Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Adaptive Fault-Tolerant Control of a Class of Nonlinear MIMO Systems

Xiaodong Zhang, Marios M. Polycarpou, and Thomas Parisini

Abstract— **This paper presents an integrated fault diagnosis and fault-tolerant control methodology for a class of nonlinear MIMO systems. Based on the fault information obtained during the diagnostic procedure, a fault-tolerant control component is designed to compensate for the effect of faults. In the presence of a fault, a baseline controller guarantees the boundedness of all system signals until the fault is detected. Then the controller is reconfigured after fault detection to compensate for the fault using online fault diagnostic information. Under certain assumptions, the stability and tracking performances of the closed-loop system are rigorously investigated. It is shown that, the system signals always remain bounded, and the output tracking error converges to a neighborhood of the origin of the state space.**

I. INTRODUCTION

During the past two decades, there has been significant research activity in the areas of fault diagnosis (see, for instance, [2], [3]) and fault-tolerant control (see, for instance, [1]). Early fault detection and isolation (FDI) can potentially avoid the development of more serious faults and malfunctions. Detailed fault information acquired by the fault diagnosis procedure is very valuable to fault-tolerant control design, since the key objective of fault-tolerant control is to compensate for the effect of such faults. However, links between fault diagnosis and fault-tolerant control are still limited, especially for nonlinear uncertain systems [1].

In previous work [10], the authors presented a unified methodology for detecting, isolating, and accommodating faults in a class of nonlinear uncertain dynamical systems. A fault diagnosis component is used for fault detection and isolation. Based on the fault information provided by the fault diagnosis procedure, a fault-tolerant control component is designed to compensate for the effect of faults. In this paper, we extend the results of [10] by considering multi-input-multi-output (MIMO) nonlinear uncertain system, while in [10] the fault-tolerant control design was analyzed for the single-input-single-output (SISO) case. Moreover, the new fault-tolerant control design presented in this paper removes an important assumption regarding uniform boundedness of modeling uncertainty that was used in [10].

X. Zhang is with the Department of Electrical Engineering, Wright State University, Dayton, OH 45435, USA xiaodong.zhang@wright.edu. His work is supported by the Research Challenge Fund of Wright State University.

M. M. Polycarpou is with Department of Electrical and Computer Engineering, University of Cyprus, Nicosia 1678, Cyprus. mpolycar@ucy.ac.cy. His work is supported by the Research Promotion Foundation of Cyprus.

T. Parisini is with Department of Electrical, Electronic and Computer Engineering University of Trieste, 34100 Trieste, Italy t.parisini@paperplaza.net

We consider a class of MIMO nonlinear systems in parametric-strict-feedback form [5], subject to unstructured (possibly nonlinear) modeling uncertainty and nonlinear faults. Specifically, the fault is assumed to be an unknown nonlinear function of the system states. The proposed fault-tolerant control scheme consists of two main components: an *on-line health monitoring (fault diagnosis) module* and a *controller (fault accommodation) module*. First, the closed-loop system stability in the presence of a fault, but before its detection, is investigated. Then in order to compensate for the effect of the fault, a fault-tolerant controller is designed, which is used after fault detection but before isolation. It is shown that the system signals always remain bounded, and the output tracking error converges to a neighborhood of the origin of the state space.

II. PROBLEM FORMULATION

A. Plant Model

We consider a class of nonlinear MIMO system given by

$$\begin{aligned} \dot{x} &= \bar{\phi}(x) + G(x)u + \eta(x,u,t) + \mathcal{B}(t-T_0)f(x) \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^m$ is the control input vector, $y \in \Re^q$ is the output vector, $\bar{\phi}$, $f : \Re^n \mapsto \Re^n$, $h : \Re^n \mapsto \Re^q$, and $\eta : \Re^n \times \Re^m \times \Re^+ \mapsto \Re^n$ are smooth vector fields, and $G(x) = [g_1(x) \,|\, g_2(x) \,|\, \cdots \,|\, g_m(x)] : \Re^n \mapsto \Re^{n\times m}$. The state equations

$$\begin{aligned} \dot{x} &= \bar{\phi}(x) + G(x)u \\ y &= h(x) \end{aligned}$$

represent the *known nominal* system dynamics, while η represent the modeling uncertainty. The changes in the system dynamics as a result of faults are characterized by the term $\mathcal{B}(t-T_0)f(x)$. Specifically, the matrix $\mathcal{B}(t-T_0)$ characterizes the time profile of a fault which occurs at some *unknown* time T_0. We let the fault time profile $\mathcal{B}(\cdot)$ be a diagonal matrix of the form

$$\mathcal{B}(t-T_0) \stackrel{\triangle}{=} \text{diag}\,[\beta_1(t-T_0), \cdots, \beta_n(t-T_0)]\ ,$$

where $\beta_i : \Re \mapsto \Re$ is a function representing the time profile of a fault affecting the i–th state equation, for $i = 1, \cdots, n$. More specifically, we consider faults with time profiles modeled by:

$$\beta_i(t-T_0) = \begin{cases} 0 & \text{if} \quad t < T_0 \\ 1 - e^{-a_i(t-T_0)} & \text{if} \quad t \geq T_0\,, \end{cases} \tag{2}$$

where the scalar $a_i > 0$ denotes the unknown fault evolution rate. Small values of a_i characterize slowly developing

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

faults, known as *incipient faults*. For large values of a_i, the time profile β_i approaches a step function, which models *abrupt faults*. Note that the fault time profile given by (2) only reflects the developing speed of the fault, while its other basic features are captured by the nonlinear function $f(x)$.

