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Mean Square Stability of Consensus over Fading Networks with Nonhomogeneous Communication Delays

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Abstract— **In this paper, we propose a discrete time consensus protocol which can solve the consensus problem in the network with nonhomogeneous communication delays. We give the sufficient conditions to reach consensus and provide a closed form formula for the consensus value. Furthermore, we investigate the mean square stability (MSS) of our protocol when each link of the network can break with a given probability at each time interval. The condition for checking MSS is equivalent to checking the spectral radius of a positive matrix. To gain more insight, we further restrict our attention to spatially invariant network structure and develop a more efficient expression to check the MSS. We derive a closed form formula to determine the MSS in the limit of large delays, get useful lower and upper bounds and analyze their implications for large classes of network topologies. We find that the consensus protocol is robust to link failures in the sense the system is always mean square stable if we put restrictions on the propagation gain.**

Keywords: network consensus, mean-square stability robustness, spatially invariant systems.

I. Introduction

Consensus problem has attracted much attention among researchers in the control community recently [5], [6], [8], [10], [11], [12]. Generally speaking, consensus means that all the agents of the network agree to a common value without recourse to a central coordinator or global communication. The consensus protocols are distributed feedback control laws based on local information that allow the coordination of multi-agent networks.

One particular interesting area of consensus problem is to study consensus in the presence of propagation delays [4], [8], [12], [16]. In particular, [8] provided a linear consensus protocol which can solve the consensus problem with a nice property that the consensus value is independent of the delays, but this property is at the price of assuming reciprocal channels and equal delays. In [12] and [16], the authors proposed a continuous time consensus protocol which can make the networked system reach consensus in the presence of nonhomogeneous delays and nonreciprocal channels. In both papers, the authors did not provide a close form formula for the consensus value, which we will investigate in this paper.

In this paper, we study the consensus problem in the network with link failures and communication delays. We first assume the network channels are reliable but subject to nonhomogeneous propagation delays. We study a discrete time consensus protocol and show that the consensus value is not the average of the initial values but has a bias term introduced by the delays.

We next study the consensus problem when each link has communication delays and can break stochastically. Deterministic network topology changes are considered in [8], [12]. In contrast, we consider the case where the network is stochastically fading due to the nature of the communication links. In [14], the authors investigate the MSS of a consensus protocol under the assumption that the network topology is undirected and the channels have no delay. Their problem can be fitted into our framework and the results in that paper can be recovered by our method. Moreover, we consider directed network topology and nonhomogeneous delays which is more general than the problem considered in [14]. This work can be also seen as the extension of the work in [3], where the communication delays are not considered. We adopt the framework of [1] and [2] which interprets the variance of random variables as the size of a stochastic uncertainty in an otherwise deterministic model, the mean network. We derive the condition for checking MSS as checking the spectral radius of a positive matrix. To gain more insight, we further restrict our attention to the case that the mean network is spatially invariant. The extra structure in the network leads to a more efficient computation of the spectral radius condition for MSS. To understand how the system parameters affect the MSS of the networked system, we investigate the situation when the propagation delays among the network channels are very large. Based on the generating function for Legendre Polynomial, we derive a simple formula for checking the MSS, as well as upper and lower bounds on the stability condition to derive properties and limitation of the algorithm which are not specific to the interconnection topology and size of the network.

[3] identified complex behaviors induced by power law distributions or heavy tails in networked control systems. In our simulation studies, we also find that if the networked system transits from Mean Square stable to Mean Square unstable, the state trajectory will exhibit a quite interesting complex collective behavior.

II. Setup and Preliminary Results

We consider a set of n identical Linear Time Invariant (LTI) discrete time systems called nodes, which are connected over a network. The interaction among the nodes is properly described by a weighted directed graph (or digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with the set of nodes $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$, set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and a weighted adjacency matrix

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$\mathcal{A} = [a_{ij}]$. An edge of $\mathcal{G}$ is denoted by $e_{ij} = (v_i, v_j)$ which means that there exists a channel from v_j to v_i. The adjacency elements associated with the edges of the graph are equal to 1, i.e. $e_{ij} \in \mathcal{E} \iff a_{ij} = 1$. Moreover, we assume $a_{ii} = 0$ for all i. For any node $v_i \in \mathcal{V}$, we define the information neighbor of v_i as

$$N_i = \{v_j \in \mathcal{V} : e_{ij} = (v_i, v_j) \in \mathcal{E}\}. \quad (1)$$

The set N_i represents the set of nodes sending data to node i and $|N_i|$ denotes the number of neighbors of node i.

