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Optimal path coordination problems

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***Abstract*— Optimal path coordination problems for multi-vehicle systems are formulated in the framework of hybrid systems and solved using dynamic programming techniques. In these problems, the path cost for one vehicle is a discontinuous function of the distances to other vehicles. This leads to a non-standard optimal control problem.**

I. INTRODUCTION

Problems of collaborative multi-vehicle control are posing new challenges to control. In some problems, cooperation concerns distributing similar vehicles over an area to optimize the rate of coverage in surveillance missions. In other problems, heterogeneous vehicles with complementary capabilities can be used more advantageously when other forms of cooperation take place. One such example arises when planing operations of unmanned air vehicles (UAV) in hostile air spaces. The probability of survival of an UAV is directly proportional to the value of the path integral taken with respect to some risk function [5]; the level of risk is significantly reduced when the UAV flies under the protection of an UAV carrying a jamming device. This is an example of a collaborative control problem where vehicles interact to improve individual or group performance.

The interesting questions are: how is optimal vehicle control related to optimal group control? What is the value of cooperation? These questions are better understood in the framework of dynamic programming (DP) [2]. DP approaches the problem of optimizing the behavior of a dynamic system with respect to some cost function by introducing a value function which gives, at each point of the state space, the optimal cost to go for the system. When the optimization problem is properly formulated (see [8] for details), the value function satisfies an equation which is derived from the Principle of Optimality, which basically states that in an optimal sequence of decisions or choices, each subsequence must also be optimal.

Here we discuss part of our research on DP for collaborative control problems with the help of a simple two-vehicle optimal path coordination control problem (see [7] for related work on DP for collaborative control). This problem is representative of more general optimal coordination problems.

Vehicle v_1 has to find the optimal trajectory from some initial location α to some destination γ. The instantaneous path cost for v_1 is reduced by a fixed amount l when the position of this vehicle "coincides" with the position of another vehicle, v_2; this means that the path cost for v_1 is a discontinuous function of the relative positions of the two vehicles. v_2 has a limited amount of fuel; it departs from $\beta \neq \alpha$ and is required to return to β before it runs out of fuel. The vehicles are allowed to met once and move together up to the point where v_2 has enough fuel to return to β.

We formulate the collaborative control problem for v_1 and v_2 as an optimal control problem for a hybrid automaton with three discrete states (the hybrid automaton models the combinatorial aspects of the problem) and find the structure of the solution using DP techniques. In this formulation, the state of the two-vehicle system has two components: a memoryless component, given by the continuous state, and a component with memory, given by the discrete state which describes the history of motions up to the current discrete state. This is because the system has to "remember" if the vehicles met at a given point, to prevent them from meeting again (as required). The jump sets are given by the set reachable by v_2 for a round trip from β (see [9] for details on dynamic optimization techniques for reachability analysis).

The motivation for our formulation comes from two problems of motion coordination discussed in [10] to illustrate the use Ordered Upwind Methods for solving optimal hybrid control problems. The first problem consists of finding an optimal trajectory on a surface, given that there are discrete transitions between a finite number of points on the continuous state-space. This problem can be interpreted as one of motion coordination between a person and a bus running between two or more bus stops: in some cases it may be better to take the bus. The directed discrete links change only the position in the continuous state space, but not the underlying dynamics. The problem is solved with the help of one value function defined on the continuous state-space. The second problem consists of finding an optimal trajectory for a person walking on a varied landscape and carrying a pair of inline roller skates. The person has the option to switch between walking and skating by paying a time penalty. This is modeled with two discrete states and two copies of the continuous-time state-space. The problem is solved with the help of a value function defined on the hybrid state-space.

The paper is organized as follows. In section II we provide some background on dynamic optimization for hybrid systems. In section III we state and formulate the path coordination problem in the framework of hybrid systems. In section IV we use DP techniques to characterize the solution to the problem. In section V we discuss optimal strategies and in section VI we present a numerical example. In section VII we draw the conclusions.

