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Synthesis of Absolutely Stabilizing Controllers

Waqar A. Malik, Sincheon Kang, Swaroop Darbha and S. P. Bhattacharyya

Abstract— **In this paper, we consider the synthesis of fixed order controllers for nonlinear systems with sector bounded nonlinearities. We construct an inner and outer approximation of the set of absolutely stabilizing linear controllers by casting the closed loop system as a Lure-Postnikov system. The inner approximation is based on the well-known sufficient conditions that require Strict Positive Realness (SPR) of open loop transfer function (possibly with some multipliers) and a characterization of SPR transfer functions that require a family of complex polynomials to be Hurwitz. The outer approximation is based on the condition that the open loop transfer function must have infinite gain margin, which translates to a family of real polynomials being Hurwitz. We illustrate the proposed methodology through the construction of an inner and outer approximation of absolutely stabilizing controllers for a mechanical system.**

I. Introduction

Absolute stability of Lure-Postnikov systems have been studied quite extensively, see the books of Aizerman [1], Popov [2], Siljak [3], Narendra and Taylor [4] and Safanov [5]. The problem of absolute stability is that of ensuring the asymptotic stability in the large of a nonlinear system of the form given in Fig. I for every nonlinearity in the first and third quadrants.

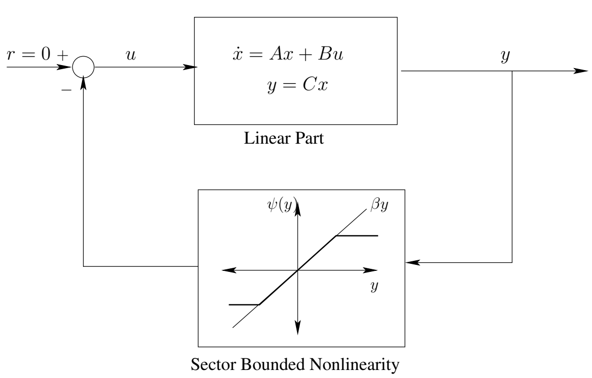

Fig. 1. Lure-Postnikov system

The seminal result of Popov subsumes earlier results of Lure and others concerning the problem and all subsequent results on this problem have the same flavor of requiring a transfer function, which is usually the product of the transfer function of the linear part of the Lure-Postnikov system and an appropriate multiplier to be strictly positive real.

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The problem of the synthesis of absolutely stabilizing controllers is important for two reasons - absolute stability naturally comes with a robustness guarantee that the zero solution of the closed loop is asymptotically stable for every nonlinearity satisfying the sector condition. In some nonlinear systems, the nonlinearity in the system is provided in terms of empirical data and only crude information about the nonlinearity is available, i.e., that it lies in the first and third quadrants. In such a case, the problem of synthesis of absolutely stabilizing controllers is relevant while being conservative. The reason for conservatism is that one is designing a controller that stabilizes the closed loop for every nonlinearity in the first and third quadrants as opposed to the specific nonlinearity provided in terms of empirical data. In some applications, the assumptions involved in developing a lumped model of a system, render the coefficient of a nonlinearity parametrically uncertain. The classic example is that of a pendulum - whether one *assumes the mass of the pendulum lumped or uniformly distributed*, the structure of the resulting equations is similar; while the nonlinearity is sector bounded, its coefficient may not be known.

In the case when the nonlinearity is known, but the coefficient is not exactly known, the situation may be remedied using nonlinear design techniques developed in [6], [7] to design a nonlinear controller which are tailor-made for the specific nonlinearity. However, the constraint on the order of the controller cannot be handled by the existing design techniques. In this paper, we will explore the synthesis of linear absolutely stabilizing controllers of a given order. Although the procedure adopted here is conservative and applies only to systems with sector bounded nonlinearities, the proposed method allows for imposing structure (such as the order) on the controller.

The problem of synthesizing absolutely stabilizing controllers has been considered in the literature, for example, see [8]–[10]. In [8], *a controller* is synthesized in terms of the solution to coupled Riccati and Lyapunov equations, while in [9], [10], the focus was on the use of LMIs to synthesize *a controller*. In this paper, we consider the problem of constructing an inner and an outer approximation of the set of stabilizing controllers of fixed order/structure for Lure-Postnikov systems. The construction of an approximation of *set* of stabilizing controllers is accomplished through the use of Hermite-Biehler theorem and a characterization of strict positive real transfer functions through the requirement of a one-parameter family of complex polynomials being Hurwitz [11]. The novelty of the paper lies in the construction of an approximation to the set of absolute stabilizing controllers as a control engineer can restrict the search for a controller

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satisfying multiple objectives from the given set.

