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A Q, L Factorization of Norm-Optimal Iterative Learning Control

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Abstract—**In this paper we consider the Norm-Optimal Iterative Learning Control (ILC) problem for discrete-time linear multi-input, multi-output systems. The solution to this problem is well known and naturally factors into a form with a filter on the previous control, $\mathbf{L_u}$ and a filter on the previous error, $\mathbf{L_e}$. We show that this solution can always be factored into a Q,L form where Q filters the previous control and QL filters the previous error. This latter form is popularized with frequency domain ILC designs, and this common factorization suggests some general relationships between Norm-Optimal and frequency domain design, which are explored. Although the Q,L factorization is well known for some special cases, the results here are general and include differently dimensioned control and observation windows.**

I. Introduction

ITERATIVE learning control (ILC) [1],[2] is used to improve the performance of systems that repeat the same operation many times. ILC uses the tracking errors from previous iterations of the repeated motion to generate a feedforward control signal for subsequent iterations. Convergence of the learning process results in a feedforward control signal that is customized for the repeated motion, yielding very low tracking error.

Many ILC design methods for linear discrete-time, time-invariant (LTI) systems can be categorized as either frequency domain methods [3]-[5] ([6],[7] for continuous-time frequency domain) or Norm-Optimal methods [8]-[10]. Frequency domain design methods often assume a learning algorithm of the form [3],[5],

$$u_{j+1} = \mathcal{Q}\left(u_j + \mathcal{L}e_j\right) \tag{1}$$

where u_j is the Fourier Transform (FT) of the control signals, e_j is the FT of the error signal, and $\mathcal{Q}$ and $\mathcal{L}$ are LTI filters. Although designed in the frequency domain, these algorithms are typically applied as finite-horizon time-domain filters. In the time-domain (1) can be written as [11],

$$\mathbf{u}_{j+1} = \boldsymbol{\mathcal{Q}}\left(\mathbf{u}_j + \boldsymbol{\mathcal{L}}\mathbf{e}_j\right), \tag{2}$$

where $\mathbf{u}_j$ is a vector of control inputs, $\mathbf{e}_j$ is a vector of error outputs, and $\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{L}}$ are lifted [11] time-domain constructions of $\mathcal{Q}$, $\mathcal{L}$ respectively.

Norm-Optimal ILC is analogous to linear quadratic optimal control for feedback systems. In this method the designer selects several time-domain weighting matrices in a particular quadratic cost function [8]-[10]. The optimal solution to this cost function is well known [9],[10] and naturally takes the form,

$$\mathbf{u}_{j+1} = \boldsymbol{\mathcal{L}}_{\mathbf{u}}\mathbf{u}_j + \boldsymbol{\mathcal{L}}_{\mathrm{e}}\mathbf{e}_j, \tag{3}$$

where $\boldsymbol{\mathcal{L}}_{\mathbf{u}}$ and $\boldsymbol{\mathcal{L}}_{\mathrm{e}}$ are given later in this article.

For some special cases, a factorization of the $\left\{\boldsymbol{\mathcal{L}}_{\mathbf{u}}, \boldsymbol{\mathcal{L}}_{\mathrm{e}}\right\}$ form into a $\left\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{L}}\right\}$ form may be obvious [2]. However, in the general case the existence of such a factorization may not be obvious, for example, when $\mathbf{u}_j$ and $\mathbf{e}_j$ are differently dimensioned. The difference in the structure of the $\left\{\boldsymbol{\mathcal{L}}_{\mathbf{u}}, \boldsymbol{\mathcal{L}}_{\mathrm{e}}\right\}$ Norm-Optimal solution and $\left\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{L}}\right\}$ frequency domain solution creates a discontinuity that makes bridging these two design approaches challenging. While in Norm-Optimal framework we may build intuition regarding $\left\{\boldsymbol{\mathcal{L}}_{\mathbf{u}}, \boldsymbol{\mathcal{L}}_{\mathrm{e}}\right\}$, it is unclear how this transmits to our intuition of $\left\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{L}}\right\}$.