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II. Background

We briefly review the literature on DP for optimal hybrid control problems.

A full-fledged hybrid system model, which subsumed previous models, was introduced by M. Branicky in [4]. The model includes autonomous and controlled jump sets and destination sets. Controlled jump sets model "lazy" transition systems in the sense that the controller can decide to jump or not to jump in these sets – this is the "lazy" transition semantics in the terminology of computer science. The transition maps associated to each jump may introduce discontinuities in state and time. The dimension of the continuous-time state space is allowed to change with the discrete state. Branicky introduces an optimal control problem over an infinite horizon with three terms discounted over time: running cost, transition cost and impulse cost. The transition maps and the cost functions are assumed to be bounded, uniformly continuous, and the vector fields associated to each discrete state are assumed to be bounded and uniformly Lipschitz in the state. The distances between autonomous and controlled jump sets (and also between autonomous jump and destination sets) are assumed to be strictly positive to prevent the occurrence of multiple transitions in zero time. The flow lines are assumed to be transversal to the boundaries of the autonomous and controlled jump sets, and the vector field is not allowed to vanish in these boundaries. This is required to prove continuity from the right of the value function for the optimal control problem. The consideration of DP techniques leads to a system of Quasi Variational Inequalities (QVI). No further analysis is carried out concerning the solution of the QVI. In [6], the value function is proved to be the "viscosity" solution to this system of QVI. The transversality assumptions lead to two modeling difficulties: 1) the state of the system is supposed to "freeze" during the time jump; however this is not possible at the boundary of the autonomous and controlled jump sets; and 2) when the state enters a controlled jump set it can only leave the set through a discrete transition, which was supposed to be optional (cf. [12]).

A set of QVI conditions similar to those presented in [4] is presented in [3]. The viscosity solution to the Hamilton-Jacobi-Bellman (HJB) is discussed. This is because under their assumptions the value function is continuous. The problem is that the value function for general hybrid control problems may be discontinuous (this is mainly due to the forced jumps, controlled jumps and discontinuous jump relations). This problem is studied in [12]. In this case, the value function is not continuous and the solution of the QVI is interpreted in the discontinuous viscosity setting.

A simplified version of the hybrid system model introduced by Branicky is presented in [11]. The keys simplification are: 1) the state is kept continuous at switching times; and 2) the dimension of the continuous-time state space is kept constant. There is a discrete transition map which defines, at each discrete state, the discrete states that can be reached in one discrete transition. The assumptions also include transversality conditions as in [4]. The author introduces a class of optimal control problems with terminal and running cost functions that depend on the discrete state; there are no switching costs. A set of necessary conditions in the form of a hybrid maximum principle are introduced. The corresponding value function is shown to be bounded and continuous. A HJB equation is derived with the help of the principle of optimality. The minimization in the HJB is taken over the continuous-time control settings and the discrete states. This is because the switching costs are zero. The HJB equation is used to establish a verification theorem for optimal control candidates, but there is no discussion on viscosity solutions. The discrete transition map is not taken into consideration as a constraint in the HJB minimization. This can only happen if all discrete states can be reached in a finite number of transitions. However, this condition is not stated in the assumptions.

III. Problem formulation

A. The system

We consider planar motion models (evolving in $\mathbb{R}^2$) for $v_i, i = 1, 2$

$$\dot{x_i}(t) = f_i(x_i, u_i), u_i \in U_i, t \geq 0$$
$$x_1(0) = \alpha, x_2(0) = \beta$$

where u_i are the controls and U_i are closed sets.