In this paper, the state vector $x(t)$ is assumed to be measurable, and the control objective is to control the output vector $y(t)$ to track a given reference vector $y_m(t)$. Throughout the paper the following assumption will be used:

Assumption 1. *The unstructured modeling uncertainty, represented by η in (1), is an unknown nonlinear function of x, u, and t, but bounded by some known function $\bar{\eta}$. Specifically, each component η_i of η is assumed to satisfy*

$$|\eta_i(x,u,t)| \leq \bar{\eta}_i(x,u,t), \quad \forall x \in \Re^n, \forall u \in \Re^m, \forall t \in \Re^+ , \tag{3}$$

where, for each $i = 1, \cdots, n$, the bounding function $\bar{\eta}_i(x,u,t) \geq 0$ is known and continuous.

Note that, in Assumption 1, the uniform boundedness assumption on $\bar{\eta}_i(x,u,t)$ made in [10] is removed.

B. Fault–Tolerant Controller

For clarification of the design of the fault-tolerant controller, let us define three important time–instants: T_0 is the time–instant when a fault occurs, $T_d > T_0$ is the time–instant when the monitoring system provides a fault detection decision, and $T_{\text{isol}} > T_d$ is the time–instant when the monitoring system provides a fault isolation decision, that is, which particular fault in a partially known fault class has actually occurred. In the case that a fault is not detected, then T_d and T_{isol} are set to ∞, respectively. The structure of the fault-tolerant controller takes on the following general form:

$$\dot{v} = \begin{cases} g_0(v,x,y_m,t)\,, & \text{for} \quad t < T_d \\ g_D(v,x,y_m,t)\,, & \text{for} \quad T_d \leq t < T_{\text{isol}} \\ g_I(v,x,y_m,t)\,, & \text{for} \quad t \geq T_{\text{isol}} \end{cases} \tag{4}$$

$$u = \begin{cases} u_0(v,y_m,t)\,, & \text{for} \quad t < T_d \\ u_D(v,y_m,t)\,, & \text{for} \quad T_d \leq t < T_{\text{isol}} \\ u_I(v,y_m,t)\,, & \text{for} \quad t \geq T_{\text{isol}} \end{cases}$$

where v is the state vector of the controller and $y_m \in \Re^q$ denotes a reference vector to be tracked by the controlled system output vector. The functions g_0, g_D, g_I and u_0, u_D, u_I are nonlinear functions to be designed according to the following objectives:

- Under normal operating conditions (i.e., for $t < T_0$), a baseline controller described by g_0, u_0 is designed to guarantee system stability and robust tracking performance in the presence of modeling uncertainty η.
- When a fault occurs at time T_0, the baseline controller should guarantee some basic stability property such as system signal boundedness, until the fault is detected, i.e., for $T_0 < t < T_d$.
- After fault detection (i.e., for $T_d \leq t < T_{\text{isol}}$) the baseline controller is *reconfigured* to compensate for the effect of the (yet unknown) fault, that is the controller described by functions g_D and u_D is designed in such a way to exploit the information that a fault occurred to recover some tracking performances.
- If the fault is isolated (i.e., for $t \geq T_{\text{isol}}$), then the controller is reconfigured again. The functions g_I and u_I are designed using the information of the fault type that has been isolated so as to enhance tracking performance (see [9], [10] for the details of the fault isolation problem).

Note that the proposed active fault-tolerant control structure given by (4) takes into account several important effects of the fault diagnosis procedure on the closed-loop system, including fault detection time/delay, fault isolation time/delay, and the occurrence of an *unanticipated* fault whose functional structure is completely unknown *a priori*.

In the sequel, we shall refer to the control laws (4) in the three different cases, by simply making reference to the control variables $u_0(t)$, $u_D(t)$, and $u_I(t)$, respectively. In this paper, we concentrate on the design and analysis of controllers $u_0(t)$, and $u_D(t)$.

III. Fault Detection Scheme

Based on the system representation (1), a fault detection and approximation estimator (FDAE) is chosen as [9]

$$\dot{\hat{x}}^0 = -\Lambda^0(\hat{x}^0 - x) + \bar{\phi}(x) + G(x)u + \hat{f}(x, \hat{\theta}^0) \tag{5}$$

where $\hat{x}^0 \in \Re^n$ is the estimated state vector, $\hat{f} : \Re^n \times \Re^p \mapsto \Re^n$ is an on-line approximation model, $\hat{\theta}^0 \in \Re^p$ represents a vector of adjustable weights of the on-line approximator, and $\Lambda^0 = \text{diag}(\lambda_1^0, \cdots, \lambda_n^0)$, where $-\lambda_i^0 < 0$ is the i-th estimator pole. The initial weight vector, $\hat{\theta}^0(0)$ is chosen such that $\hat{f}\big(x, u, \hat{\theta}^0(0)\big) = 0$, which corresponds to the case where the system is in "healthy" (no fault) condition.

Let $\epsilon^0(t) \triangleq x(t) - \hat{x}^0(t)$ be the state estimation error. An adaptive threshold $\bar{\epsilon}_i^0(t)$ for fault detection is chosen as:

$$\bar{\epsilon}_i^0(t) \triangleq \int_0^t e^{-\lambda_i^0(t-\tau)} \bar{\eta}_i(x,u,\tau) d\tau + |\epsilon_i^0(0)| e^{-\lambda_i^0 t}. \tag{6}$$

After a fault occurs, but before its detection (i.e., for $T_0 \leq t < T_d$), the state estimation remains below its adaptive threshold, and hence the fault is not detected yet, we have

$$|\epsilon_i^0(t)| \leq \bar{\epsilon}_i^0(t)\,, \quad \text{for } T_0 \leq t < T_d\,. \tag{7}$$

Clearly, the occurrence of a fault may affect the stabilizing property of the baseline controller. In this respect, to investigate the system stability property in the presence of a fault but before its detection, we first need the following basic results concerning *fault detection time*:

Lemma 1. *Suppose that a fault occurs at some time T_0. Moreover, assume that there exist a time interval $[T_1, T_2]$, a scalar $M > 0$, and an index $i \in \{1, \cdots, n\}$, such that for all $t \in [T_1, T_2]$,*

$$|\eta_i(x,u,t) + \beta_i(t - T_0) f_i(x)| \geq M + \bar{\eta}_i(x,u,t)\,, \tag{8}$$

where $T_1 > T_0$, $T_2 > T_1 + D(M)$, *and* $D(M)$ *is a time period defined as*

$$D(M) \triangleq \frac{1}{\lambda_i^0} \ln\left(1 + \frac{2\lambda_i^0 \bar{\epsilon}_i^0(T_1)}{M}\right) .$$

Then, an upper bound for the fault detection time is $T_1 + D(M)$.

Due to space limitation, the proof of Lemma 1 is omitted. In the presence of a fault but before its detection,the baseline controller may lose its stabilizing capability [1]. In qualitative terms, the key issue to be addressed is to guarantee that the faulty behavior is detected before the possible occurrence of an unbounded growth of some state variables. Therefore, in the following analysis, a contradiction logic will be exploited. Let us assume that the quantity $|\eta_i(x,u,t) + \beta_i(t - T_0) f_i(x(t))| - \bar{\eta}_i(x,u,t)$ has some *finite escape time* T_e before fault detection , i.e., $\lim_{t \to T_e^-} (|\eta_i + \beta_i f_i)| - \bar{\eta}_i) = \infty$, for $T_0 < T_e < T_d$; more specifically, $\forall M > 0$, $\exists\, \delta(M) > 0$, such that

$$|\eta_i + \beta_i f_i| - \bar{\eta}_i > M , \quad \forall\, t \in [T_e - \delta(M), T_e] . \tag{9}$$

Clearly, given a value of M, $\delta(M)$ is related to the rate of growth of the quantity $|\eta_i + \beta_i f_i| - \bar{\eta}_i$ before fault detection. In the rest of the paper, the following assumption is made:

Assumption 2. *There exists some finite scalar* M, *such that* $\delta(M) > D(M)$, *where* $\delta(M)$ *is defined in (9) and the function* $D(M)$ *is defined in Lemma 1.*

Assumption 2 is needed in order to ensure the proposed fault detection scheme is capable of timely detecting the faulty behavior before the possible occurrence of an unbounded growth of system variables [10].

Then, we have the following result:

Lemma 2. *Suppose that a fault occurs at time* T_0, *and that it is detected at some finite time* $T_d > T_0$. *Then, the quantity* $|\eta_i + \beta_i f_i| - \bar{\eta}_i$ *remains bounded before the fault is detected, i.e.,*

$$|\eta_i + \beta_i f_i| - \bar{\eta}_i \le B , \quad i = 1, \cdots, n;\ t \in (T_0, T_d) , \tag{10}$$

for some finite positive constant B, *for all* $t \in (T_0, T_d)$, *and for all* $1 \le i \le n$.

The proof of Lemma 2 follows a similar reasoning logic reported in [10], and thus is omitted here.

IV. Controller Module

In this section, we present the fault-tolerant control design. In order to control q output variables, the number of control inputs m in (1) should satisfy $m \ge q$. Without loss of generality, we assume $m = q$. Note that if $m > q$, then more design redundancy is available, and the m control inputs can be grouped into q virtual actuator groups based on actuator characteristics [8]. Then, after the q virtual control inputs are designed using the method presented below, a control allocation scheme can be used to allocate each virtual control input to the actuators in the corresponding group. To facilitate the analysis of the feedback control systems, in the sequel, the following specific class of nonlinear MIMO systems will be considered:

$$\left\{\begin{array}{rcl} \dot{x}_{11} &=& x_{12} + \phi_{11}(\bar{x}_{11}) + \eta_{11}(\bar{x}_{11},t) + \beta_{11} f_{11}(\bar{x}_{11}) \\ \dot{x}_{12} &=& x_{13} + \phi_{12}(\bar{x}_{12}) + \eta_{12}(\bar{x}_{12},t) + \beta_{12} f_{12}(\bar{x}_{12}) \\ \vdots && \\ \dot{x}_{1\rho_1} &=& \phi_{1\rho_1}(x) + \sum_{l=1}^{q} g_{1l}(x)\, u_l + \eta_{1\rho_1}(x,t) \\ && + \beta_{1\rho_1} f_{1\rho_1}(x) \end{array}\right.$$

$$\vdots$$

$$\left\{\begin{array}{rcl} \dot{x}_{q1} &=& x_{q2} + \phi_{q1}(\bar{x}_{q1}) + \eta_{q1}(\bar{x}_{q1},t) + \beta_{q1} f_{q1}(\bar{x}_{q1}) \\ \vdots && \\ \dot{x}_{q\rho_q} &=& \phi_{q\rho_q}(x) + \sum_{l=1}^{q} g_{ql}(x)\, u_l + \eta_{q\rho_q}(x,t) \\ && + \beta_{q\rho_q} f_{q\rho_q}(x) \end{array}\right.$$

$$y = [x_{11},\, x_{21}\,, \cdots,\, x_{q1}]^\top . \tag{11}$$

where

$$x \triangleq \mathrm{col}\left(x_{11}, \cdots, x_{1\rho_1}, x_{21}, \cdots, x_{2\rho_2}, \cdots, x_{q1}, \cdots, x_{q\rho_q}\right)$$

is the state vector, $u \in \Re^q$ denotes the control input vector, $y \in \Re^q$ denotes the output vector. The integers ρ_j, $j = 1, \cdots, q$, are the so-called *control characteristic indices* [6]. Without loss of generality, here we assume $\rho_1 \ge \rho_2 \ge \cdots \ge \rho_q$. For $r = 1, \cdots, q$, and $b_r = 1, 2, \cdots, \rho_r$, the functions ϕ_{rb_r}, η_{rb_r}, f_{rb_r}, and g_{rl} are generic smooth functions, and

$$\begin{array}{rcl} \bar{x}_{rb_r} & \triangleq & \mathrm{col}(x_{11}, \cdots, x_{1(\rho_1 - \rho_r + b_r)}, \cdots, x_{r1}, \cdots, x_{rb_r}, \\ && \cdots, x_{q1}, \cdots, x_{q(\rho_q - \rho_r + b_r)}) . \end{array}$$