In this paper we show that the Norm-Optimal algorithm always admits a solution that factors into $\left\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{L}}\right\}$. We find several properties of Norm-Optimal filters $\left\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{L}}\right\}$ and discuss some relationships to frequency domain filters $\left\{\mathcal{Q}, \mathcal{L}\right\}$. It is not our intention to advocate one design method over the other, but rather to find similarities between the methods in order to bridge the gap. Therefore, our comparison focuses on structural similarities between the Norm-Optimal $\left\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{L}}\right\}$ and frequency-domain $\left\{\mathcal{Q}, \mathcal{L}\right\}$, instead of performance metrics.

The remainder of this paper is organized as follows. In Section II we present Norm-Optimal ILC. The $\left\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{L}}\right\}$ factorization of Norm-Optimal ILC is developed in Section III. Properties of the Norm-Optimal $\left\{\boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{L}}\right\}$ are presented and discussed in Section IV. Finally, conclusions are given in Section V.

II. Norm-Optimal ILC

Consider the discrete-time multi-input multi-output (MIMO) LTI dynamic system,

$$e_j(k) = P(z)u_j(k) + e_0(k), \quad j = 1, 2, \ldots \tag{4}$$

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where k is the time index, $j = 1,2,\ldots$ is the iteration index, $u_j(k) \in \mathbb{R}^n$ is the control input, $e_j(k) \in \mathbb{R}^m$ is the tracking error, and $e_0(k) \in \mathbb{R}^m$ is the initialization error signal. Let z^{-1} be the backward time-shift operator $z^{-1}x(k) = x(k-1)$. We assume that $P(z)$ is an n-input, m-output asymptotically stable system that can be written as

$$P(z) = \sum_{k=0}^{\infty} p(k) z^{-k}, \tag{5}$$

where

$$p(k) = \begin{bmatrix} p^{11}(k) & \cdots & p^{1n}(k) \\ \vdots & & \vdots \\ p^{m1}(k) & \cdots & p^{mn}(k) \end{bmatrix}$$

and $p(k) \in \mathbb{R}^{m\times n}$. $p(0), p(1), \ldots$ are the Markov parameters of $P(z)$ and the sequence $\{p(0), p(1), p(2), \ldots\}$ is the impulse response. Repeating disturbances [11], repeated nonzero initial conditions [4], and systems augmented with feedback and feedforward control [11] can be captured in $e_0(k)$.

Consider the control window $u_j(0), \ldots, u_j(N_u)$ and the observation window $e_j(o), \ldots, e_j(o+N_e)$, where $o \in \mathbb{Z}$ is the offset, and assume that $u_j(k) = 0$ for $k < 0$ and $k > N_u$. Now consider the vector description of u and e as,

$$\mathbf{u}_j = \begin{bmatrix} u_j^T(0) & \cdots & u_j^T(N_u) \end{bmatrix}^T,$$
$$\mathbf{e}_j = \begin{bmatrix} e_j^T(o) & \cdots & e_j^T(o+N_e) \end{bmatrix}^T.$$

Then (4) can be written compactly as a lifted system [12],

$$\mathbf{e}_j = \mathbf{P}\mathbf{u}_j + \mathbf{e}_0, \tag{6}$$

where $\mathbf{P} \in \mathbb{R}^{mN_e \times nN_u}$ is the convolution matrix composed of the Markov parameters $p(0), p(1), \ldots$. The reader may be most familiar with the special case $o = 0$ and $N_u = N_e = N$, where,

$$\mathbf{P} = \begin{bmatrix} p_0 & 0 & 0 & \cdots & 0 \\ p_1 & p_0 & 0 & \ddots & \vdots \\ p_2 & p_1 & p_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ p_N & \cdots & p_2 & p_1 & p_0 \end{bmatrix}. \tag{7}$$