Consider v_1. The cost of a path joining α and γ is

$$J_1(u_1(.), \gamma) = \int_0^{t_f} l(x_1, x_2) \cdot k_1(x_1, u_1) ds \qquad (1)$$

where $k_1(.,.) \geq 0$, $l : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, 1]$ is a piecewise constant function ($l = c, 0 < c < 1$ if $x_1 = x_2$ and $l = 1$ otherwise) and t_f is the first time when $x_1(t_f) = \gamma$ under the control function $u_1(.)$. The function l models the fact that the path cost for v_1 is reduced when the positions of v_1 and v_2 coincide.

v_2 is fuel constrained. The model of fuel consumption is captured by an additional state variable $c_2 \in \mathbb{R}$ (indicating the amount of fuel in the fuel tank)

$$\dot{c_2}(t) = g_2(x_2, u_2) = \begin{cases} w_2(x_2, u_2) & \text{if } c_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$
$$c_2(0) = \theta$$

where $w_2(.,.) \leq 0$.

We associate the cost function J_2 to the fuel remaining in v_2 when it reaches x at time t under the control $u_2(.)$

$$J_2(u_2(.), x) = c_2(t) \qquad (2)$$

The standing assumptions are:

A1) $f_i, w_2 : \mathbb{R}^2 \times U_i \to \mathbb{R}^2$ are uniformly Lipschitz in x and uniformly continuous in the control variable. This condition ensures existence and uniqueness of solutions for the differential equations.

A2) There exist $K_1 < \infty$ and $1 \leq \varsigma_1 < \infty$ such that $\|l(x_1, x_2) \cdot k_1(x_1, u_1)\| \leq K_1(1 + \|(x_1, x_2)\|)^{\varsigma_1}$ for $(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2, u_1 \in U_1$.

A3) There exist $K_2 < \infty$ and $1 \leq \varsigma_2 < \infty$ such that $\|g_2(x_2, u_2)\| \leq K_2(1 + \|x\|)^{\varsigma_2}$ for $x \in \mathbb{R}^2, u_2 \in U_2$. This assumption and the previous are related to the existence of solution to the problem.

A4) $0 \in \text{int } f_i(x_i, U_i)$. This means that each vehicle is locally controllable.

A5) $f_1(x, U_1) \subseteq f_1(x, U_2)$. This means that v_2 is capable of replicating the motions of v_1.

B. The case for coordination

The optimal path planning problem for v_1 when operating in isolation is ($l = 1$) is

Problem 1: [Uncoordinated] Find

$$\inf_{u_1(.)} J_1(u_1(.), \gamma) \tag{3}$$

The path planning problem becomes more interesting when the two vehicles are allowed to coordinate their motions. We consider the following operational constraints: 1) if v_2 leaves β, then it must return to β; and 2) the vehicles are allowed to meet only once and then move together up to the point where v_2 returns to β (this precludes behaviors where the vehicles move together and separate repeatedly).

In what follows, and to simplify the analysis of the problem, we introduce assumption A6. This assumption means that the problem is symmetric in the terminology of [1].

A6) The fuel optimal paths for v_2 are also fuel optimal for the path traveled in the opposite direction.

Let R denote the set of point reachable by v_2 for a round trip from β under fuel budget θ. This is the set of points where the two vehicles can meet at one point. A characterization of R is in order. For this purpose we introduce a value function for the problem of minimizing the fuel consumption for vehicle v_2

$$V_2(x) = \max_{u_2(.)} J_2(u_2(.), x)$$
$$V_2(\beta) = \theta$$

Proposition 3.1: Under the standing assumptions the value function V_2 is continuous in x.

The proof is standard and we omit it.

Proposition 3.2: R is a closed set given by

$$R = \{x : V_2(x) \leq \frac{\theta}{2}\} \tag{4}$$

Proof. The expression for R follows from the consideration of Assumption A6. The fact that R is closed follows from the continuity of V_2. □

It may be worthwhile for v_1 to deviate from the optimal path for Problem 1 to join v_2 at a point in R, before reaching γ. The following example illustrates this point.