The paper is organized as follows: In section II, we provide the necessary mathematical preliminaries and in section III, we provide the main results concerning the construction of absolutely stabilizing linear controllers of fixed order. In section IV, we provide a corroboration of the developed methodology.

II. Mathematical Preliminaries

In this section, we present the results that have been well established in the literature and are required for arriving at the main result discussed in the next section. Lemmas 1 and 2 are concerned with conditions for guaranteeing absolute stability of a Lure-Postnikov system. In Theorem 1, we present the well-known sufficient condition for absolute stability through the requirement of a transfer function being strictly positive real. The conditions of Lemmas 1 and 2 and Theorem 1 require a family of real or complex polynomials to be Hurwitz. For this reason, the rest of the section deals with our previous results in [12] concerning a systematic procedure for constructing sets of stabilizing controllers that render a family of real or complex polynomials Hurwitz.

Consider the problem of synthesizing an absolutely stabilizing controller for the following system:

$$\dot{x} = Ax - B_1 u - B_2 \phi(y), \quad (1)$$
$$y = Cx, \quad (2)$$

where the nonlinear function $\phi(y)$ satisfies $0 \leq y\phi(y)$ for all $y \in \Re$.

Let $G_1(s) := C(sI-A)^{-1}B_1$ and $G_2 := C(sI-A)^{-1}B_2$. Consider a controller $G_c(s) = \frac{N_c(s)}{D_c(s)}$, where the polynomial, $D_c(s)$ is monic and of degree r, while the degree of the polynomial $N_c(s)$ is assumed to be at most r. Let (A_c, B_c, K_1, K_2) be a minimal realization of $G_c(s)$. Hence, $G_c(s) = K_1(sI - A_c)^{-1}B_c + K_2$. We will assume that they are in the controllable canonical form. The coefficients of the polynomials $N_c(s)$ and $D_c(s)$ are free parameters that must be chosen so as to make the zero solution of the closed loop absolutely stable:

$$\dot{x} = Ax - B_1 u - B_2 \phi(y), \quad (3)$$
$$\dot{x}_c = A_c x_c + B_c y, \quad (4)$$
$$y = Cx, \quad (5)$$
$$u = (K_1 x_c + K_2 y). \quad (6)$$

In the above equation, $x_c(t) \in \Re^r$ represents the state of the controller.

The closed loop system may be expressed as a Lure-Postnikov system as follows:

$$\dot{z} = A_{cl} z - B_{cl} \phi(y), \quad (7)$$
$$y = C_{cl} z, \quad (8)$$

for some A_{cl}, B_{cl} and C_{cl} which constitute a realization of the transfer function $H(s) = (1 + G_1 G_c(s))^{-1} G_2$. If one were to write $G_1(s) = \frac{N_1(s)}{D_p(s)}$, $G_2(s) = \frac{N_2(s)}{D_p(s)}$ and $G_c(s) = \frac{N_c(s)}{D_c(s)}$, then the transfer function $H(s)$ may be expressed as $\frac{N_2(s)D_c(s)}{D_p(s)D_c(s)+N_1(s)N_c(s)}$. It is clear that the coefficients of the numerator and denominator of $H(s)$ are affine in the parameters of the controller.

Let the transfer function $H(s) = \frac{N_{cl}(s)}{D_{cl}(s)}$ for some co-prime polynomials, where $N_{cl}(s) = N_2(s)D_c(s)$ and $D_{cl}(s) = D_p(s)D_c(s) + N_1(s)N_c(s)$.

Lemma 1. *The requirement that $D_{cl}(s) + \lambda N_{cl}(s)$ be Hurwitz for every $\lambda \geq 0$ is a necessary condition for the absolute stability of the zero solution of the Lure-Postnikov System considered above.*

A sufficient condition for absolute stability is given in terms of Popov's criterion [1].

Theorem 1. *If there exists a $q \geq 0$ such that $(1+qs)H(s)$ is Strictly Positive Real (SPR), then the zero solution of the Lure-Postnikov system is absolutely stable. This is also a necessary and sufficient condition for the existence of a Lyapunov function of the form $x^T Px + q\int_0^y \phi(\eta)d\eta$.*

A proof of the theorem can be found in [1]–[5].

The following characterization of SPR functions [11] for reducing the problem of synthesizing SPR functions to that of controllers rendering a family of polynomials Hurwitz:

Lemma 2. *A rational transfer function $\frac{N(s)}{D(s)}$ is SPR if and only if*

1) $\frac{N(0)}{D(0)} > 0$
2) *The polynomials $N(s)$ and $D(s)$ are Hurwitz,*
3) *The family of complex polynomials, $D(s) + j\alpha N(s)$, $\alpha \in \Re$ is Hurwitz.*

As these necessary and sufficient conditions respectively involve a family of real and complex polynomials being Hurwitz, we will provide a characterization of Hurwitz polynomials suitable for the construction of outer and inner approximation using Linear Programming (LP) techniques. Such techniques have been developed in [12]. A characterization of Hurwitz polynomials via the Hermite-Biehler theorem will be used for the synthesis of stabilizing controllers. Such a characterization for real and complex polynomials respectively [3], [11], [12], are provided.