The goal of Norm-Optimal ILC is to select $\mathbf{u}_{j+1}$ as

$$\arg\min_{\mathbf{u}_{j+1}} \mathbf{J}_{j+1}, \tag{8}$$

where

$$\mathbf{J}_{j+1} = \mathbf{e}_{j+1}^T \mathbf{Q} \mathbf{e}_{j+1} + \left(\mathbf{u}_{j+1} - \mathbf{u}_j\right)^T \mathbf{R} \left(\mathbf{u}_{j+1} - \mathbf{u}_j\right) + \mathbf{u}_{j+1}^T \mathbf{S} \mathbf{u}_{j+1}, \tag{9}$$

and $\mathbf{Q} = \mathbf{Q}^T \geq 0$, $\mathbf{R} = \mathbf{R}^T \geq 0$, and $\mathbf{S} = \mathbf{S}^T \geq 0$ are appropriately sized real matrices. The solution to this problem is well known [9],[10] and is obtained by substituting system dynamics from (6) into (9),

$$\begin{aligned} \mathbf{J}_{j+1} &= \left(-\mathbf{P}\mathbf{u}_{j+1} + \mathbf{e}_0\right)^T \mathbf{Q} \left(-\mathbf{P}\mathbf{u}_{j+1} + \mathbf{e}_0\right) \\ &\quad + \left(\mathbf{u}_{j+1} - \mathbf{u}_j\right)^T \mathbf{R} \left(\mathbf{u}_{j+1} - \mathbf{u}_j\right) + \mathbf{u}_{j+1}^T \mathbf{S} \mathbf{u}_{j+1}, \\ &= \mathbf{u}_{j+1}^T \left(\mathbf{P}^T \mathbf{Q} \mathbf{P} + \mathbf{R} + \mathbf{S}\right) \mathbf{u}_{j+1} - 2\mathbf{u}_{j+1}^T \mathbf{P}^T \mathbf{Q} \mathbf{e}_0 \\ &\quad + \mathbf{e}_0^T \mathbf{Q} \mathbf{e}_0 - 2\mathbf{u}_{j+1}^T \mathbf{R} \mathbf{u}_j + \mathbf{u}_j^T \mathbf{R} \mathbf{u}_j. \end{aligned}$$

and setting $\partial \mathbf{J}_{j+1} / \partial \mathbf{u}_{j+1} = 0$. If $\mathbf{P}^T\mathbf{Q}\mathbf{P} + \mathbf{R} + \mathbf{S}$ is nonsingular, the optimal control is then given by,

$$\mathbf{u}_{j+1} = \left(\mathbf{P}^T \mathbf{Q} \mathbf{P} + \mathbf{R} + \mathbf{S}\right)^{-1} \left(\mathbf{R}\mathbf{u}_j + \mathbf{P}^T \mathbf{Q} \mathbf{e}_0\right). \tag{10}$$

Rewriting (6) as $\mathbf{e}_0 = -\mathbf{P}\mathbf{u}_j + \mathbf{e}_j$ and substituting into (10), we get

$$\mathbf{u}_{j+1} = \left(\mathbf{P}^T \mathbf{Q} \mathbf{P} + \mathbf{R} + \mathbf{S}\right)^{-1} \left(\left(\mathbf{P}^T \mathbf{Q} \mathbf{P} + \mathbf{R}\right) \mathbf{u}_j + \mathbf{P}^T \mathbf{Q} \mathbf{e}_j\right). \tag{11}$$

The result in (11) can naturally be written as separate filters on $\mathbf{u}_j$ and $\mathbf{e}_j$ as in (3) where,

$$\begin{aligned} \mathcal{L}_{\mathbf{u}} &= \left(\mathbf{P}^T \mathbf{Q} \mathbf{P} + \mathbf{R} + \mathbf{S}\right)^{-1} \left(\mathbf{P}^T \mathbf{Q} \mathbf{P} + \mathbf{R}\right), \\ \mathcal{L}_{\mathrm{e}} &= \left(\mathbf{P}^T \mathbf{Q} \mathbf{P} + \mathbf{R} + \mathbf{S}\right)^{-1} \mathbf{P}^T \mathbf{Q}. \end{aligned} \tag{12}$$

In the following section we factor $\{\mathcal{L}_{\mathbf{u}}, \mathcal{L}_{\mathrm{e}}\}$ into $\{\mathcal{Q}, \mathcal{L}\}$.