Example 3.1: Consider Figure 1. Let:
$\dot{x}_i(t) \in B_0, i = 1, 2$ (B_0 is the closed unit ball in $\mathbb{R}^2$).
$\alpha = (0, 0), \beta = (50, 40), \gamma = (100, 0)$.
$\eta = (39.2000, 24.1254), \mu = (60.7999, 24.1254)$.
$c_2(0) = \theta = 12$.
$k_1(x_1, u_1) = 1, -w_2(x_2, u_2) = 0.2, l(x, x) = 0.1$.
R is the circle of radius 30 with center β (the optimal fuel cost of the round trip from β to the boundary of the circle is $60 \times 0.2 = 12 = \theta$). This is because this system satisfies the assumption A6: 1) the cost function does not depend on the direction of motion; and 2) the system dynamics are reversible. Observe that this is the set of points where the two vehicles can start to move together.

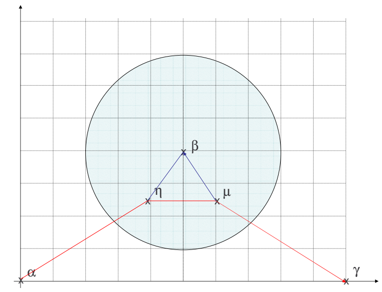


Fig. 1. Example of coordinated paths.

The fuel optimal paths for v_2 are straight lines. The same happens with the optimal paths for v_1 (for fixed values of l). This is because we have simple dynamics and piecewise constant cost functions. The straight line joining α and γ is the optimal path for Problem 1; the optimal cost is 100. The cost of the path $(\alpha, \eta, \mu, \gamma)$, where v_1 deviates from the original optimal path to benefit from a cost reduction in the segment (η, μ), is 94.2182. v_2 complies with the constraints by taking a loop (triangle) from β, with fuel cost 12.0000 (within the fuel budget).

Remark 1: We briefly discuss the structure of the solution in the previous example. Consider, for the sake of our discussion, that the optimal coordinated path for v_1 is $(\alpha, \eta, \mu, \gamma)$. Then the two path segments (α, η) and (μ, γ) are optimal with respect to the uncoordinated cost function. Otherwise we could pick other paths to connect these points with a lower cost. This is impossible since the path $(\alpha, \eta, \mu, \gamma)$ is optimal under our assumption. This means that up to the point η, the path optimization for v_1 is independent of what v_2 does. The same happens with v_2 for the path segments (β, η) and (μ, β). On the other hand, when the two vehicles meet at point η, the path optimization for both vehicles is no longer decoupled. Here, we need a third state variable to describe the evolution of the system. This is because the motions of the vehicles coincide, and because we need to keep track of the fuel consumption for v_2. This means that, from the perspective of v_1, all that really matters in what concerns v_2 is: 1) the point where the meeting takes place; and 2) the amount of the fuel remaining in the fuel tank of v_2. We observe that the amount of the fuel in v_2 at the meeting point should be optimal (otherwise this vehicle spent more fuel than what was needed to reach that point).

C. Hybrid model

The formulation of the coordinated optimal path planning problem for vehicle v_1 requires the consideration of a state variable that keeps track of what each vehicle does. We do this with a 3-state hybrid automaton. The hybrid state space is $S = \bigcup_{v \in \{a,b,c\}} (S_v \times v)$. v_1 evolves in $S_a = \mathbb{R}^2$ after departing from α. The positions of the two vehicles coincide in the discrete state b. We need an additional variable to keep track of the fuel consumption for v_2; this is why $S_b = \mathbb{R}^2 \times \mathbb{R}_0^+$. v_1 moves in $S_c = \mathbb{R}^2$ after taking the transition from discrete state b to discrete state c (after leaving v_2).

There is a controlled vector field f_v associated to each discrete state, where $f_a = f_c = f_1$ and $f_b = \{f_1, g_2\}$. The control constraints are $U_a = U_1, U_b = U_1 \times U_2$ and $U_c = U_1$. In the terminology of [4], associated to each discrete state v there are autonomous jump sets $A_{v,v'}$, controlled jump sets $C_{v,v'}$ and jump destination sets $D_{v,v'}$. The trajectory of the system jumps from S_v to $S_{v'}$ upon hitting the autonomous jump set $A_{v,v'}$; it may or may not leave S_v upon hitting the controlled jump set $C_{v,v'}$ and it can leave S_v at any point in $C_{v,v'}$; the destination of a jump is $D_{v,v'}$.