Let $P(s,K)$ be an n^{th} degree *real* polynomial whose coefficients are affinely dependent on the vector of design parameters, K. Write $P(jw,K) := P_e(w^2,K) + jwP_o(w^2,K)$, where P_e and P_o are polynomials with real coefficients. The degrees of polynomials P_e and P_o are n_e and n_o respectively in w^2; if n is odd, $n_e = n_o = \frac{n-1}{2}$ and if n is even, $n_e = \frac{n}{2}$ and $n_o = n_e - 1$. Let $w_{e,i}$, $w_{o,i}$ denote the i^{th} positive real roots of P_e and P_o respectively.

The Hermite-Biehler theorem for real polynomials may be stated as follows; for the sake of clarity, and for the general case, the dependence on K is suppressed.

Hermite-Biehler Theorem for real polynomials. *A real polynomial $P(s)$ is Hurwitz iff*

1) *The constant coefficients of $P_e(w^2)$ and $P_o(w^2)$ are of the same sign,*

2) *All roots of $P_e(w^2)$ and $P_o(w^2)$ are real and distinct; the positive roots interlace according to the following:*
   - *if n is even:*
   $$0 < w_{e,1} < w_{o,1} < \cdots < w_{o,n_e-1} < w_{e,n_e}$$
   - *if n is odd:*
   $$0 < w_{e,1} < w_{o,1} < \cdots < w_{e,n_e} < w_{o,n_e}$$

A proof of this theorem can be found in [3], [11], [13].

The following generalization of Descartes' Rule of signs by Poincaré [14] will be used in this paper; for the sake of clarity, and for the general case, the dependence on K is suppressed.

Poincaré's generalization. *The number of sign changes in the coefficients of $(s+1)^k P(s)$ is a non-increasing function of k; for a sufficiently large k, the number of sign changes in the coefficients exactly equals the number of real, positive roots of $P(s)$.*

An application of Poincaré's generalization only accounts for the requirement of real, positive roots of the even and odd parts of $P(s,K)$. However, it does not account for the interlacing requirement of roots of the even and odd parts of $P(s,K)$ as required by the Hermite-Biehler theorem. The following lemma, given in [12], converts the requirement of interlacing of roots to checking the signs of the coefficients of a family of polynomials.

Lemma 3. *Suppose $Q(\lambda) = q_0 + q_1\lambda + \ldots + q_m\lambda^m$ and $R(\lambda) = r_0 + r_1\lambda + \ldots + r_m\lambda^m$ are polynomials of the same degree m. Let $\mu_1, \ldots \mu_m$ be the roots of $Q(\lambda)$ and $\xi_1, \ldots, \xi_m$ be the roots of $R(\lambda)$. Consider a one-parameter family of polynomials: $\tilde{Q}(\lambda,\eta) = Q(\lambda) - \eta R(\lambda)$. Then, the following two statements are equivalent:*

1) *The roots of $Q(\lambda)$ and $R(\lambda)$ are real and interlacing, i.e., $\mu_1 < \xi_1 < \mu_2 < \xi_2 < \ldots < \mu_m < \xi_m$.*
2) *For any $\eta \in \Re$, the roots of $\tilde{Q}(\lambda,\eta)$ are all real.*

Let $Q(s)$ be a complex polynomial of degree n. We will write $Q(jw) = Q_r(w) + jQ_i(w)$. If its degree is n, then for a sufficiently large w^*, the Mikhailov plot of $Q(jw)$ will lie entirely in one quadrant for all $w < -w^*$; we will say that $q_{r,n} + jq_{i,n}$ defined through $\lim_{w\to-\infty} \frac{Q(jw)}{|w|^n}$ belongs to the same quadrant. We will assume without any loss of generality that $q_{r,n}q_{i,n} \neq 0$; in fact, if $q_{r,n}q_{i,n} = 0$, one can consider $(1 + j\tau)Q(jw)$; for this polynomial, the leading coefficients of the corresponding real and imaginary polynomials are different from zero whenever $\tau \neq 0$ and the location of the roots of this polynomial being the same as that of $Q(jw)$. Let C_k, S_k, $k = 1,2,3,4$ be diagonal matrices of dimension $2n$; the $(m+1)^{st}$ diagonal elements of these matrices are respectively the signs of $\cos((2k-1)\frac{\pi}{4} + m\frac{\pi}{2})$ and $\sin((2k-1)\frac{\pi}{4} + m\frac{\pi}{2})$.