III. $\{\mathcal{Q}, \mathcal{L}\}$ FACTORIZATION OF THE NORM-OPTIMAL ALGORITHM.

Before showing the $\{\mathcal{Q}, \mathcal{L}\}$ equivalency of the $\{\mathcal{L}_{\mathbf{u}}, \mathcal{L}_{\mathrm{e}}\}$ algorithm in (3), (12), we present some definitions and supporting lemmas. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times m}$, $\mathbf{B} = \mathbf{B}^T \geq 0$. We define $\mathbf{B}^{1/2}$ as the factorization of $\mathbf{B}$ satisfying $\mathbf{B} = \left(\mathbf{B}^{1/2}\right)^T \mathbf{B}^{1/2}$. $\mathbf{A}^\dagger$ is the pseudo-inverse of $\mathbf{A}$ [13]. The null space of $\mathbf{A}$ is defined as $\mathcal{N}(\mathbf{A}) = \{\mathbf{x} : \mathbf{A}\mathbf{x} = 0, \mathbf{x} \in \mathbb{R}^n\}$ and the orthogonal complement of the null space is defined as $\mathcal{N}(\mathbf{A})^\perp$.

Lemma 1: For $\mathbf{P} \in \mathbb{R}^{m\times n}$, $\mathbf{Q} \in \mathbb{R}^{m\times m}$, $\mathbf{Q} = \mathbf{Q}^T \geq 0$, then,

$$\mathbf{P}^T\mathbf{Q} = \mathbf{P}^T\mathbf{Q}\mathbf{P}\left(\mathbf{Q}^{1/2}\mathbf{P}\right)^\dagger \mathbf{Q}^{1/2} \tag{13}$$

Proof: $\mathbf{Q} = \left(\mathbf{Q}^{1/2}\right)^T \mathbf{Q}^{1/2}$ because $\mathbf{Q} = \mathbf{Q}^T \geq 0$, so,

$$\mathbf{P}^T\mathbf{Q} = \left(\mathbf{Q}^{1/2}\mathbf{P}\right)^T \mathbf{Q}^{1/2}.$$

For a real matrix $\mathbf{A}$, we have $\mathbf{A}^T = \mathbf{A}^T\mathbf{A}\mathbf{A}^\dagger$, and thus we can write

$$\begin{aligned} \mathbf{P}^T\mathbf{Q} &= \left(\mathbf{Q}^{1/2}\mathbf{P}\right)^T \left(\mathbf{Q}^{1/2}\mathbf{P}\right) \left(\mathbf{Q}^{1/2}\mathbf{P}\right)^\dagger \mathbf{Q}^{1/2}, \\ &= \mathbf{P}^T\mathbf{Q}\mathbf{P}\left(\mathbf{Q}^{1/2}\mathbf{P}\right)^\dagger \mathbf{Q}^{1/2}, \end{aligned}$$

which completes our proof. ■

Lemma 2: For $\mathbf{P} \in \mathbb{R}^{m\times n}$, $\mathbf{Q} \in \mathbb{R}^{m\times m}$, $\mathbf{Q} = \mathbf{Q}^T \geq 0$, $\mathbf{R} \in \mathbb{R}^{n\times n}$, $\mathbf{R} = \mathbf{R}^T \geq 0$, then,

$$\mathbf{P}^T\mathbf{QP} = \left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)^{\dagger}\mathbf{P}^T\mathbf{QP}. \quad (14)$$