For instance,

$$\begin{array}{rcl} \bar{x}_{11} & \triangleq & \mathrm{col}(x_{11}, x_{2(\rho_2 - \rho_1 + 1)}, \cdots, x_{q(\rho_q - \rho_1 + 1)}) , \\ \bar{x}_{12} & \triangleq & \mathrm{col}(x_{11}, x_{12}, x_{21}, x_{2(\rho_2 - \rho_1 + 2)}, \cdots, x_{q1}, \\ && x_{q(\rho_q - \rho_1 + 2)}) . \end{array}$$

The control objective is to control the output vector $y(t)$ to track a given reference vector $y_m(t)$. We assume that each reference signal $y_{mr}(t)$, $r = 1, \cdots, q$, and its first ρ_r derivatives are known, piecewise continuous, and bounded.

Remark 1. The system model (11) is in the parametric-strict-feedback form [5]. In the SISO case, it defines a lower triangular form. The existence conditions of a diffeomorphism, which transforms the general nonlinear system model (1) into the specific form (11), have been discussed in [8], [6]. The problem of actuator fault accommodation for a similar class of nonlinear MIMO systems has been investigated in [8] using direct adaptive control methods, without the use of on-line fault diagnostic information.

Remark 2. Each *control characteristic index* ρ_j, $j = 1, \cdots, q$, is the least order of the time derivative of the output y_j that is directly affected at least by some input u_j [6]. It is worth noting that, for the sake of notational simplicity, here we assume $\rho_1 + \rho_2 + \cdots + \rho_q = n$. In the case of $\rho_1 + \rho_2 + \cdots + \rho_q < n$, the transformed system (11) will have another part describing the zero dynamics [6], [8]. Then, with an additional assumption on input-to-state stability of the zero dynamics, similar analytical results can be obtained.

A. Baseline Controller and System Stability Before Fault Detection

In this section, we design the baseline controller and investigate the system stability before fault detection. Using the backstepping methodology [5], a new state vector z is defined recursively by the following coordinate transformation: for $r = 1, \cdots, q$, and $b_r = 1, \cdots, \rho_r$,

$$z_{rb_r} = x_{rb_r} - \alpha_{r(b_r-1)}\left(\bar{x}_{r(b_r-1)}, \bar{y}_{mr}^{(b_r-2)}\right) - y_{mr}^{(b_r-1)} \quad (12)$$

where the intermediate control functions α_{rb_r} are recursively given by

$$\begin{aligned} \alpha_{r0} &= 0 \\ \alpha_{r1} &= -c_{r1}z_{r1} - \phi_{r1} + \chi_{r1}(\bar{x}_{r1}, y_{mr}) \\ \alpha_{rb_r} &= -c_{rb_r}z_{rb_r} - z_{r(b_r-1)} - \phi_{rb_r} + \chi_{rb_r}(\bar{x}_{rb_r}, \bar{y}_{mr}^{(b_r-1)}) \\ &\quad + \sum_{j=1}^{q}\sum_{k=1}^{(\rho_j-\rho_r+b_r-1)}\left\{\frac{\partial\alpha_{r(b_r-1)}}{\partial x_{jk}}(x_{j(k+1)} + \phi_{jk})\right\} \\ &\quad + \sum_{d=1}^{b_r-1}\left(\frac{\partial\alpha_{r(b_r-1)}}{\partial y_{mr}^{(d-1)}} y_{mr}^{(d)}\right), \\ &\qquad \text{for } b_r = 2, \cdots, \rho_r, \end{aligned} \quad (13)$$

where c_{r1} and c_{rb_r} are design constants, $\bar{y}_{mr}^{(b_r)} \triangleq \text{col}\,(y_{mr}, y_{mr}^{(1)}, \ldots, y_{mr}^{(b_r)})$, and χ_{rb_r} is a smooth function to be defined later on using bounding control techniques [4].

We make the following assumption for the design and analysis of the proposed fault-tolerant control scheme.

Assumption 3. *The following actuation matrix $G(x)$ is nonsingular*

$$G(x) = \begin{bmatrix} g_{11} & \cdots & g_{1q} \\ \cdots & \cdots & \cdots \\ g_{q1} & \cdots & g_{qq} \end{bmatrix}. \quad (14)$$

Now, we analyze the stability and tracking properties of closed-loop system before the detection of a fault. We apply the backstepping design and design the baseline controller as

$$u_0(t) = G(x)^{-1}\begin{bmatrix} \alpha_{1\rho_1} + y_{m1}^{\rho_1} \\ \vdots \\ \alpha_{q\rho_q} + y_{mq}^{\rho_q} \end{bmatrix}. \quad (15)$$

Due to space limitation, the details of the procedure is omitted.