Hermite-Biehler Theorem for complex polynomials. *Let $Q(s)$ be a complex polynomial of degree n with $q_{r,n}q_{i,n} \neq 0$; the following statements are equivalent:*

1) *$Q(s)$ is Hurwitz.*
2) *All roots of $Q(jw)$ have positive imaginary parts.*
3) *All roots of the polynomials $Q_r(w)$ and $Q_i(w)$ are real and interlace; specifically, there exists a set of $(2n-1)$ real frequencies satisfying $-\infty < w_1 < w_2 < \ldots < w_{2n-1} < \infty$ that separates the roots of the real polynomials in such a way that for exactly one of $k = 1,2,3,4$, the following conditions hold:*

$$C_k \begin{bmatrix} q_{r,n} \\ Q_r(w_1) \\ Q_r(w_2) \\ \vdots \\ Q_r(w_{2n-1}) \end{bmatrix} > 0, \qquad S_k \begin{bmatrix} q_{i,n} \\ Q_i(w_1) \\ Q_i(w_2) \\ \vdots \\ Q_i(w_{2n-1}) \end{bmatrix} > 0$$

III. Main Results

Using the results of the earlier section, we will get the inner and outer approximation of the set of absolutely stabilizing controllers. The closed loop may be expressed as a linear system with transfer function, $H(s)$, perturbed by a sector-bounded non-linearity, $\phi(y)$, in the feedback path and hence may be treated as Lure-Postnikov system.

Let $N_c(s) = n_0 + n_1 s + \cdots + n_r s^r$ and $D_c(s) = d_0 + d_1 s + \cdots + d_{r-1}s^{r-1} + s^r$. Let K be the $(2r+1)-$tuplet, $(n_0,\ n_1,\ \cdots, n_r,\ d_0,\ d_1, \cdots, d_{r-1})$. Let $\Delta_1(s,K) = N_1(s)N_c(s) + D_p(s)D_c(s)$, where the coefficients of $\Delta_1(s,K)$ are affine functions of K. For a given $\mu \in [0,1]$, let $\Delta_2(s,K,\mu) = \mu\Delta_1(s,K) + (1-\mu)D_c(s)N_2(s)$ and let $Q(s,K,\mu)$ denote a one-parameter family of polynomials as μ varies from 0 to 1. Let $\mathscr{A}$ be the set of all K that render the closed loop absolutely stable. If $\mathscr{A}_{outer}$ is any set containing $\mathscr{A}$, we refer to $\mathscr{A}$ as an outer approximation and similarly if $\mathscr{A}_{inner}$ is a set contained in $\mathscr{A}$, it will be referred to as an inner approximation.

Based on the results in the earlier section, we now provide a way to construct an outer approximation of $\mathscr{A}$ below. Let $\Delta_2(jw,K,\mu) = \delta_r(w^2,K) + jw\delta_i(w^2,K)$ for some real polynomials δ_r and δ_i. Let the degrees of the polynomials $\delta_r(\lambda,K)$ and $\delta_i(\lambda,K)$ be n_r and n_i respectively.

Lemma 4. *Let $\mathscr{S}(p,\mu)$ be the set of K satisfying the following conditions for a given non-negative integer p and a $\mu \in [0,1]$: The number of sign changes in the coefficients of the polynomials $(1+\lambda)^p\delta_r(\lambda,K)$ and $(1+\lambda)^p\delta_i(\lambda,K)$ are respectively n_r and n_i. Then, $\mathscr{S}(p,\mu)$ is an outer approximation of $\mathscr{A}$ for every p and for every $\mu \in [0,1]$. Moreover, $\mathscr{S}(p+1,\mu) \subset \mathscr{S}(p,\mu)$.*

Proof. If K is any absolutely stabilizing controller, then for every μ, the polynomial $\Delta_2(s,K,\mu)$ is Hurwitz. By the Hermite-Biehler theorem, the polynomials δ_r, δ_i must have n_r, n_i real, positive respectively. By the generalization of the Descartes' rule of signs, for any p, the polynomials $(1+\lambda)^p\delta_r$, $(1+\lambda)^p\delta_i$ must have exactly n_r and n_i sign changes respectively in their coefficients. Hence, $K \in \mathscr{S}(p,\mu)$. Therefore, $\mathscr{A} \subset \mathscr{S}(p,\mu)$.