Proof: Let $\mathbf{U\Sigma U}^T$ be the singular value decomposition (SVD) of $\mathbf{P}^T\mathbf{QP}+\mathbf{R}$, where $\mathbf{U}=\begin{bmatrix}\mathbf{U}_1 & \mathbf{U}_2\end{bmatrix}$ and $\mathbf{U}_2$ is the basis for $\mathcal{N}\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)$. Then, the SVD of $\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)^{\dagger}$ is $\mathbf{U\Sigma}^{\dagger}\mathbf{U}^T$, from which we conclude that

$$\mathcal{N}\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right) = \mathcal{N}\left(\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)^{\dagger}\right).$$

Furthermore,

$$\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)^{\dagger} = \mathbf{U}^T\mathbf{\Sigma\Sigma}^{\dagger}\mathbf{U},$$

where

$$\mathbf{\Sigma\Sigma}^{\dagger} = \begin{bmatrix}\mathbf{I}_r & 0\\ 0 & 0\end{bmatrix},$$

$\mathbf{I}_r$ is the $r\times r$ identity matrix and $r = rank\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)$. Therefore,

$$\mathcal{N}\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right) = \mathcal{N}\left(\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)^{\dagger}\right),$$

and

$$\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)^{\dagger}x = x \text{ for } x\in\mathcal{N}\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)^{\perp}.$$

Finally, because

$$\begin{aligned}\mathcal{N}\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right) &= \mathcal{N}(\mathbf{R})\cap\mathcal{N}\left(\mathbf{P}^T\mathbf{QP}\right)\\ &\subseteq \mathcal{N}\left(\mathbf{P}^T\mathbf{QP}\right),\end{aligned}$$

we conclude that

$$\mathbf{P}^T\mathbf{QP}x = \left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)^{\dagger}\mathbf{P}^T\mathbf{QP}x$$

for all $x\in\mathcal{N}\left(\mathbf{P}^T\mathbf{QP}\right)^{\perp}$, which proves (14). ∎

We are now able to show that the Norm-Optimal $\{\mathcal{L}_{\mathbf{u}},\mathcal{L}_{\mathrm{e}}\}$ algorithm in (3), (12) can always be factored into the $\{\mathcal{Q},\mathcal{L}\}$ algorithm in (2).

Theorem 1: Given $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$ as defined above, the learning algorithm given in (3), (12) is equivalent to the learning algorithm in (2), where,

$$\begin{aligned}\mathcal{Q} &= \left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}+\mathbf{S}\right)^{-1}\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right),\\ \mathcal{L} &= \left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)^{\dagger}\mathbf{P}^T\mathbf{QP}\left(\mathbf{Q}^{1/2}\mathbf{P}\right)^{\dagger}\mathbf{Q}^{1/2}.\end{aligned} \quad (15)$$

Proof: The proof follows from the substitutions provided by lemmas 1 and 2. From (3), (12) and Lemma 1 we have,

$$\begin{aligned}\mathbf{u}_{j+1} = &\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}+\mathbf{S}\right)^{-1}\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)\mathbf{u}_j\\ &+\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}+\mathbf{S}\right)^{-1}\mathbf{P}^T\mathbf{QP}\left(\mathbf{Q}^{1/2}\mathbf{P}\right)^{\dagger}\mathbf{Q}^{1/2}\mathbf{e}_j.\end{aligned}$$

Using Lemma 2,

$$\begin{aligned}\mathbf{u}_{j+1} = &\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}+\mathbf{S}\right)^{-1}\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)\mathbf{u}_j\\ &+\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}+\mathbf{S}\right)^{-1}\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}\right)^{\dagger}\\ &\cdot\mathbf{P}^T\mathbf{QP}\left(\mathbf{Q}^{1/2}\mathbf{P}\right)^{\dagger}\mathbf{Q}^{1/2}\mathbf{e}_j.\end{aligned}$$

The factorization into $\{\mathcal{Q},\mathcal{L}\}$ follows directly. ∎

IV. Some Properties of the Norm-Optimal $\mathcal{Q}$ and $\mathcal{L}$.

In this section, we present and discuss some properties of Norm-Optimal filters $\mathcal{Q}$ and $\mathcal{L}$. When possible, we relate these properties to properties of frequency-domain designs for $\mathcal{Q}$ and $\mathcal{L}$. We begin with a discussion of the Norm-Optimal $\mathcal{Q}$.