Let us consider a Lyapunov function candidate of the form $V = \frac{1}{2}\sum_{r=1}^{q}\sum_{b_r=1}^{\rho_r}(z_{rb_r})^2$. After some algebraic manipulations, it can be shown that the time–derivative of V satisfies

$$\begin{aligned} \dot{V} \le & \sum_{r=1}^{q}\sum_{b_r=1}^{\rho_r}\Bigg\{-c_{rb_r}z_{rb_r}^2 + |z_{rb_r}|\Bigg[|\eta_{rb_r} + \beta_{rb_r}f_{rb_r}| \\ & + \sum_{j=1}^{q}\sum_{k=1}^{(\rho_j-\rho_r+b_r-1)}\left(\left|\frac{\partial\alpha_{r(b_r-1)}}{\partial x_{jk}}\right||\eta_{jk} + \beta_{jk}f_{jk}|\right)\Bigg] \\ & + z_{rb_r}\chi_{rb_r}\Bigg\}. \end{aligned} \quad (16)$$

Next, we consider the design of bounding control functions χ_{rb_r}. In the sequel, the following property of the hyperbolic tangent function is used [7]: for any $\epsilon > 0$ and for any $q \in \Re$,

$$0 \le |q| - q\tanh\left(\frac{q}{\epsilon}\right) \le \bar{k}\epsilon, \quad (17)$$

where $\bar{k}$ is a constant that satisfies $\bar{k} = e^{-(\bar{k}+1)}$; i.e., $\bar{k} \simeq 0.2785$. Let us choose

$$\begin{aligned} \zeta_{rb_r} &= \bar{\eta}_{rb_r} + \textstyle\sum_{j=1}^{q}\sum_{k=1}^{(\rho_j-\rho_r+b_r-1)}\left(\left|\frac{\partial\alpha_{r(b_r-1)}}{\partial x_{jk}}\right|\bar{\eta}_{jk}\right), \\ \chi_{rb_r} &= -\zeta_{rb_r}\tanh\left(\frac{z_{rb_r}\zeta_{rb_r}}{\epsilon}\right). \end{aligned} \quad (18)$$

From Lemma 2, before the detection of the possible presence of a fault (i.e., for $0 \le t < T_d$), we have

$$|\eta_{rb_r} + \beta_{rb_r}f_{rb_r}| \le \bar{\eta}_{rb_r} + B, \quad (19)$$

for some finite positive constant B. By using (16), (17), (18), and (19), it can be shown that

$$\dot{V} \le \sum_{r=1}^{q}\sum_{b_r=1}^{\rho_r}\left(-c_{rb_r}z_{rb_r}^2 + \bar{k}\epsilon + |z_{rb_r}|\,\bar{B}\right),$$

where $\bar{B} \triangleq B + \sum_{j=1}^{q}\sum_{k=1}^{(\rho_j-\rho_r+b_r-1)}\left(\left|\frac{\partial\alpha_{r(b_r-1)}}{\partial x_{jk}}\right|B\right)$. By completing the squares, we obtain

$$\dot{V} \le -cV + b, \quad (20)$$

where $b \triangleq n\bar{k}\epsilon + \sum_{r=1}^{q}\sum_{b_r=1}^{\rho_r}\left(\frac{1}{2c_{rb_r}}\bar{B}^2\right)$ and $c \triangleq \min_{\substack{1\le r\le q \\ 1\le b_r\le\rho_r}}\{c_{rb_r}\}$.

Now, if we let $\bar{\kappa} \triangleq b/c > 0$, we obtain

$$0 \le V(t) \le \bar{\kappa} + [V(0) - \bar{\kappa}]e^{-ct}. \quad (21)$$

Therefore, $z(t)$, and $x(t)$ are uniformly bounded. Furthermore, note that before the occurrence of any faults (i.e., for $t < T_0$), we have $\beta_{rb_r}f_{rb_r} = 0$, which implies $B = 0$ and $\bar{B} = 0$. Therefore, given any $\bar{\epsilon} > \sqrt{2\bar{\kappa}}$, where $\bar{\kappa} = n\bar{k}\epsilon/c$, there exists some finite time T, such that for all $t \ge T$ the output vector y satisfies $|y(t) - y_m(t)| \le \bar{\epsilon}$. Note that $\bar{\kappa}$ can be made small by using a small value of ϵ.

The aforementioned design and analysis procedure is summarized in the following result:

Theorem 1. (Stability before fault detection) *Suppose that a fault occurs at time T_0, and consider the time window $[0, T_d)$. Then, the baseline controller given by (15) and (18) guarantees that*

1) *In the absence of faults, given any* $\bar{\epsilon} > \sqrt{2\bar{\kappa}}$, *there exists* $T(\bar{\epsilon})$ *such that* $|y(t) - y_r(t)| \leq \bar{\epsilon}$, *for all* $T(\bar{\epsilon}) < t < T_0$.
2) *All the signals are uniformly bounded before and after fault occurrence, i.e.,* $z(t)$ *and* $x(t)$ *are bounded for all* $t \in [0, T_d)$;

B. Controller reconfiguration before fault isolation

Let us assume that a fault is detected at time $t = T_d$. Starting from T_d, as inequality (10) is no longer satisfied, therefore, the baseline controller $u_0(t)$ is reconfigured to ensure signal boundedness and some tracking performance given the fact that the diagnostic system has detected a fault. In this section, we describe the design of the fault-tolerant controller $u_D(t)$ defined in (4), using stable adaptive tracking techniques [4], [5].