We will first observe that the maximum number of sign changes in the coefficients of $(1+\lambda)^p\delta_r$ is n_r as the number of real positive roots can at most be n_r. By the generalization of Descartes' rule of signs, the number of sign changes in the coefficients of $(1+\lambda)^p\delta_r$ is a non-increasing function of p. Therefore, if the number of sign changes in the coefficients of $(1+\lambda)^{p+1}\delta_r$ is n_r, it must follow that the number of sign changes in the coefficients of $(1+\lambda)^p\delta_r$ must also be n_r. A similar case holds for $(1+\lambda)^p\delta_i$. Hence, if $K \in \mathscr{S}(p+1,\mu)$, it must be that $K \in \mathscr{S}(p,\mu)$ for every non-negative integer p. □

The set $\mathscr{S}(p,\mu)$ can be refined further taking into account the requirement of interlacing in the following way: Let $K \in \mathscr{S})(p,\mu)$ be such that $\Delta_2(s,K,\mu)$ is not Hurwitz. Then $\mathscr{S}(p,\mu)$ can be refined in the following steps:

1) Find a $\eta > 0$ such that $Q(\lambda) = \delta_r(\lambda,K) - \eta\delta_i(\lambda,K)$ does not have all real roots.
2) Find a q such that $(1+\lambda)^q Q(\lambda)$ has fewer than n_r changes in the sign of its coefficients.
3) Consider the LPs associated with requiring the coefficients of $(1+\lambda)^p Q(\lambda,K)$ to have n_r sign changes subject to $K \in \mathscr{S}(p,\mu)$. Let the corresponding set be $\mathscr{S}_{ref}$. The set $\mathscr{S}_{ref}$ is a refinement of $\mathscr{S}(p,\mu)$ and is an outer approximation.

One can construct outer approximations corresponding to various values of $\mu \in [0,1]$. Since each of them is an outer approximation, $\cap_\mu \mathscr{S}_{ref}(\mu)$, is also an outer approximation. Such an outer approximation is a refinement of the outer approximation obtained for each μ and can again be determined using a linear programming approach.

We will use the characterization of SPR transfer functions given by Lemma 2 and the characterization of Hurwitz polynomials given by the Hermite-Biehler theorem to obtain an inner approximation for $\mathscr{A}$. The transfer function $G_T(s,K) = \frac{N_T(s,K)}{D_T(s,K)} = (1+qs)H(s,K) = (1+qs)\frac{N_{cl}(s,K)}{D_{cl}(s,K)}$ is required to be SPR. Consider the complex polynomial $\Delta_c(s,K,\alpha) := D_T(s) + j\alpha N_T(s)$ and further let $\Delta_c(jw,K,\alpha) := \Delta_r(w,K) + jD_i(w,K)$ for some real polynomials $\Delta_r(w,K)$ and $\Delta_i(w,K)$. Let the degree of $\Delta_c(s,K,\alpha)$ be $N = n+r$. Further, let $\Delta_r(w,K) = \delta_{r,N}w^N + \delta_{r,N-1}w^{N-1} + \cdots + \delta_{r,0}$ and similarly, let $\Delta_i(w,K) = \delta_{i,N}w^N + \delta_{i,N-1}w^{N-1} + \cdots + \delta_{i,0}$.

Theorem 2. *There exists a controller, $C(s)$, of order r, that renders the transfer function $G_T(s,K)$ SPR if and only if there exists a K such that*

1) $H(0,K) = \frac{N_{cl}(0,K)}{D_{cl}(0,K)} > 0$,
2) *the polynomials $N_{cl}(s,K)$ and $D_{cl}(s,K)$ are Hurwitz,*
3) *for every $\alpha \in \Re$, there exists a set of $2N-1$ frequencies, $-\infty < w_1(\alpha) < w_2(\alpha) < \cdots < w_{2N-1}(\alpha) < \infty$ such that K is a feasible solution of one of the following four linear programs:*

$$C_k \begin{bmatrix} \delta_{r,N}(\alpha) \\ \Delta_r(w_1,\alpha) \\ \Delta_r(w_2,\alpha) \\ \vdots \\ \Delta_r(w_{2N-1},\alpha) \end{bmatrix} > 0, \qquad S_k \begin{bmatrix} \delta_{i,N}(\alpha) \\ \Delta_i(w_1,\alpha) \\ \Delta_i(w_2,\alpha) \\ \vdots \\ \Delta_i(w_{2N-1},\alpha) \end{bmatrix} > 0.$$

These conditions are obtained by applying Lemma 2 to the transfer function $G_T(s,K)$ and application of the Hermite-Biehler theorem for complex polynomials to render the family of complex polynomials to be Hurwitz. By Theorem 1, the set K obtained as the solution to the above theorem, is an inner approximation for the set of absolutely stabilizing controllers.