A. The Norm-Optimal $\mathcal{Q}$

The Norm-Optimal $\mathcal{Q}$ has the following two properties.

P$\mathcal{Q}$1) $\mathcal{Q}=\mathcal{Q}_2\mathcal{Q}_1$, where $\mathcal{Q}_1=\mathcal{Q}_1^T>0$, $\mathcal{Q}_2=\mathcal{Q}_2^T\geq 0$.

P$\mathcal{Q}$2) $0\leq\sigma_i\leq 1$, $i=1,2,\ldots,mN_e$, where σ_i is the i^{th} singular value of $\mathcal{Q}$.

P$\mathcal{Q}$1) follows directly by selecting $\mathcal{Q}_2=\left(\mathbf{P}^T\mathbf{QP}+\mathbf{R}+\mathbf{S}\right)^{-1}$ and $\mathcal{Q}_1=\mathbf{P}^T\mathbf{QP}+\mathbf{R}$. The bound on σ_i in P$\mathcal{Q}$2) follows from the fact that $\mathbf{P}^T\mathbf{QP}+\mathbf{R}+\mathbf{S}\geq\mathbf{P}^T\mathbf{QP}+\mathbf{R}$ and positive semi-definiteness of $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$. We discuss some analogous properties for frequency-domain $\mathcal{Q}$ in the following two subsections.

1) Symmetric $\mathcal{Q}$ and Zero-Phase $\mathcal{Q}$

In frequency-domain design it is sometimes advocated [4] that the $\mathcal{Q}$-filter be designed with zero-phase to prevent time shifts in the control signal that may lessen performance. One popular method of achieving zero-phase shift is the so-called *filtfilt* method [4]. In the filtfilt method a signal is filtered through a causal, nonzero-phase filter, then reversed and filtered again through the same filter, and finally reversed again. The forward and then backward filtering results in zero phase shift of the signal. In the time domain, the filtfilt process can be represented by the product of a lower-triangular matrix and its upper-triangular transpose. Using the Cholesky factorization, we can find lower triangular $\mathcal{Q}_{C,1}$ and $\mathcal{Q}_{C,2}$ such that $\mathcal{Q}_1=\mathcal{Q}_{C,1}^T\mathcal{Q}_{C,1}$ and $\mathcal{Q}_2=\mathcal{Q}_{C,2}^T\mathcal{Q}_{C,2}$. Therefore, $\mathcal{Q}=\left(\mathcal{Q}_{C,2}^T\mathcal{Q}_{C,2}\right)\left(\mathcal{Q}_{C,1}^T\mathcal{Q}_{C,1}\right)$ is a filtfilt operation with $\mathcal{Q}_{C,1}$ and another filtfilt operation with $\mathcal{Q}_{C,2}$.

