(x) \leq c_1, V_0(\sigma) \leq \mu^2 c_2, V_s \leq c_3\}$$

for some $c_3 > 0$. The zero-error manifold is rewritten as

$$\mathcal{N}_\mu = \{x = 0, \sigma = \bar{\sigma}, \eta_s = 0\}$$

Let the initial states $(x(0), \sigma(0), \eta_s(0)) \in \mathcal{G}$ and $\eta_f(0) \in \mathcal{H}$, where $\mathcal{G}$ is a compact set which contains $\mathcal{N}_\mu$. From [1], [2], it can be shown that there is a neighborhood $\mathcal{M}$ of the origin of the system (54)-(56), independent of ε, and $\varepsilon_1 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_1$, the origin is exponentially stable and every trajectory in $\mathcal{M}$ converges to the origin as $t \to \infty$. From [1], [2], there is $\varepsilon_2 > 0$ such that for every $0 < \varepsilon \leq \varepsilon_2$, the solutions starting in $\mathcal{G} \times \mathcal{H}$ enter $\mathcal{M}$ in finite time. Hence, for every $0 < \varepsilon \leq \varepsilon_3 = min\{\varepsilon_1, \varepsilon_2\}$, the origin is exponentially stable and $\mathcal{G} \times \mathcal{H}$ is a subset of the region of attraction. Thus, for sufficiently small ε, the closed-loop system (53), under the output feedback controller (42), is uniformly exponentially stable with respect to the set $\mathcal{N}_\mu \times \{\eta_f = 0\}$. Hence, $\lim_{t\to\infty} e(t) = 0$.

VII. SIMULATION EXAMPLE

Consider a minimum-phase SISO linear system, that corresponds to (14) with

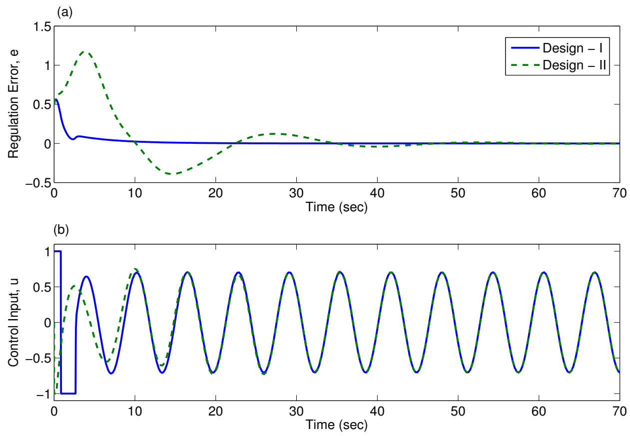

Fig. 1. Performance comparison of the two control designs (a) Regulation error 'e' during the transient period (b) Corresponding control input, 'u'

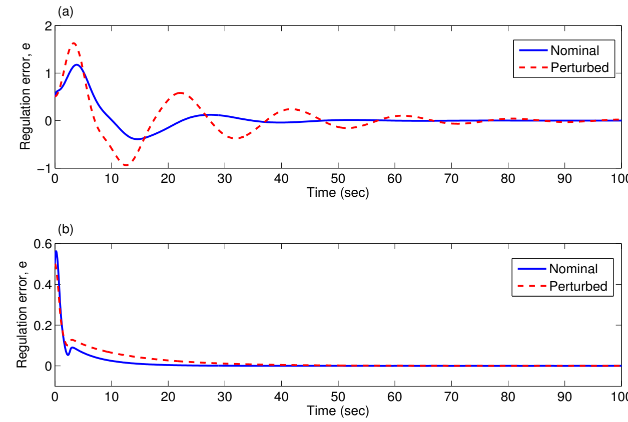

Fig. 2. Transient performance of the two control designs when control coefficient is perturbed by 40 percent (a) Design II - Nominal vs Perturbed (b) Design I - Nominal vs Perturbed

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}, \; F = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

with the signal w generated by the exosystem

$$\dot{w} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} w, \quad w^T(0) = [0, w_0]$$

We show the performance of two designs: Design I incorporates the saturated high-gain feedback using a conditional servocompensator and the full-order observer (48)-(49). For this design, K_1 is chosen so as to assign the eigenvalues of $S - JK_1$ at -0.5 and -1, and the observer gain $L(\varepsilon)$ is designed such that the eigenvalue of (48) is assigned at the location of the invariant zero of the triplet (A, B, C), i.e. at -1, and $L_f = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ is chosen such that the polynomial $\varpi^2 + g_1\varpi + g_2$ is Hurwitz. Design II is based on the linear observer-based error-feedback control approach [8], reviewed in Section III. For this design, a fifth-order linear observer of the form (18) is constructed where the matrices L_A and L_S are chosen such as to assign the eigenvalues of the matrix $\bar{A}$ at $[-22, -23, -24, -25, -26]$. We use the following numerical values in the simulation: $\omega = 1$ rad/s, $w_0 = 0.5$, $\mu = 0.1$, $g_1 = 2$, $g_2 = 1$, $\lambda = 0.05$ and $\varepsilon = 0.05$.

Figure 1(a) shows the regulation error during the transient period for the two designs and Figure 1(b) shows the corresponding control input. The regulation error goes to zero sharply in the case of Design I, where as in Design II, the same oscillates before eventually converging to zero. Note that due to the fact that a higher dimensional (fifth-order) observer is used in Design II, in order to achieve reasonable performance, the observer gains were required to be pushed very high e.g. $O(10^7)$, in contrast to $O(10^3)$ for those in Design I. Figure 2 shows the performance of the two control designs when the control coefficient is perturbed from 1 to 1.4. The results suggest that Design I may be more robust than Design II, however, further investigation is needed to confirm this observation.

VIII. Conclusions

This paper studies the output regulation problem of linear systems subject to input constraints. We presented a novel control design that includes a conditional servocompensator, introduced via Lyapunov redesign and saturated high-gain feedback. The use of a conditional servocompensator enables us to achieve zero steady-state regulation error, without degrading the transient response. The output feedback control is implemented using a two-time-scale observer design of [3] and the performance recovery is shown using the separation principle of [1], [2]. The performance of the control design is demonstrated by a simulation example.

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