IV. Illustrative Example

We will consider a one-link robot with a flexible joint, shown in Fig. 2, as an example for absolute stabilization. The governing equations of motion may be written as:

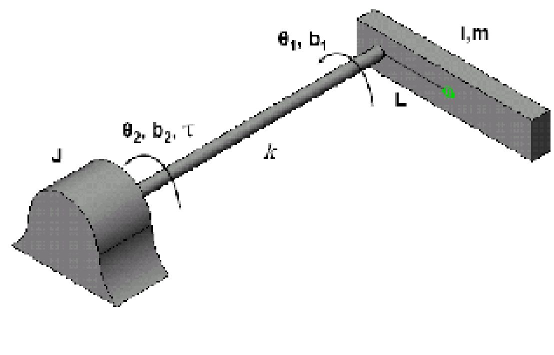

Fig. 2. One-Link Robot with a Flexible Joint

$$\begin{aligned} I_1\ddot\theta_1 + b_1\dot\theta_1 + mgL\sin\theta_1 + k(\theta_1-\theta_2) &= 0, \\ J\ddot\theta_2 + b_2\dot\theta_2 + k(\theta_2-\theta_1) &= \tau \end{aligned} \tag{9}$$

We can obtain a state space representation of the system (9) by choosing state variables :

$$x_1 = \theta_1, \quad x_2 = \dot\theta_1, \quad x_3 = \theta_2 \quad x_4 = \dot\theta_2$$

The state space representation is:

$$\begin{aligned} \dot x &= Ax + B_1 u - B\psi(y) \\ y &= Cx, \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{I} & -\frac{b_1}{I} & \frac{k}{I} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J} & 0 & -\frac{k}{J} & -\frac{b_2}{J} \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\psi(y) = \frac{mgL}{I}\sin y, \; u = \frac{\tau}{J}$$

The system parameters are given as follows :

$$J = 0.5kg\cdot m^2, \; b1 = 0.0Nm\cdot s/rad, \; k = 50.0Nm/rad$$
$$I = 25.0kg\cdot m^2, \; b2 = 1.0Nm\cdot s/rad, \; m = 1.0kg, \; L = 5.0m$$

A. PID Controller

Let us consider a PID controller :

$$\begin{aligned} C(s) &= k_p + \frac{k_i}{s} + k_d s \\ u &= k_p(r-y) + k_d(\dot r - \dot y) + k_i w \\ \dot w &= r - y, \end{aligned}$$

where $C(s)$ is the PID controller, w is the integral of the error and r is reference which is set to be 0. Fig. 3 shows a control structure for the one-link robot with a flexible joint which has a sector-bounded nonlinearity.

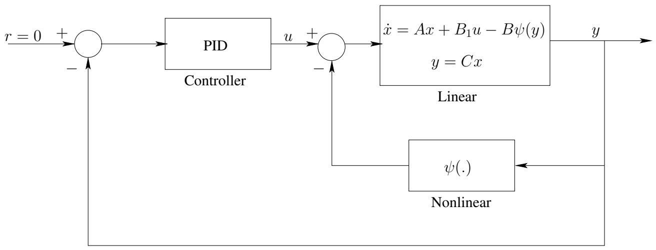

Fig. 3. Control Structure of One-Link Robot with a Flexible Joint

The closed loop system can be represented as an augmented system as follows :

$$\begin{aligned} \dot{z} &= \mathbf{A}z - \mathbf{B}\psi(y) \\ y &= \mathbf{C}z, \end{aligned}$$

where $z = \begin{bmatrix} x & w \end{bmatrix}'$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{k}{I} & -\frac{b_1}{I} & \frac{k}{I} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{k}{J} - k_p & -k_d & -\frac{k}{J} & -\frac{b_2}{J} & k_i \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\psi(y) = \frac{mgL}{I} \sin y,$$

which constitutes a realization of the transfer function,

$$\begin{aligned} G(s) =& \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B} \\ =& \frac{N_{cl}(s)}{D_{cl}(s)} \\ =& \frac{s^3 + 2s^2 + 100s}{s^5 + 2s^4 + 102s^3 + (4 + 2k_d)s^2 + 2k_p s + 2k_i} \end{aligned}$$

Note that the nonlinearity, $\psi(.) = \frac{mgL}{I} sin(.)$ though sector bounded, is not restricted to the first and third quadrants. A nonlinearity, $\psi(.)$ is said to belong to a sector $[\alpha, \beta]$, if the graph of this function belongs to a sector whose boundaries are the lines αy and βy. Fig. 4 shows the nonlinearity, $\psi(y) = \frac{mgL}{I} \sin y$ and the associated sector. From the figure it is clear that $\psi(.) \in [\alpha, \beta]$, where $\beta = \frac{mgL}{I}$ and $\alpha = \beta \cos(y^*)$, $y^* \approx 1.5\pi$ is the solution of the equation $y\cos(y) - \sin(y) = 0$.