2) Unit norm-bounded $\mathcal{Q}$ and $\mathcal{Q}$

The unit norm bound in P$\mathcal{Q}$2) identifies that $\mathcal{Q}$ is an attenuating filter. In frequency-domain design $\mathcal{Q}$ is also an attenuating filter with $\mathcal{H}_\infty$-norm less than one. $\mathcal{Q}$ is usually a lowpass filter that is designed for robust learning. Because $\mathcal{Q}$ has very low gain at high frequencies, high-frequency learning is effectively disabled and robust to high-frequency model uncertainty that may otherwise cause instability. $\mathcal{Q}$ may also serve in a similar robustifying role. Consider the SVD of $\mathcal{Q}=\mathbf{V}_{\mathcal{Q}}\Sigma_{\mathcal{Q}}\mathbf{U}_{\mathcal{Q}}^T$, where $\Sigma_{\mathcal{Q}}=diag\{\Sigma_{\mathcal{Q},1},\Sigma_{\mathcal{Q},2}\}$, $\Sigma_{\mathcal{Q},2}\approx 0$ are small singular values, and $\mathbf{U}_{\mathcal{Q}} = \left[\mathbf{U}_{\mathcal{Q},1} \quad \mathbf{U}_{\mathcal{Q},2}\right]$, where $\mathbf{U}_{\mathcal{Q},2}$ are the directions associated with $\Sigma_{\mathcal{Q},2}$. Then, control signals in $span\left\{\mathbf{U}_{\mathcal{Q},2}\right\}$ are essentially disabled. When $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{S}$ are selected in such a way that $span\left\{\mathbf{U}_{\mathcal{Q},2}\right\}$ maps to high frequency control, $\mathcal{Q}$ acts in a similar role as the frequency domain $\mathcal{Q}$.

B. *The Norm-Optimal* $\mathcal{L}$.

In this section we begin by presenting a further factorization of the learning filter $\mathcal{L}$ that will be useful for relating to the frequency domain. We factor $\mathcal{L}$ as,

$$\mathcal{L} = \mathbf{H}\mathbf{P}_{inv} \tag{16}$$

where,

$$\mathbf{H} = \left(\mathbf{P}^T\mathbf{Q}\mathbf{P} + \mathbf{R}\right)^{\dagger}\mathbf{P}^T\mathbf{Q}\mathbf{P}, \tag{17}$$

is $nN_u \times nN_u$ and

$$\mathbf{P}_{inv} = \left(\mathbf{Q}^{1/2}\mathbf{P}\right)^{\dagger}\mathbf{Q}^{1/2} \tag{18}$$

is $nN_u \times mN_e$. Here, we call $\mathbf{H}$ the *learning rate* filter and $\mathbf{P}_{inv}$ the *weighted system inverse* filter.

1) $\mathbf{P}_{inv}$ *and Model Inversion*

We begin with $\mathbf{P}_{inv}$ and consider the weighted least-squares model-inversion problem,

$$\operatorname{argmin}_{\mathbf{x}} \left\|\mathbf{P}\mathbf{x} - \mathbf{b}\right\|_{\mathbf{Q}}. \tag{19}$$

Using relationships between the pseudo-inverse and the least-squares problem, we find that

$$\begin{aligned}\operatorname{argmin}_{\mathbf{x}} \left\|\mathbf{P}\mathbf{x} - \mathbf{b}\right\|_{\mathbf{Q}} &= \operatorname{argmin}_{\mathbf{x}} \left\|\mathbf{Q}^{1/2}\mathbf{P}\mathbf{x} - \mathbf{Q}^{1/2}\mathbf{b}\right\| \\ &= \left(\mathbf{Q}^{1/2}\mathbf{P}\right)^{\dagger}\mathbf{Q}^{1/2}\mathbf{b} \\ &= \mathbf{P}_{inv}\mathbf{b},\end{aligned}$$

and therefore $\mathbf{P}_{inv}$ is the optimal filter for the $\mathbf{Q}$-weighted inverse of $\mathbf{P}$.

Consider now a special case where $\mathbf{Q}$ is full rank and $\mathbf{P}$ is full row rank. We interpret this as the fully controllable case because we can reach any $\mathbf{e}_j$ with $\mathbf{u}_j$, and fully weighted because no directions of $\mathbf{Q}$ have zero weighting. The following corollary shows that for this case a simpler form of $\mathbf{P}_{inv}$ can be obtained.