At this stage, since no information about the specific fault type is available (see [9], [10] for the details of the fault isolation problem), adaptive approximators such as neural network models can be used to estimate the unknown fault function $\beta_{rb_r} f_{rb_r}$. Specifically, we consider linearly parameterized networks (for example, radial basis function networks with fixed centers and variances) described as

$$\hat{f}_{rb_r}(\bar{x}_{rb_r}, \hat{\theta}_{rb_r}) = (\hat{\theta}_{rb_r})^\top \varphi_{rb_r}(\bar{x}_{rb_r}) , \qquad (22)$$

where $\hat{\theta}_{rb_r}$ denotes the adjustable weights of the linearly parameterized adaptive approximation model, and $\varphi_{rb_r}(\bar{x}_{rb_r})$ represents the network basis functions. Therefore, the system model (11) can be rewritten as follows, for $r = 1, \cdots, q$, and $b_r = 1, \cdots, \rho_r - 1$:

$$\begin{aligned}
\dot{x}_{rb_r} &= x_{r(b_r+1)} + \phi_{rb_r}(\bar{x}_{rb_r}) + \hat{f}_{rb_r}(\bar{x}_{rb_r}, \theta_{rb_r}) \\
&\quad + \beta_{rb_r}\delta_{rb_r}(\bar{x}_{rb_r}) + \eta_{rb_r}(\bar{x}_{rb_r}, t) \\
&\quad + (\beta_{rb_r} - 1)\hat{f}_{rb_r}(\bar{x}_{rb_r}, \theta_{rb_r}) \\
&\vdots \\
\dot{x}_{r\rho_r} &= \phi_{r\rho_r}(x) + \sum_{l=1}^{q} g_{rl}(x)\, u_l + \hat{f}_{r\rho_r}(x, \theta_{r\rho_r}) \\
&\quad + \beta_{r\rho_r}\delta_{r\rho_r}(x) + \eta_{r\rho_r}(x, t) \\
&\quad + (\beta_{r\rho_r} - 1)\hat{f}_{r\rho_r}(x, \theta_{r\rho_r}) \qquad (23)
\end{aligned}$$

where

$$\delta_{rb_r}(\bar{x}_{rb_r}) \triangleq f_{rb_r}(\bar{x}_{rb_r}) - \hat{f}_{rb_r}(\bar{x}_{rb_r}, \theta_{rb_r})$$

is the network approximation error, and θ_{rb_r} is the optimal weight vector [4]. For each network, we make the following assumption on the network approximation error:

Assumption 4. *For each* $r = 1, \cdots, q$, *and* $b_r = 1, \cdots, \rho_r$

$$|\delta_{rb_r}(\bar{x}_{rb_r})| \leq \psi_{rb_r} s_{rb_r}(\bar{x}_{rb_r}) , \qquad (24)$$

where $\psi_{rb_r} \geq 0$ *are unknown bounding parameters and* $s_{rb_r} : \Re^{rb_r} \mapsto \Re^+$ *are known smooth bounding functions.*

The system described by (23) and (24) is characterized by two types of uncertainty: (i) parametric uncertainty, which arises due to the unknown network weights θ_{rb_r}; (ii) bounding uncertainty that arises due to the unknown bounding parameters ψ_{rb_r} and unknown incipient fault time profile β_{rb_r}. We let $\tilde{\theta}_{rb_r}(t) \triangleq \hat{\theta}_{rb_r}(t) - \theta_{rb_r}$ denote the network weight estimation error, and $\tilde{\psi}(t) \triangleq \hat{\psi}(t) - \psi_m$ represent the corresponding bounding parameter estimation error, where ψ_m is an unknown constant defined as follows

$$\psi_m \triangleq \sup_{\substack{1\leq i\leq q \\ 1\leq b_r\leq \rho_r}} \max\{\beta_{rb_r}(t-T_0)\psi_{rb_r}, |(\beta_{rb_r}(t-T_0)-1)\theta_{rb_r}|\} .$$

Note that the fault time profile $\beta_{rb_r}(t - T_0)$ satisfies $0 \leq \beta_{rb_r} \leq 1$. Then such a finite constant ψ_m always exists.

Now, we proceed to present the design of the fault-tolerant controller $u_D(t)$ described in (4), which consists of two components: backstepping design and adaptive bounding design. For the sake of compactness of notation, we denote $\bar{a} \triangleq (\rho_r - \rho_1 + a)$ and

$$\begin{aligned}
\bar{\theta}_{r\bar{a}} \triangleq \operatorname{col}\Big(&\hat{\theta}_{11}, \cdots, \hat{\theta}_{1(a-1)}, \cdots, \hat{\theta}_{r1}, \cdots, \hat{\theta}_{r(\bar{a}-1)}, \\
&\cdots, \hat{\theta}_{q1}, \cdots, \hat{\theta}_{q(\rho_q-\rho_1+a-1)}\Big) .
\end{aligned}$$

Due to space limitation, below we only give the basic idea of the design. The details of intermediate steps are omitted.

Step 1. Backstepping design procedure

Consider a new state vector z defined by the following change of coordinates: for $1 \leq a \leq \rho_1$, $1 \leq r \leq q$

$$z_{r\bar{a}} = x_{r\bar{a}} - y_{mr}^{(\bar{a}-1)} - \alpha_{r(\bar{a}-1)} , \qquad (25)$$

where the intermediate control functions are given by:

$$\begin{aligned}
\alpha_{r0} &= 0 \\
\alpha_{r1} &= -c_{r1}z_{r1} - \phi_{r1} - (\hat{\theta}_{r1})^\top \varphi_{r1} + \chi_{r1} \\
\alpha_{r\bar{a}} &= -c_{r\bar{a}}z_{r\bar{a}} - z_{r(\bar{a}-1)} - \phi_{r\bar{a}} - (\hat{\theta}_{r\bar{a}})^\top \varphi_{r\bar{a}} \\
&\quad + \sum_{j=1}^{q} \sum_{k=1}^{(\rho_j-\rho_1+a-1)} \Big\{ \frac{\partial \alpha_{r(\bar{a}-1)}}{\partial x_{jk}} \Big(x_{j(k+1)} + \phi_{jk} \\
&\quad + (\hat{\theta}_{jk})^\top \varphi_{jk} \Big) \Big\} + \sum_{d=1}^{\bar{a}-1} \left(\frac{\partial \alpha_{r(\bar{a}-1)}}{\partial y_{mr}^{(d-1)}} y_{mr}^{(d)} \right) \\
&\quad + \sum_{j=1}^{q} \sum_{k=1}^{(\rho_j-\rho_1+a-1)} \left(\frac{\partial \alpha_{r(\bar{a}-1)}}{\partial \hat{\theta}_{jk}} \tau_{jk}^{(\rho_j-\rho_1+a)} \right) \\
&\quad - \sum_{j=1}^{q} \sum_{k=1}^{(\rho_j-\rho_1+a-1)} \Big\{ \frac{\partial \alpha_{r(\bar{a}-1)}}{\partial x_{jk}} \varphi_{jk} \Gamma_{jk} \cdot \\
&\qquad \sum_{l=1}^{q} \sum_{m=k}^{(\rho_l-\rho_1+a-2)} \left(\frac{\partial \alpha_{lm}}{\partial \hat{\theta}_{jk}} z_{l(m+1)} \right) \Big\} \\
&\quad + \chi_{r\bar{a}}(\bar{x}_{r\bar{a}}, \hat{\psi}, y_{mr}^{(\bar{a}-1)}), \text{ for } 2 \leq \bar{a} \leq \rho_r \qquad (26)
\end{aligned}$$

In (25) and (26), $c_{r\bar{a}}$ are design constants, $\chi_{r\bar{a}}$ denotes smooth functions to be defined later on by adaptive bounding control techniques, and the intermediate adaptive functions

$\tau_{jk}^{(\rho_j-\rho_1+a)}$ are recursively updated as follows:

$$\begin{aligned}\tau_{j(\rho_j-\rho_1+a)}^{(\rho_j-\rho_1+a)} &= \Gamma_{j(\rho_j-\rho_1+a)}\big(\varphi_{j(\rho_j-\rho_1+a)} z_{j(\rho_j-\rho_1+a)} \\ &\quad -\sigma(\hat{\theta}_{j(\rho_j-\rho_1+a)} - \theta^0_{j(\rho_j-\rho_1+a)})\big)\,, \\ \tau_{jk}^{(\rho_j-\rho_1+a)} &= \tau_{jk}^{(\rho_j-\rho_1+a-1)} - \Gamma_{jk}\sum_{l=1}^{q}\bigg(\frac{\partial \alpha_{l(\rho_l-\rho_1+a-1)}}{\partial x_{jk}} \\ &\quad \cdot\varphi_{jk} z_{l(\rho_l-\rho_1+a)}\bigg) \\ &\quad \text{for } 1\le k\le \rho_j-\rho_1+a-1\,, \end{aligned} \tag{27}$$

where σ and $\theta^0_{j(\rho_j-\rho_1+a)}$ are design constants.

The backstepping procedure [5] is applied to the state variables according to the order of $\{z_{11}, z_{21}, \cdots, z_{r(\rho_r-\rho_1+1)}, \cdots, z_{1a}, \cdots, z_{r(\rho_r-\rho_1+a)}, \cdots, z_{q(\rho_q-\rho_1+a)}, \cdots, z_{1\rho_1}, \cdots, z_{q\rho_q}\}$. At the last stage of the design procedure, we choose the fault-tolerant controller $u_D(t)$ defined in (4) as

$$u_D(t) = G(x)^{-1}\begin{bmatrix}\alpha_{1\rho_1}+y_{m1}^{\rho_1}\\ \vdots \\ \alpha_{q\rho_q}+y_{mq}^{\rho_q}\end{bmatrix}, \tag{28}$$

where the matrix G is defined in (14), and the intermediate control functions $\alpha_{r\rho_r}$, $r=1,\cdots,q$, are given in (26). Now we consider the overall Lyapunov function candidate

$$V=\sum_{j=1}^{q}\sum_{k=1}^{\rho_j}\left(\frac{1}{2}(z_{jk})^2+\frac{1}{2}(\tilde{\theta}_{jk})^\top\Gamma_{jk}^{-1}\tilde{\theta}_{jk}\right)+\frac{1}{2\gamma_\psi}\tilde{\psi}^2\,.$$

On the basis of (25), (26), (27), and (28), and by choosing the adaptive laws for updating $\hat{\theta}_{jk}(t)$ as

$$\dot{\hat{\theta}}_{jk}=\tau_{jk}^{\rho_j},\quad 1\le j\le q,\, 1\le k\le\rho_j\,, \tag{29}$$

it can be shown that the time derivative of V is given by

$$\dot{V} = \sum_{j=1}^{q}\sum_{k=1}^{\rho_j}\Big(-c_{jk}(z_{jk})^2-\sigma(\tilde{\theta}_{jk})^\top(\hat{\theta}_{jk}-\theta^0_{jk})\Big)+\Lambda_{q\rho_q}\,, \tag{30}$$

where

$$\begin{aligned}\Lambda_{q\rho_q} &\triangleq \Lambda_{(q-1)\rho_{(q-1)}} + z_{q\rho_q}\bigg\{\chi_{q\rho_q}+\eta_{q\rho_q}+\beta_{q\rho_q}\delta_{q\rho_q} \\ &\quad +(\beta_{q\rho_q}-1)(\theta_{q\rho_q})^\top\varphi_{q\rho_q}-\frac{\partial\alpha_{q(\rho_q-1)}}{\partial\hat{\psi}}\dot{\hat{\psi}} \\ &\quad +\sum_{j=1}^{q}\sum_{k=1}^{\rho_j-1}\bigg(\frac{\partial\alpha_{q(\rho_q-1)}}{\partial x_{jk}}\big(\eta_{jk}+\beta_{jk}\delta_{jk}+(\beta_{jk}-1) \\ &\quad (\theta_{jk})^\top\varphi_{jk}\big)\bigg)\bigg\}\,. \end{aligned} \tag{31}$$

Step 2. Adaptive bounding design procedure

We now consider the recursive design of the bounding control function $\chi_{r\bar{a}}$, $1\le r\le q$, $\bar{a}\triangleq\rho_r-\rho_1+a$, and the adaptive law for the bounding estimate $\hat{\psi}(t)$.