Since the theory developed in this paper requires the nonlinearity to lie in the first and third quadrants, we need to transform the above system to the appropriate form. The following loop transformation (see Ex.6.1 in [15]) basically transforms the nonlinearity $\psi(.) \in [\alpha, \beta]$ to a case where the nonlinearity belongs to the sector $[0, \infty]$ (Fig. 5).

The nonlinearity $\bar{\psi}(.)$ now lies in the sector $[0, \infty]$. The modified plant is given by

$$\bar{G}(s) = \frac{\bar{N}(s,K)}{\bar{D}(s,K)} = 1 + (\beta - \alpha)\frac{G(s)}{1 + \alpha G(s)} = \frac{1 + \beta G(s)}{1 + \alpha G(s)}$$

The results developed in this paper are now applied to this modified system.

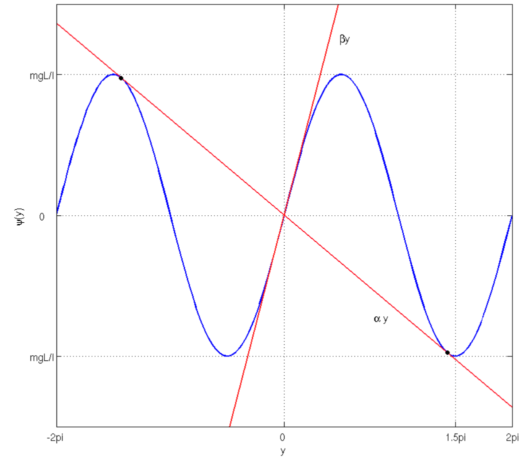

Fig. 4. Plot of the nonlinearity, $\psi(y) = \frac{mgL}{I} sin(y)$

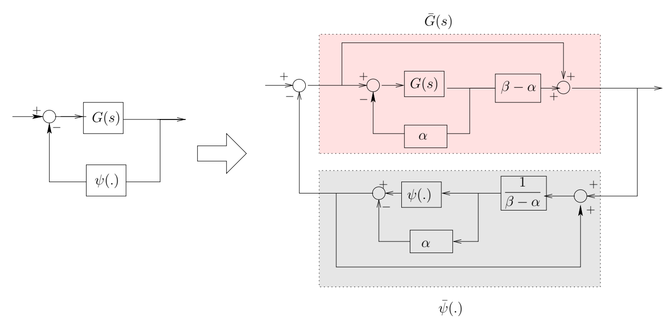

Fig. 5. Loop transformation

1) Outer Approximation: For a given $\mu \in [0,1]$, let $\Delta(s,K,\mu) = \mu\bar{N}(s,K) + (1-\mu)\bar{D}(s,K)$ and let $Q(s,K,\mu)$ denote a one-parameter family of polynomials as μ varies from 0 to 1.

Let $\Delta(jw,K,\mu) = \delta_r(w^2,K,\mu) + jw\delta_i(w^2,K,\mu)$.

To generate the outer approximation, we consider different values of $\mu \in [01]$, and require the polynomials $\delta_r(w^2,K)$ and $\delta_i(w^2,K)$ to have exactly two sign changes. Application of Lemma 4, generates an outer approximation to the set of absolutely stabilizing controllers.

2) Inner Approximation: Using Theorem 1, the system is absolutely stable if there is $q \geq 0$ such that $G_T(s,K) = \frac{N_{GT}(s,K)}{D_{GT}(s,K)} = (1+qs)\bar{G}(s,K)$ is SPR.

For strictly positive realness of the $G_T(s)$, the following conditions should be satisfied: (Theorem 2)

1) $G_T(0) = \frac{N_{GT}(0)}{D_{GT}(0)} > 0$,
2) $N_{GT}(s,K)$ and $D_{GT}(s,K)$ are Hurwitz for some $q \geq 0$,
3) $P(s,K) = D_{GT}(s,K) + j\alpha N_{GT}(s,K)$ is Hurwitz for some $q \geq 0$, $\forall \alpha \in \Re$.

We will illustrate how to find the set of all controllers so that above SPR conditions satisfy under $q = 1$.

1) For condition 1: We find that $G_T(0) = \frac{k_i}{k_i} = 1 > 0$.
2) For condition 2: $N_{GT}(s) = (1+qs)\bar{N}(s)$ is Hurwitz if

$\bar{N}(s)$ is Hurwitz. The real and imaginary parts of the $\bar{N}(s)$ at $s = jw$ are given by

$$\bar{N}(jw,K) = (2w^4 - (2k_d + 4.392)w^2 + 2k_i) + \\ + jw(w^4 - 102.2w^2 + 2k_p + 19.62)$$