We assume that each node is Single Input Single Output (SISO), strictly proper, and is governed by the following difference equation:

$$P: \quad \begin{array}{rcl} x_p^+ & = & A_p x_p + B_p u_p \\ y_p & = & C_p x_p, \end{array} \quad (2)$$

where $x_p \in \mathbb{R}^m$, x_p^+ denote the system state at next discrete time, $A_p \in \mathbb{R}^{m \times m}$, $B_p \in \mathbb{R}^{m \times 1}$ and $C_p \in \mathbb{R}^{1 \times m}$. We assume the plant is stabilizable.

Although the nodes have the same dynamics, they set their input u_p differently based on their neighbors in the network. It is convenient to introduce the following diagonal transfer matrix to represent the dynamics of all the nodes. Let $u = [u_{p1}, \ldots, u_{pn}]^T$ and $y = [y_{p1}, \ldots, y_{pn}]^T$. Let $\hat{G}$ be the transfer matrix from u to y. $\hat{G}$ is diagonal with $G_{ii} = P$, for $i = 1, \ldots, n$, and $G_{ij} = 0$ for $i \neq j$. The state space representation of G is given by $G = (A, B, C, 0)$, where A, B and C are block diagonal with n blocks and each block being equal to A_p, B_p and C_p respectively.

In this paper we will study the network where all its channels are packet drop channels. The stochastic set-up in this paper is motivated by the presence of stochastic drop-out links and it is different from recent studies on deterministic\arbitrary switching [8].

A. Packet Drop Channel Model

Our simplified model of a packet drop channel neglects the quantized nature of the packet and focuses on the unreliability of the connection. The PD channel is a memoryless map $PD : \mathbb{R} \to \mathbb{R} \times \{0, 1\}$ which has one input v' and two outputs ξ' and η' defined as $\eta' = \xi' v'$, where ξ' denotes the channel state and it is a Bernoulli IID random variable with probability e of being equal to 0.

Note that $\xi' = 1$ means that the channel input is correctly received and provided to the plant while $\xi' = 0$ implies that the message is lost.

B. Fading Networks

The general framework introduced in [1] and [2] allows to easily model the presence of multiple PD channels as a fading network.

Definition 2.1: [3]An analog Fading Network is composed by two parts:

1) The Mean Network $\mathcal{N}$
2) The stochastic perturbation, Δ.

The Mean Network is a deterministic LTI system described

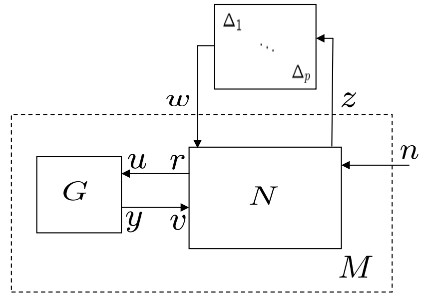

Fig. 1. General robust control framework with stochastic perturbations.

by the following state space realization:

$$\mathcal{N}: \quad \begin{bmatrix} \eta^+ \\ z \\ r \end{bmatrix} = \begin{bmatrix} F & G_n & G_w & G_v \\ H_z & L_{zn} & L_{zw} & L_{zv} \\ H_r & L_{zn} & L_{rw} & L_{rv} \end{bmatrix} \begin{bmatrix} \eta \\ n \\ w \\ v \end{bmatrix}, \quad (3)$$

where η is the Network state, v is the network input vector (which corresponds to the vector y in our case), r is the Network output vector (which corresponds to the plant input u in our case). $w \in \mathbb{R}^p$ and $z \in \mathbb{R}^p$. Δ maps $z \to w$ and is defined as $\Delta = \text{diag}(\Delta_i, i = 1, \ldots, p)$. For each $i = 1, \ldots, p$, $\Delta_i(0), \ldots, \Delta_i(k), \ldots$ are IID random variable with

$$E\{\Delta_i(k)\} = 0 \qquad and \qquad E\{(\Delta_i(k))^2\} = \sigma^2 \qquad \forall k \geq 0.$$