In what follows, x^i represents the i-th component of x.

The autonomous and controlled jump sets for the system are respectively $A = \bigcup_{v,v'} A_{v,v'}$ and $C = \bigcup_{v,v'} C_{v,v'}$. The jump set is $J = A \bigcup C$. These are given by

$$\begin{aligned} C_{a,b} &= R \\ A_{b,c} &= \{(x^1, x^2, x^3) : x^3 = V_2(x^1, x^2)\} \\ D_{a,b} &= \{(x^1, x^2, x^3) : x^3 \geq V_2(x^1, x^2)\} \\ D_{b,c} &= S_c \end{aligned}$$

with R given by equation 4. The transition maps are

$$\begin{aligned} G_{a,b} : C_{a,b} \rightarrow D_{a,b}, G_{a,b}(x) = (x, \theta - V_2(x)) \\ G_{b,c} : A_{b,c} \rightarrow D_{b,c}, G_{b,c}(x) = (x^1, x^2) \end{aligned}$$

The interpretation is as follows. v_1 starts moving in S_a; if $x_1(.)$ enters $C_{a,b}$ then it may continue in S_a, or take a controlled jump to S_b. In the case of a controlled jump, the transition map $G_{a,b}$ maps the current state of v_1 to a state extended to include the optimal amount of fuel remaining in v_2 at the same location after departing from β with an initial amount of fuel θ. In S_b, the positions of the two vehicles coincide; there is an autonomous jump from S_b to S_c when the trajectory of the system hits $A_{b,c}$. This means that v_2 had to leave, since there was just enough fuel to go back to β. The jump relation consists of eliminating the third component of the state. The transition maps imply that v_2 uses fuel optimal strategies to travel to the meeting point and to reach β after leaving v_1. One could ask why is it necessary to include the discrete state c in the model (instead of having the autonomous jump from discrete state b to discrete state a). An autonomous transition from b to a could lead to trajectories in the controlled jump set $C_{a,b} = R \subset S_a$. But this jump can only be taken once. We need to keep track of the jump. We do this with the discrete state c.

In what follows we adopt the notation from [12]. Time is measured continuously with a real variable t in $[0, +\infty)$ and the state variable is (x, v). Trajectories are piecewise continuous in x and are normalized to be right-continuous. The hybrid control input is $I = (\{t_0, u_{v(0)}(.)\}\{t_i, u_{v(i)}\}_1^N), N \in \{0, 1, 2\}$, where $t_i \leq t_{i+1} (t_0 = 0)$ gives the sequence of times selected to switch the discrete dynamics. The activation of hybrid control input can only take place in the set C, or in the boundary of the set A. This spatial dependence translates to time dependence as follows.

Given (x, v) and $u(.)$, define the hitting times of A and J as

$$\begin{aligned} T^A(x, v, u(.)) = \inf\{t \geq 0 : (x(t), v) \in A\} \\ T^J(x, v, u(.)) = \inf\{t \geq 0 : (x(t), v) \in J\} \end{aligned}$$

where $x(.)$ is the trajectory departing from (x, v) under the control function $u(.)$.

Definition 3.1: Given a hybrid state (x, v) a hybrid control I is called an admissible control with respect to (x, v) if:

- $0 = t_0, t_i \leq t_{i+1}$
- $T^J(x(t_i^+), v, u(.)) \leq t_{i+1} - t_i \leq T^A((t_i^+), v, u(.))$

This means that between discrete jumps the trajectory may evolve in J. Jumps may take place in C and must take place in ∂A (the boundary of A).