Corollary 1: If $rank(\mathbf{Q}) = mN_e$ and $rank(\mathbf{P}) = mN_e$, then

$$\mathbf{P}_{inv} = \mathbf{P}^{\dagger}.$$

Proof: Because $\mathbf{Q}$ is full rank and $\mathbf{P}$ is full row rank, then $\mathbf{Q}^{1/2}\mathbf{P}$ is full row rank. For a real matrix $\mathbf{A}$ with full row rank we have $\mathbf{A}^{\dagger} = \mathbf{A}^T\left(\mathbf{A}\mathbf{A}^T\right)^{-1}$. Therefore, we find that,

$$\begin{aligned}\left(\mathbf{Q}^{1/2}\mathbf{P}\right)^{\dagger}\mathbf{Q}^{1/2} &= \left(\mathbf{Q}^{1/2}\mathbf{P}\right)^T\left(\mathbf{Q}^{1/2}\mathbf{P}\left(\mathbf{Q}^{1/2}\mathbf{P}\right)^T\right)^{-1}\mathbf{Q}^{1/2} \\ &= \mathbf{P}^T\left(\mathbf{Q}^{1/2}\right)^T\left(\mathbf{Q}^{1/2}\right)^{-T}\left(\mathbf{P}\mathbf{P}^T\right)^{-1}\left(\mathbf{Q}^{1/2}\right)^{-1}\mathbf{Q}^{1/2} \\ &= \mathbf{P}^T\left(\mathbf{P}\mathbf{P}^T\right)^{-1} = \mathbf{P}^{\dagger}.\end{aligned} \qquad \blacksquare$$

When $\mathbf{P}$ is square, Corollary 1 further simplifies to $\mathbf{P}_{inv} = \mathbf{P}^{-1}$. Therefore, we see that the Norm-Optimal algorithm is a general type of model-inversion algorithm [2]. Many frequency domain designs can be interpreted as model-inversion approaches, albeit oftentimes with reduced or simplified models of the inverse.

2) $\mathbf{H}$ *and Learning Rate*

$\mathbf{H}$ has the following two properties similar to those we found for $\mathcal{Q}$.

P $\mathbf{H}$ 1) $\mathbf{H} = \mathbf{H}_2\mathbf{H}_1$, where $\mathbf{H}_1 = \mathbf{H}_1^T > 0$, $\mathbf{H}_2 = \mathbf{H}_2^T > 0$.

P $\mathbf{H}$ 2) $0 \le \sigma_i \le 1$, $i = 1, 2, \ldots, mN_e$, where σ_i is the i^{th} singular value of $\mathbf{H}$.

As with $\mathcal{Q}$, we can interpret $\mathbf{H}$ as being analogous to a frequency domain filter that has zero-phase and is attenuating. Similar filters are used in frequency domain design to slow the rate of learning for robustness purposes and also to limit noise sensitivity. Most commonly, frequency domain designs use a scalar to slow the learning rate, as in $\mathcal{L} = \eta\mathcal{P}_{inv}$, where $0 < \eta \le 1$ and $\mathcal{P}_{inv}$ is an approximation of the inverse system. In some frequency domain designs, unit-bounded, zero-phase filters are used in the design of $\mathcal{L}$ [4].

V. Conclusions

In this work we showed that the Norm-Optimal ILC algorithm, which is naturally written in an $\{\mathcal{L}_{\mathbf{u}}, \mathcal{L}_{\mathbf{e}}\}$ form, can always be factored into the $\{\mathcal{Q}, \mathcal{L}\}$ form popularized by frequency domain ILC designs. Explicit formulas for the Norm-Optimal $\mathcal{Q}$ and $\mathcal{L}$ are given. We find that several properties of the $\mathcal{Q}$ and $\mathcal{L}$ filters are analogous to properties of the $\mathcal{Q}$ and $\mathcal{L}$ obtained using frequency domain designs. The discovered similarity in filters helps to bridge the oft dissociated Norm-Optimal and frequency domain ILC design approaches. This may lead to new design insights and combined Norm-Optimal-plus-frequency-domain design strategies.

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