Moreover, $\Delta_1(k), \ldots, \Delta_p(k)$ are independent for each k, although not necessarily identically distributed. Δ acts as multiplication operator on z to provide w; i.e., $w_i(k) = \Delta_i(k) z_i(k)$ for $i = 1, \ldots, p$, $\forall k \geq 0$. Finally n is a vector of white noise signals independent from each other and independent from Δ.

C. Feedback Over Analog Fading Networks

Next, we consider the framework described in Figure 1. Since we are interested in stabilization, we assume that n is not present.

Let $M = \mathcal{F}(G, N) = (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ denote the minimal realization of feedback interconnection of G with N. Following the setup shown in Figure 1, the LTI discrete-time system M has $p-$inputs and $p-$outputs and is in feedback with the diagonal uncertainty Δ described in Definition 2.1.

Let $H = \mathcal{F}(\Delta, M)$ denote the feedback interconnection of Δ and M, with $\chi_0 = \chi(0)$ independent of Δ. We assume the feedback interconnection of Δ and M is well-posed, namely that the solution to system H exists for any realization of Δ.

Definition 2.2: System H is mean square stable if $Q(k) = E\{\chi(k)\chi(k)^{'}\}$ is well defined for all k and $\lim_{k\to\infty} Q(k) = 0$, where $E\{\cdot\}$ denotes the expectation of a random variable.

When n is present, H has n as an input and the definition of MSS requires that $\lim_{k\to\infty} Q(k) < \infty$. At any rate, since the noise n enters the equation linearly, the results does not change.

Theorem 2.3: [2] Assume that $M = \mathcal{F}(G, N) = (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is stable and that $\mathcal{D}$ is either strictly lower triangular or strictly upper triangular. The feedback interconnection of M and Δ is Mean Square stable iff $\sigma^2 < \frac{1}{\rho(\hat{M})}$ where $\rho(\cdot)$ denotes the spectral radius and $\hat{M} = \begin{bmatrix} \|M_{11}\|_2^2 & \dots & \|M_{1p}\|_2^2 \\ \vdots & \dots & \vdots \\ \|M_{p1}\|_2^2 & \dots & \|M_{pp}\|_2^2 \end{bmatrix}$.

Remark 2.4: We would like to point out that an advantage of the above condition, with respect to the standard state space contraction condition in [7] is that it is expressed in terms of input output maps and their $\mathcal{H}_2$ norms. This allows us to handle the presence of (large) delays more effectively directly in the frequency domain. We refer the reader to [1] for complete discussion and reference to related work.

D. Graph Theory and Circulant Matrix

In this subsection, we will review some notations and previous results of graph theory and circulant matrix that will be used later in this paper.

Definition 2.5: The degree matrix D of a digraph $\mathcal{G}$ is a diagonal matrix with diagonal entries $[D]_{ii} = d_i$, where $d_i = \sum_{j=1}^n a_{ij}$ is the in degree of node i.

Note that for $0-1$ adjacency elements, $d_i = |N_i|$. We also define $d = \max\{d_1, d_2, \dots, d_n\}$, which is the maximum in-degree of all the nodes.

Definition 2.6: The (weighted) Laplacian $L = [l_{ij}]$ of a digraph $\mathcal{G}$ is defined as

$$l_{ij} \triangleq \begin{cases} \sum_{k=1\neq i}^n a_{ik} & \text{if } j = i \\ -a_{ij} & \text{if } j \neq i \end{cases} \tag{4}$$

we can also equivalently define the Laplacian as $L \triangleq D - \mathcal{A}$. According to the above definition of graph Laplacian, all the row-sums of L are zero, therefore L always has an eigenvalue of zero corresponding to the eigenvector $\mathbf{1} = (1, \dots, 1)^T$, i.e. $L\mathbf{1} = 0$. Invoking the Geršgorin disk theorem [17], all the other eigenvalues of L have positive real parts.