In our model $D_{a,b} \cap A_{b,c} \neq \emptyset$ and $D_{a,b}$ is not a closed set. This makes it possible for an instantaneous jump from discrete state a to c to occur: first as a controlled jump from a to b at the points in ∂R, and then as an autonomous jump to c. This problem can be solved by changing these sets to impose a strictly positive distance between them.

Let $I(x, v)$ denote an admissible control with respect to (x, v) and $\Lambda(x, v)$ denote the set of all admissible controls.

Proposition 3.3: Given an initial hybrid state (x, v) the hybrid system possesses a unique hybrid execution.

Proof. The proof follows standard arguments from [11]. □

D. Optimal collaborative control

Now consider the running cost maps $k_v : S_v \times U_v \rightarrow \Re^+$:

$$\begin{aligned} k_a(x, u) &= k_1(x, u) \\ k_b(x, u) &= \sigma l(x, x) k_1((x^1, x^2), u_1) - \\ &\qquad (1 - \sigma) g_2((x^1, x^2), u_2) \\ k_c(x, u) &= k_1(x, u) \end{aligned}$$

where $\sigma \in [0, 1]$. An explanation for the definition of k_b (and σ) is in order. The positions of the two vehicles coincide in the discrete state b. However, the minimization of the path cost for v_1 may not be compatible with the minimization of the fuel consumption for v_2. The problem is that v_2 is fuel constrained. The longer the fuel lasts, the longer v_1 benefits from the path coordination. We model this trade-off with $k_b(x, u)$ which is a convex combination of the two other cost functions.

Consider the coordinated path optimization problem for v_1. The cost of a path joining (α, a) and (γ, v) is

$$\tilde{J}_1((I(\alpha, a), (\gamma, v), \sigma) = \sum_{i=0}^{N} \int_{t_i}^{t_{i+1}} k_{v(i)}(x(s), u_{v(i)}(s)) ds \tag{5}$$

where $N \leq 2, t_{N+1} = t_f$ and $x(t_f) = \gamma$.

We introduce the explicit dependence on σ to remind us that the optimal solution depends on this parameter.

Problem 2: [Coordinated] Find

$$\inf_{I(\alpha,a)\in\Lambda(\alpha,a)} \tilde{J}_1(I(\alpha,a),(\gamma,v),\sigma) \tag{6}$$

Let T denote the set of points reachable by v_2 in S_b under the fuel constraint θ for a round-trip from β. T is the set of all $(x^1,x^2,x^3)\in S_b$ such that the first two components (x^1,x^2) are in R and the last component (x^3) satisfies the fuel constraint:

$$T=\{x\in S_b:(x^1,x^2)\in R\wedge(x^3\geq V_2(x^1,x^2))\wedge ((\theta-V_2(x^1,x^2))\geq x^3)\}$$

Remark 2: $M=\{S_b\backslash T,b\}$ is not reachable in S.

IV. Dynamic programming

In the spirit of DP we embed Problem 2 in a family of optimization problems where the final position varies. Introduce the value function

$$V(x,v,\sigma)=\inf_{I(\alpha,a)\in\Lambda(\alpha,a)} \tilde{J}_1(I(\alpha,a),(x,v),\sigma)$$
$$V(\alpha,a,\sigma)=0$$

where $\forall x\in(S_b\backslash T):V(x,b,\sigma)=+\infty$.

The fact that not all points in S_b are reachable under the constraints imposed on v_2 leads to this extended-valued value function.

In what follows we drop the explicit dependence of V on σ to simplify the notation.

The following theorem, presented without proof, states two important properties of the value function.

Theorem 1: The value function $V(x,v)$ is bounded and continuous in $S\backslash\, M$.

The following theorems can be proved with the help of the results from [12].

Theorem 2: The value function $V(x,v)$ satisfies the principle of optimality for every $v\in\{a,b,c\}$.