The adaptive bounding design [4] is recursively applied to $\Lambda_{r\bar{a}}$ according to the order of $\{\Lambda_{11}, \Lambda_{21}, \cdots, \Lambda_{r(\rho_r-\rho_1+1)}, \cdots, \Lambda_{1a}, \cdots, \Lambda_{r(\rho_r-\rho_1+a)}, \cdots, \Lambda_{q(\rho_q-\rho_1+a)}, \cdots, \Lambda_{1\rho_1}, \cdots, \Lambda_{q\rho_q}\}$. At the last stage of the design procedure, we have

$$\begin{aligned}\Lambda_{q\rho_q} &\le n\bar{k}\epsilon+n\bar{k}\epsilon\psi_m-\sigma\tilde{\psi}(\hat{\psi}-\psi^0)+(\dot{\hat{\psi}}-\nu_{q\rho_q}) \\ &\quad\cdot\left[\gamma_\psi^{-1}\tilde{\psi}-\sum_{j=1}^{q}\sum_{k=1}^{\rho_j-1}\left(\frac{\partial\alpha_{jk}}{\partial\hat{\psi}}z_{j(k+1)}\right)\right],\end{aligned}$$

where

$$\nu_{q\rho_q}\triangleq\gamma_\psi\left[\sum_{j=1}^{q}\sum_{k=1}^{\rho_j}(z_{jk}\omega_{jk})-\sigma(\hat{\psi}-\psi^0)\right].$$

Therefore, by choosing the adaptive law

$$\dot{\hat{\psi}}=\nu_{q\rho_q}\,, \tag{32}$$

we have

$$\Lambda_{q\rho_q}\le n\bar{k}\epsilon+n\bar{k}\epsilon\psi_m-\sigma\tilde{\psi}(\hat{\psi}-\psi^0)\,. \tag{33}$$

By substituting (33) into (30), and completing the squares for each parameter estimate, it can be shown that the Lyapunov function V satisfies the following inequality

$$\dot{V}\le -cV+b\,, \tag{34}$$

where $c\triangleq\{2c_{jk},\,\sigma/(\lambda_{\min}(\Gamma_{jk}^{-1})),\,\sigma\gamma_\psi\}$, and $b\triangleq n\bar{k}\epsilon\psi_m+n\bar{k}\epsilon+\frac{\sigma}{2}\Big(\sum_{j=1}^{q}\sum_{k=1}^{\rho_j}|\theta_{jk}-\theta^0_{jk}|^2+|\psi_m-\psi^0|^2\Big)$. Now (34) is in the same form of (20). The following important results follows from the proof of Theorem 1:

Theorem 2. *Suppose the bounding Assumption 4 holds globally. Then, if a fault is detected, the adaptive fault-tolerant control law (28), the weight parameter adaptive law (29) and the bounding parameter adaptive law (32) guarantee that:*

1) *all the signal and parameter estimates are uniformly bounded, i.e.,* $z(t)$, $\hat{\theta}(t)$, $\hat{\psi}(t)$ *and* $x(t)$ *are bounded for all* $t\in(T_0,\,T_d)$;
2) *given any* $\bar{\epsilon}>(2b/c)^{\frac{1}{2}}$, *there exists* $T(\bar{\epsilon})$ *such that* $|y(t)-y_r(t)|\le\bar{\epsilon}$, *for all* $t>T(\bar{\epsilon})$.

References

[1] M. Blanke, M. Kinnaert, J. Lunze, and M. Staroswiecki. *Diagnosis and Fault-Tolerant Control*, Springer, Berlin, 2006.

[2] J. Chen and R. J. Patton. *Robust Model-Based Fault Diagnosis for Dynamic Systems*. Kluwer Academic Publishers, London, 1999.

[3] J. J. Gertler, *Fault Detection and Diagnosis in Engineering Systems*. Marcel Dekker, New York, 1998.

[4] J. Farrell and M. M. Polycarpou. *Adaptive Approximation Based Control*. Hoboken, NJ: J. Wiley, 2006.

[5] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design*. Wiley, NY, 1995.

[6] R. Marino and P. Tomei. *Nonlinear Control Design*. Prentice Hall, Englewood Cliffs, NJ, 1995.

[7] M. M. Polycarpou and P. Ioannou, "A robust adaptive nonlinear control design," *Automatica*, vol. 32, no. 3, pp. 423-427, March 1996.

[8] X. Tang, G. Tao, and S. M. Joshi, "Adaptive actuator failure compensation for nonlnear MIMO systems with an aircraft control application," *Automatica*, vol. 43, pp. 1869–1883, 2007.

[9] X. Zhang, M. M. Polycarpou, and T. Parisini, "A robust detection and isolation scheme for abrupt and incipient faults in nonlinear systems," *IEEE Trans. on Automatic Control*, vol. 47, pp. 576-593, 2002.

[10] X. Zhang, T. Parisini, and M. M. Polycarpou, "Adaptive fault-tolerant control of nolinear uncertain systems: a diagnostic information based approach," *IEEE Transactions on Automatic Control*, Vol. 49, no. 8, pp. 1259-1274, 2004.