Definition 2.7: A digraph is *strongly connected* (SC) if any two distinct nodes of the digraph can be connected through a path that follows the direction of the edges of the digraph.

The strongly connected digraph has the following property.

Lemma 2.8: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted digraph with Laplacian L. If $\mathcal{G}$ is strongly connected, then $\text{rank}(L) = n-1$ and L has a positive left eigenvector associated with the eigenvalue of zero.

The proof of this lemma can be found in [18]. Since $\text{rank}(L) = n-1$, the previous lemma implies that the Laplacian of a strongly connected digraph has a simple eigenvalue of zero.

Given an n-dimensional real vector, $[c_0, c_1, \dots, c_{n-1}]$, the associated circulant matrix is given by $\bar{L} = \text{circul}[c_0, c_1, \dots, c_{n-1}]$. Circulant matrices have the following properties [19]:

1) The eigenvalues of a circulant matrix are given by $\lambda_k = \sum_{i=0}^{n-1} c_i \rho_k^i$, where $\rho_k = e^{j2\pi k/n}$ is one of the n roots of -1.

2) The $n \times n$ Fourier matrix V, whose k^{th} column

$$v_k = n^{-1/2} \cdot [1, \rho_k, \rho_k^2, \dots, \rho_k^{n-1}]$$

diagonalizes any circulant matrix of size n. Furthermore, $VV^* = V^*V = I$ and $\det(V) = 1$, where $\det(\cdot)$ denotes the determinant of a matrix.

III. A Consensus Protocol

We first consider the consensus problem in the network whose channels are reliable but subject to nonhomogeneous communication delays. We assume the agents are very simple and have minimal intelligence. We consider the following discrete-time protocol

$$x_i(k+1) = x_i(k) + \beta \sum_{j \in N_i} [x_j(k - \tau_{ij}) - x_i(k)] \tag{5}$$

where $\beta > 0$ is the propagation gain and τ_{ij} is the delay in the channel e_{ij}. Now taking z transform of (5), we get

$$z x_i(z) - z x_i(0) = x_i(z) + \beta \sum_{j \in N_i} [x_j(z) \cdot z^{-\tau_{ij}} - x_i(z)].$$

We put the above equations into a compact form as

$$x(z) = z[zI - I + \beta L(z)]^{-1} x(0), \tag{6}$$

where $x(z) = [x_i(z), x_2(z), \dots, x_n(z)]$ and $L(z) = [l_{ij}(z)]_{n \times n}$ with $l_{ii}(z) = d_i$ and $l_{ij}(z) = -z^{-\tau_{ij}}$ $\forall j \in N_i$. We need to note that $L(1)$ is a graph Laplacian. The following result directly follows from Theorem 3.9 in [4].

Corollary 3.1: Consider a networked system with its channels being subject to non-homogeneous communication delays, suppose the network topology is strongly connected, then protocol (5) will solve the consensus problem if $\beta < 1/d$. Furthermore, the consensus value is given by

$$\alpha = \frac{\sum_{i=1}^n \gamma_i x_i(0)}{\sum_{i=1}^n \gamma_i + \beta \cdot \sum_{i=1}^n \sum_{j \in N_i} \gamma_i \tau_{ij}},$$

where γ is the left eigenvector corresponding to the zero eigenvalue of L , i.e. $\gamma^T L = 0$, and $x_i(0)$ is the initial condition of the i-th agent for $i = 1, 2, \dots, n$.

Remark 3.2: For the discrete time consensus protocol in [9], the authors show that the consensus value is not affected by β. We need point out that there is no confliction between these two results since in their case, the channel has no dynamics, thus $\tau_{ij} = 0$ $\forall i, j$ and the β does not affect the final value. In particular, their protocol can be seen as a special case in our framework when each channel has no delay. As can be seen from Corollary 3.1, when the delays in the channel are very large, the consensus value will be almost zero, although it still relates to the average of the nodes initial condition.