For the polynomial $\bar{N}$ to be Hurwitz, there must exist a set of frequencies $0 = w_0 < w_1 < w_2 < w_3 < w_4$ for which at least one of the following two LPs is feasible $k = (1,3)$:

$$C_k \begin{bmatrix} 1 & 0 & 0 \\ 1 & w_1^2 & w_1^4 \\ 1 & w_2^2 & w_2^4 \\ 1 & w_3^2 & w_3^4 \\ 1 & w_4^2 & w_4^4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 0 \\ -4.3924 & 0 & 0 & -2 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_p \\ k_i \\ k_d \end{bmatrix} > 0,$$

$$S_k \begin{bmatrix} 1 & 0 & 0 \\ 1 & w_1^2 & w_1^4 \\ 1 & w_2^2 & w_2^4 \\ 1 & w_3^2 & w_3^4 \\ 1 & w_4^2 & w_4^4 \end{bmatrix} \begin{bmatrix} 19.62 & 2 & 0 & 0 \\ -102.1962 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_p \\ k_i \\ k_d \end{bmatrix} > 0.$$

A similar procedure is applied to find the set of controllers for which $D_T(s,K) = \bar{D}(s,K)$ is Hurwitz.

3) For condition 3: The family of polynomials $Q(s,\alpha) = D_{GT}(s) + j\alpha N_{GT}(s)$ should be Hurwitz $\forall \alpha \in \Re$. This family of polynomials can be decomposed as $Q(jw,K) = Q_r(w,K) + jQ_i(w,K)$.

For the polynomial $Q(s,K)$ to be Hurwitz, there must exist a set of frequencies $-1 = w_0 < w_1 < w_2 < \cdots < w_{10} < w_{11}$ for which at least one of the following four LPs (for $k = 1,\dots 4$) is feasible:

$$C_k \begin{bmatrix} 0 & 0\dots & -1 \\ 1 & w_1\dots & w_1^5 \\ 1 & w_2\dots & w_2^5 \\ \vdots & & \\ 1 & w_{11}\dots & w_{11}^5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 0 \\ -19.6\alpha & -2\alpha & -2\alpha & 0 \\ -3.92 & 0 & 0 & -2 \\ 106.6\alpha & 0 & 0 & 2\alpha \\ 2 & 0 & 0 & 0 \\ -3\alpha & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_p \\ k_i \\ k_d \end{bmatrix} > 0,$$

$$S_k \begin{bmatrix} 0 & 0\dots & -1 \\ 1 & w_1\dots & w_1^6 \\ 1 & w_2\dots & w_2^6 \\ \vdots & & \\ 1 & w_{11}\dots & w_{11}^6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2\alpha & 0 \\ -4.26 & 2 & 0 & 0 \\ -24.0\alpha & -2\alpha & 0 & -2\alpha \\ -102 & 0 & 0 & 0 \\ 104.2\alpha & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1\alpha & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ k_p \\ k_i \\ k_d \end{bmatrix} > 0$$

Fig. 6 shows the outer and inner approximation for the set of absolutely stabilizing controller on the same plot.

V. Conclusions

In this paper, we develop a procedure for the synthesis of fixed order controllers for nonlinear systems with sector bounded nonlinearities. We construct an inner and outer approximation of the set of absolutely stabilizing linear controllers by casting the closed loop system as a Lure-Postnikov system. The inner approximation is based on the well-known sufficient conditions that require Strict Positive Realness (SPR) of open loop transfer function (possibly with some multipliers) and a characterization of SPR transfer functions that require a family of complex polynomials to be Hurwitz. The outer approximation is based on the condition that the open loop transfer function must have infinite gain margin, which translates to a family of real polynomials being Hurwitz. We illustrate the proposed methodology through the construction of an inner and outer approximation of absolutely stabilizing controllers for a mechanical system.

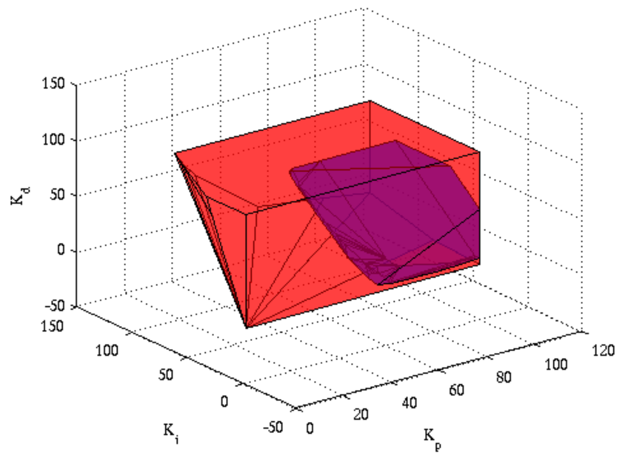

Fig. 6. Outer and inner approximation for the set of absolutely stabilizing PID controllers

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