Theorem 3: The value function $V(x,v)$ is the viscosity solution of the HJB equation.

$$V_t(x,v)+\inf_{u\in U}[V_x(x,v)\cdot f_v(x,u)-k_v(x,u)]=0$$
$$V(\alpha,a)=0$$

V. Optimal strategies

The optimal strategy for v_1 is derived from the value function $V(x,v)$. This requires some additional computations.

The position of v_1 is given by the continuous state of the hybrid automaton in the discrete states a and c, and by the first two components of the continuous state in the discrete state b; the third component, x^3, is the fuel remaining in v_2. However, the value function V in b depends not only on the position of v_1 (x^1,x^2), but also on the fuel remaining in v_2 (x^3). An additional minimization over x^3 is required. This is done next with the help of a new function, $\tilde{V}:\mathbb{R}^2\rightarrow\mathbb{R}$.

$$\tilde{V}(x,a)=V(x,a)$$
$$\tilde{V}(x,b)=\min_{x^3\in[V_2(x),\theta-V_2(x)]} V((x,x^3),b)$$
$$\tilde{V}(x,c)=V(x,c)$$

$\tilde{V}(x,a)$ is also the optimal value function for Problem 1.

Keep in mind that the discrete state keeps the history of the system. So v_1 can reach a same position in the three discrete states. To find the optimal path cost at $x\in\mathbb{R}^2$ we need to drop the dependence of $\tilde{V}$ on the discrete state with another minimization. This is done with the the help of a new function, $\overline{V}(x):\mathbb{R}^2\rightarrow\mathbb{R}$.

$$\overline{V}(x)=\min_{v\in\{a,b,c\}}\tilde{V}(x,v) \tag{7}$$

The optimal discrete state at x is given by

$$v^*=argmin_{v\in\{a,b,c\}}\tilde{V}(x,v) \tag{8}$$

Observe that v^* is not necessarily a singleton. We summarize these observations in the theorem.

Theorem 4: $\overline{V}(\gamma)$ is the optimal value for solving Problem 2. If $v^*=a$ then path coordination is not optimal.

The optimal control is given by u^* as follows

$$u^*=argmin_{u\in U}\quad V_t(x,v)+ [V_x(x,v)\cdot f_v(x,u)-k_v(x,u)] \tag{9}$$

Both the dynamics and the cost function do not depend directly on time. This simplifies the coordination of the optimal paths for the case when path coordination is the optimal solution: the vehicles are required to meet at the point where the two paths intersect for the first time.

We now study the conditions under which the solutions to Problems 1 and 2 differ. These are aimed at simplifying the process of finding numerical solutions to the coordinated problem.

Proposition 5.1: Let $\Upsilon=V(\gamma,a)$ and $Q=\{x\in S_a: V(x,a)\leq\Upsilon\}$. If $Q\cap R=\emptyset$, then the solutions of Problems 1 and 2 coincide.

*P*roof. The condition $Q\cap R=\emptyset$ means that γ can be reached with cost budget less than the one required to reach the set R, where coordination is possible. □

Proposition 5.2: The optimal cost for Problem 2 is l times the optimal cost for Problem 1 when there exists a trajectory $x_2(.)$ leaving β passing through α and γ and returning to β such that: 1) $x_2(.)$ satisfies the fuel constraint θ; and 2) the segment of $x_2(.)$ joining α and γ coincides with the optimal path for Problem 1.

Proof. Consider first that v_2 is not fuel constrained. Then, the trajectories of v_2 can be made to coincide with the trajectories of v_1 along the path for v_1. This means that: 1) there exists a path as the one in the statement of the proposition; and 2) that v_1 benefits from a constant cost reduction along its path. Now consider the case when v_2 is fuel constrained. If there is a path satisfying the conditions of the proposition, the optimal cost for v_1 cannot be further reduced from the optimal level obtained without fuel constraints.□

VI. Numerical example

Consider Example 3.1 again. The computation of the value function becomes easier because of the simplicity of the considered cost function (piecewise constant over the state and input spaces, and time-invariant) and system dynamics. A custom algorithm was especially tailored to take in account those specific assumptions. Research of algorithms for more general classes of problems is a work underway. The value function is computed over a equally spaced grid.