IV. Consensus over Fading Networks

Protocol (5) can solve the consensus problem in the network with communication delays. However, when all the channels can break with a given probability at each time interval, the system becomes stochastic, and we need to study its stochastic stability properties. We study a natural modification of protocol (5) which characterize the unreliability of the communication channels.

$$x_i(k+1) = x_i(k) + \beta \sum_{j\in N_i} \xi_{ij}(k)[x_j(k-\tau_{ij}) - x_i(k)] \quad (7)$$

for $i = 1,\ldots,n$, where $\xi_{ij}(k)$ denotes the state of the channel e_{ij} at time k. We assume

$$\xi_{ij}(k) = \begin{cases} 1 & \text{with Prob. } 1-e \\ 0 & \text{with Prob. } e \end{cases} \quad \forall i,j,k \qquad (8)$$

and $\xi_{ij}(k)$ are independent $\forall i,j,k$. Next, we define $\xi_{ij}(k) = \mu + \Delta_{ij}(k)$, where $\mu = 1-e$ and $\Delta_{ij}(k)$ is a new random variable with $E(\Delta_{ij}(k)) = 0$ and $\sigma^2(\Delta_{ij}(k)) = e(1-e)$. By this definition $\Delta_{ij}(k)$ are independent $\forall i,j,k$ too. We can rewrite (7) as

$$\begin{aligned} x_i(k+1) &= x_i(k) + \beta\mu \sum_{j\in N_i} [x_j(k-\tau_{ij}) - x_i(k)] \\ &+ \beta \sum_{j\in N_i} \Delta_{ij}(k)[x_j(k-\tau_{ij}) - x_i(k)]. \quad (9) \end{aligned}$$

A. Consensus in the Mean

We begin by giving the definition of consensus in the mean.

Definition 4.1: System (7) achieves consensus in the mean if $\lim_{k\to\infty} E\{x(k)\} = \mathbf{1}\cdot c$, where $c \in \mathbb{R}$ is a constant.

The following condition for consensus in the mean can be derived easily from Corollary 3.1.

Corollary 4.2: Consider a fading network with its channels being subject to non-homogeneous communication delays, suppose the network topology is strongly connected, protocol (7) achieves consensus in the mean if $0 < \beta\mu < 1/d$ and the consensus value is given by

$$\lim_{k\to\infty} E\{x(k)\} = \frac{\sum_{i=1}^n \gamma_i x_i(0)}{\sum_{i=1}^n \gamma_i + \beta\mu \cdot \sum_{i=1}^n \sum_{j\in N_i} \gamma_i \tau_{ij}}.$$

B. Mean Square Stability

Although we get consensus in the mean, its second moment may deteriorate the convergence. To study the MSS of the system, we seek to apply Theorem 2.3 and therefore we need to characterize the transfer function matrix M seen by the Δ. The reader can follow the main steps with the help of Figure 2. Taking the Z-transfom of (9) and putting the transformed equations together, we can get the closed-loop system dynamics as

$$zx(z) - zx(0) = (I - \beta\mu L(z) + \beta Q\Delta P(z))x(z), \quad (10)$$

where Q, $P(z)$ and Δ are defined respectively as $Q = \text{diag}\{Q_1,\ldots,Q_n\}$, $P(z) = [P_1(z),\ldots,P_n(z)]^T$ and $\Delta = \text{diag}\{\Delta_1,\ldots,\Delta_n\}$. Here, $Q_i = \underbrace{[1,\ldots,1]}_{d_i}$, $\Delta_i = \text{diag}\{\Delta_{i1},\ldots,\Delta_{id_i}\}$ and $P_i(z) = [l_{ij}(z)]^T_{j\in N_i}$ for $i = 1,\ldots,n$. Note that $l_{ij}(z)$ is a row vector with $i-$th element being 1 and $j-$th element being $-z^{-\tau_{ij}}$. We also have $\sum_{j\in N_i} l_{ij}(z) = l_i(z)$, where $l_i(z)$ is the i-th row of $L(z)$. For example, suppose $l_i(z) = [2, 0, -z^{-3}, 0 - z^{-5}]$, then we have $l_{i3}(z) = [1, 0, -z^{-3}, 0, 0]$ and $l_{i5}(z) = [1, 0, 0, 0, -z^{-5}]$.

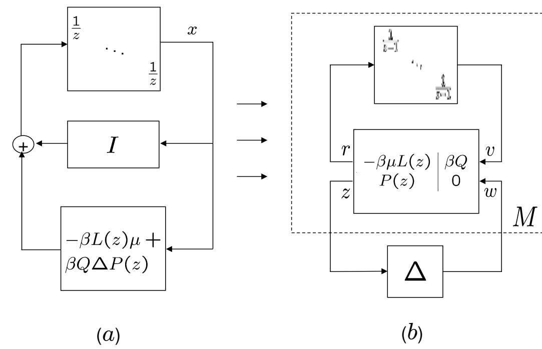

Fig. 2. a) Block diagram for protocol (7). b) Equivalent representation in terms of Mean Network framework.

The Mean Network is then obtained as follows: w is the output from Δ, from (10), we know $w = \Delta P(z)x(z)$. Since z is the input to Δ, $z = P(z)x(z)$. Thus, with $v = x(z)$, we get the mean network

$$\mathcal{N}: \quad \begin{bmatrix} r \\ z \end{bmatrix} = \begin{bmatrix} -\beta\mu L(z) & \beta Q \\ P(z) & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \qquad (11)$$

From Figure 2, the transfer matrix seen by Δ is given by the following expression:

$$M(z) = \beta P(z) H^{-1} Q$$

where $H(z) = (z-1)I + \beta\mu L(z)$ and $M(z)$ has the dimension of $\sum_{i=1}^n d_i$.

Remark 4.3: Since $H(1) = \beta\mu L$, we know $H^{-1}(z)$ has a pole at 1. $P(z)$ has a multivariable zero at 1 since $\gamma^T QP(1) = 0$. Therefore, there is a pole-zero cancellation at $z = 1$ in the product of $P(z)H^{-1}(z)$ and the direction of $\mathbf{1}$ is not observable. Therefore, we study the MSS of $\tilde{x}(k)$ (not of $x(k)$), where $\tilde{x}(k)$ has its components as

$$\tilde{x}_i(k) := x_i(k) - \frac{1}{n}\sum_{i=1}^n x_i(k)$$

and the system is mean square stable if $\lim_{k\to\infty}\{E(\tilde{x}(k)\tilde{x}'(k))\} = 0$.

The following condition for checking MSS can be derived directly form theorem 2.3.

Corollary 4.4: The networked system under (7) is mean square stable iff $\mu(1-\mu) < \frac{1}{\rho(\hat{M})}$, where $M(z) = \beta P(z)H^{-1}(z)Q$ and $\mu = 1-e$.

From Theorem 2.3, to compute $\rho(\hat{M})$, we need compute the $\mathcal{H}_2$ norm of $(\sum_{i=1}^n d_i)^2$ transfer functions. Therefore as the number of uncertain links increase, the computational complexity will increase at a speed of n_u^2. Although Corollary 4.4 gives a general computational way to check the MSS, testing such condition may be cumbersome in systems with many agents and unreliable links and little structure. This difficulty may translate into the impossibility of guaranteeing MSS in self assembled or human designed large heterogeneous systems. For this reason, we next restrict our attention to spatially invariant topologies, for which the spectral radius computation can be simpler and MSS is manageable even in large scale networks.

C. MSS over Spatially Invariant Architectures

We need the following assumption in our future development.

Assumption 4.5: $L(z)$ (in (6)) is a circulant matrix such that $L(z) = \text{circul}[q, c_2, \dots, c_n]$, where q denotes the number of neighbors of every node, $c_i = 0$ if node i is not a neighbor of node 1 and $c_i = -z^{-n_i}$ if node i is a neighbor of node 1.

Note that with Assumption 4.5, we have $d_i = q$ for all i. By Assumption 4.5, we actually restrict our attention to the spatially invariant network topology. The spatially invariant structure of the network greatly reduce the computational complexity to determine the MSS of the networked system as shown by the next result.