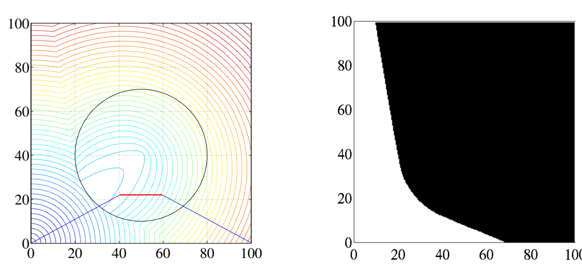
The computation of the value function is done in three stages. In the first stage, the system is in the discrete state a. Since the running cost, $k_1(x, u) = 1$, is independent of the input and vehicle's position, the optimal trajectory from the initial position to any position x is a straight line, traveled at unit speed (the maximum speed). It is trivial to note that $V(x, a) = \|x\|_2$. Therefore the exact value of $V(x, a)$ is known at the grid points.

In the second stage, the algorithm considers only the points in T For each point in T, the algorithm computes the cost to every other point (in $\mathbb{R}^3$) that can be reached respecting the fuel constraint, and updates $V(x, b)$ accordingly. The computation of $\tilde{V}(x, b)$ is straightforward.

Another version of this algorithm, which computes directly $\tilde{V}(x, b)$ performing all computations on a two dimensional grid (therefore demanding smaller computation time), was also implemented. This version considers only the points in R: for each point in R, it computes the cost to every other point that can be reached respecting the fuel constraint, and updates $\tilde{V}(x, b)$ accordingly. However, this version does not allow the determination of the optimal trajectory using Eq. 9.

Finally, in the third stage the algorithm starts from the positions where $\tilde{V}(x, b)$ is finite, computed on the previous stage, and propagates the value function. In this final stage, $\overline{V}$ is computed as defined in Eq. 7.

The level sets of $\overline{V}$ are plotted on Fig. 2(a) along with the optimal trajectory from $\alpha = (0, 0)$ to $\gamma = (100, 0)$. The circle of radius 30 centered at $(50, 40)$ delimits R. Fig. 2(b) identifies two distinct regions of the x-y plane: in white, the final destinations for which the optimal strategy is the uncoordinated motion (no collaborative operation of v_1 with v_2); in black, the final destinations for which v_1 will benefit from coordinated motion with v_2, i.e., the set of points x such that $V(x, b) < V(x, a)$ and $V(x, c) < V(x, a)$.



(a) Level sets of $\overline{V}$ and optimal trajectory of v_1 for Example 3.1.

(b) Destinations on black region benefit from coordinated motion.

Fig. 2. Optimal strategies.

VII. Conclusions

We have formulated and solved a path coordination problem to illustrate the use of DP techniques in collaborative control problems. The problem consists of minimizing the path cost for v_1 when this cost is a discontinuous function of the relative positions of the two vehicles and v_2 is required to return to its starting point. The problem is formulated as an optimal hybrid control problem. The state has a memoryless component and a component with memory. The autonomous and controlled jump sets are both given by the set reachable by v_2 when departing from β under the given fuel constraints. The optimal strategies for both vehicles are derived from value function, which depends on the location of v_1 and on the discrete state. The optimal path cost for v_1 at a given location is given by two sequential minimizations of the value function for the optimal hybrid control problem. Transitions in the hybrid automaton take place when collaboration is the optimal solution. The transition to the second state is taken by v_1 under the assumption that it meets v_2 and that v_2 followed a fuel-optimal path. This is a non-standard hybrid control problem: the jump sets are given by reach sets; and the value function for the coordinated problem assumes compatible optimal behavior by v_2 (this is given by a different value function for v_2).

Future work concerns investigating other collaborative control problems in the DP framework and removing some of the more restrictive assumptions.

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