Theorem 4.6: The networked system under Assumption 4.5 is mean square stable iff

$$\mu(1-\mu) < \frac{1}{\beta^2 \cdot \frac{1}{n}\sum_{k=1}^{n}\sum_{i\in N_1}\|f_{ki}(z)\|_2^2}, \qquad (12)$$

where $f_{ki}(z) = \frac{1-\rho_k^{i-1}\cdot z^{-n_i}}{z-(1-\beta\mu q)-\beta\mu\sum_{j\in N_1}\rho_k^{j-1}\cdot z^{-n_j}}$ with $\rho_k = e^{j\frac{2\pi}{n}k}$. Recall that $\lambda_k = \sum_{i\in N_1}\rho_k^{i-1}$ is the $k-$th eigenvalue of the adjacency matrix $\mathcal{A}$.

Theorem 4.6 shows that when we restrict the network topology to be spatially invariant, instead of computing $\mathcal{H}_2$ norm of n^2q^2 transfer functions, we only need consider nq transfer functions.

Remark 4.7: Note that (12) does not hold, and therefore the system is not MSS according to Definition 2.2, when $\mu = 0$, as $\rho(\hat{M}) = \infty$ when $\mu = 0$.

D. Limit of Large Communication Delays

The condition (12) can still be cumbersome to verify if the number of delays is large. At the same time, it is of interest to understand how large (dominant) delays affect the stability properties of the protocol. In this subsection, we study the MSS in the limit of large communication delays. This leads to a simplified formula and captures a relevant limiting case. We restrict our analysis to the case where each link has the same delay

Assumption 4.8: All the delays in the network channels are equal and denoted by n_d.

For the networked system with homogeneous time delays, we have the following results.

Theorem 4.9: Suppose $\beta\mu q < 1$ and the networked system satisfy Assumption 4.5 and 4.8, then

$$\lim_{n_d\to\infty}\sum_{i\in N_k}\|f_{ki}(z)\|_2^2 = \frac{q}{b(1-b)}\cdot\left[1-\sqrt{\frac{(1-b)^2-|c_k|^2}{(1+b)^2-|c_k|^2}}\right],$$

where $b = 1-\beta\mu q$ and $c_k = -\beta\mu\lambda_k$.

As the consequence of theorem 4.9, we have the following corollaries.

Corollary 4.10: Suppose $\beta\mu q < 1$ and the networked system satisfy Assumption 4.5 and and 4.8, then as $n_d \to \infty$, it is mean square stable iff

$$\mu(1-\mu) < \frac{\mu n b}{\beta(1-\mu)\cdot\sum_{k=1}^{n}\left[1-\sqrt{\frac{(1-b)^2-|c_k|^2}{(1+b)^2-|c_k|^2}}\right]}.$$

Note that μ can be simplified from both sides if we consider the resulting condition only for $\mu > 0$. Corollary 4.10 provides us with a condition which as expected depends for given β and e on the topology of the network through the eigenvalues of the adjacency matrix. Next, we propose upper and lower bounds which only loosely depend on the topology through the number on neighborhoods, q.

Corollary 4.11: Assume $\beta\mu q < 1$ and the networked system satisfy Assumption 4.5 and 4.8, then as $n_d \to \infty$, for $\mu > 0$

$$\frac{2\beta(1-\mu)}{2-\beta\mu q} < \sigma^2\rho(\hat{M}) \le \frac{\beta(1-\mu)}{1-\beta\mu q}$$

and the upper bound is tight iff $q = 1$.

Remark 4.12: The lower bound for $\rho(\sigma^2\hat{M})$ is not achievable. We need $c_k = 0$ for all k to achieve this bound, but $c_k = 0$ means that $\lambda_k = 0$, $\forall k$, which is a contradiction to the fact that $\mathcal{A}$ always has a eigenvalue of q since $\mathcal{A}\cdot\mathbf{1} = q\cdot\mathbf{1}$. Besides, note that both the lower and upper bound are independent of the size of the network, which means we can identify situations where the MSS is independent from the size of the network.

Remark 4.13: Assume $\beta\mu q < 1$ to ensure the mean stability, we consider the following cases:

1) The step size β is not fixed, which means each network agent has the ability to adapt to the variation of the network topology. From corollary 4.11, if $\beta < \frac{1}{1-\mu+\mu q}$ the network is mean square stable and if $\beta > \frac{2}{2-2\mu+\mu q}$ the network is mean square unstable. If the probability of link failure does not change much, as we increase the number of links by adding more neighbors to each node, β must be kept appropriately small to accommodate these incoming links, otherwise we may first lose MSS and next lose mean stability.

2) The step size β is fixed. In this case, from Corollary 4.11, if $\beta > \frac{1}{1-\mu}$, the networked system is mean square unstable. Assume $\beta < \frac{1}{1-\mu}$, if $q < \frac{1-\beta(1-\mu)}{\beta\mu}$ the network is mean square stable and if $q > \frac{1-\beta(1-\mu)}{\beta\mu}$ the network is mean square unstable. Therefore we need to keep the number of neighbors small or decrease the probability of link failure (make μ small) to maintain the MSS.

Corollary 4.14: Assume $\beta\mu q < 1$ and the networked system satisfy Assumption 4.5 and 4.8. As $n_d \to \infty$, if $\beta < 1/q$, the networked system is always mean square stable for any $0 < \mu \le 1$.

Remark 4.15: Corollary 4.14 is a consequence of Corollary 4.11 since when $\beta < 1/q$, the upper bound of $\sigma^2\rho(\hat{M})$ is always less than 1. Corollary 4.14 implies that the consensus protocol we studied is robust to the link failures in the sense that we can guarantee the MSS by choosing β properly. The constraint on β to ensure MSS is actually the same constraint

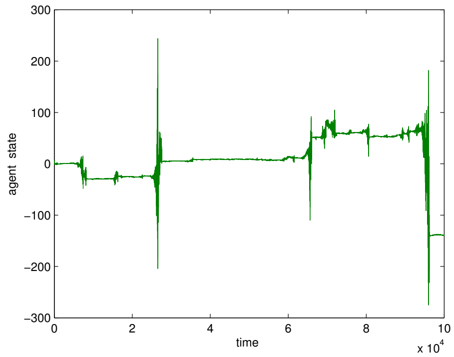

Fig. 3. Complex behavior: ring topology of 4 nodes, $n_d = 5$, $\beta = 1.06$, $\mu = 0.5$ and $\sigma^2\rho(\hat{M}) = 1.0504$.

on β to achieve consensus when the channels are reliable. However, we see that then knowledge of q is required to choose β appropriately. Although one could play safe by choosing β very small, this affects the convergence speed. Finally it is interesting that for systems under study, the lost of MSS seems to be due to the mismatch between the expected number of neighbors and the worst case number of neighbors that the channel realizations permit. In other words, the system is more stressed when all the neighbors talk to the node. Of course, this situation is specific to the case under study and is consistent with the findings of [3], [14].

E. Simulation Studies

In [3], Elia found the emergence of power laws in the networked control system when the system is subject to stochastic link failures and additive White Gaussian noise. Here we do the same and simulate. We investigate the consensus protocol (7) in the presence of a small variance additive noise. We find that a complex behavior like the one shown in Figure 3 will emerge when the system transit from mean square stable to mean square unstable. We also find that the emergence of the complex behavior is independent of whether there are delays in the channels and how many uncertain links the system has and as expected can be related to the integration of noise process of unbounded variance.

V. CONCLUSIONS

In this paper, we study the MSS of the networked system whose channels are modeled as packet drop channels. We develop a computational-efficient way to determine the mean square stability when the network topology is spatially invariant. For the case of large delays, we derive the closed form formula to determine the MSS and consequently get the lower and upper bound of $\sigma^2\rho(\hat{M})$, which is independent of the size of the network. As a result, as we increase the number of neighbors, we first lose MSS and then lose mean stability. We also identified that the consensus protocol we investigated is robust to link failures if we choose the system parameter β properly. We also presented simulations that show the emergence of a complex behavior as the system loses the MSS. Future research is needed to further characterize the behavior, as well as to identify other large scale architectures that exhibit strong robustness properties.

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