Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

An LMI Framework for Analysis and Design of Multi-dimensional Haptic Systems

Gianni Bianchini, Marcello Orlandesi, Domenico Prattichizzo

Abstract— **This paper presents a convenient framework based on passivity and Linear Matrix Inequalities (LMIs) for stability analysis and controller design for haptic systems involving multiple devices and human operators interacting with a common virtual environment. The proposed approach addresses peculiar features of the multi-dimensional scenario such as different operator-device configurations, and allows for taking into account structural constraints such as decentralized controller implementation. An LMI-based stability condition is given and a a class of stabilizing structured controllers that can be parameterized in terms of the solution of suitable LMI problems is introduced.**

I. Introduction

Stability is a key feature in haptic interaction with virtual environments, since unwanted oscillations can impair realism and, most importantly, may be potentially harmful for the human operator. The issue of stability in this context has been addressed by several authors since the early 90's [1] and involves quite a few aspects, since the systems at hand are complex and some of their components, namely the human operators, are difficult to model. Stability has been considered from multiple viewpoints, and passivity has often been exploited in this context, since it provides a powerful tool for analyzing etherogeneous interconnected systems [2]. The fundamental paper [3] and more recent works such as [4] provide different approaches to the characterization of passivity in sampled-data systems and in particular in haptics. In [5], [6], [7], [8] a discrete-time passivity framework is proposed to deal with stability analysis and controller design also in the presence of non-passive virtual environments, and in particular [7] addresses the presence of nonlinearities. In [9], an H_∞ approach to controller design for transparency is proposed. The above contributions focus on the case of a single human operator interacting with a one-degree of freedom virtual environment.

Multi-contact interaction is an important issue in haptics [10]. Researchers have investigated several aspects in this scenario such as friction modeling [11], [12] and interaction with deformable objects [13], but mostly neglected stability issues. Recently, haptic systems where many users interact with a common virtual environment [14] have been considered. The analysis of stability in the case of one or more human operators interacting with a shared virtual enviroment through multiple points of contact is a multidimensional problem that exhibits peculiar issues that need investigation.

In this paper, we deal with stability analysis in the multi-contact scenario in a passivity-based framework which is an extension of the one employed in [5], [6], [7], [8]. This framework relies on the basic assumptions that a human performing an interaction using m degrees of freedom be regarded as a $m-$input, $m-$output passive operator, and that every degree of freedom of a device is both sensed and actuated. Also, the virtual coupling approach to controller design is used.

This paper builds upon previous contributions by the authors [15], [16]. In particular, the LMI-based approach in [15] suffers from a certain amount of conservatism since it addresses multiple devices but does not distinguish among the different configurations in which one or several operators may interact with the devices themselves. As an example, the case of two haptic devices operated by a single human with two hands/fingers is considered the same as the case of two operators interacting separately with the same devices. In [16] an attempt is made at assessing stability while taking different device/operator configurations into account at the price of solving a set of bilinear matrix inequalities, which is in general not computationally efficient.

As far as virtual coupling design is concerned, particular attention has to be devoted to structural constraints that may arise. Indeed, multi-contact systems may be physically distributed, and therefore the virtual coupling may be constrained to share only limited information with the devices and the virtual environment due to decentralization and limited communication requirements. In both the above papers, stabilizing controllers are designed as virtual spring-damper networks of given structure that provide an amount of energy dissipation which is pre-computed in order to guarantee stability. Clearly, the spring-damper model is in general restrictive, and relying merely on the "implementation" of a pre-computed amount of damping is likely to introduce conservatism.

In this paper, we present two main improvements over [16]. Firstly, we propose a sufficient stability condition that can be checked by solving a single LMI problem, and therefore very efficiently. As a second contribution, we provide a parameterization of a wider class of structured stabilizing controllers which no longer relies on pre-computation of virtual damping and is characterized in terms of the solution of a sequence of LMI problems. This parameterization foreshadows the possibility of addressing several performance problems, such as device and controller transparency, within the proposed controller class in a computationally efficient way.

The paper is organized as follows. In Section II we report some preliminary results; in Section III we formalize the problem and introduce some specific results on passivity-based analysis of multi-dimensional haptic systems; in Section IV we derive the sought LMI stability condition, while in Section V we address controller parameterization. Section VI reports an illustrative application example and conclusions are drawn in Section VII.

Notation

For a square matrix X, $X>0$ ($X<0$) denotes positive (negative) definiteness, X^T denotes transpose and $\|X\|$ denotes some matrix norm of X; I_m is the $m \times m$ identity matrix. $X = \text{blockdiag}(X_1,\ldots,X_N)$ denotes a block-diagonal matrix with diagonal blocks $X_1,\ldots,X_N$. With $\mathscr{BD}(m;m_1,\ldots,m_N)$ we denote the set of $m\times m$ block-diagonal matrices whose N blocks have dimensions $m_1\times m_1,\ldots,m_N\times m_N$, with $\sum_{i=1}^N m_i = m$. The latter notation is also used without ambiguity for block-diagonal transfer matrices of $m-$input, $m-$output linear systems and, more generally, of $m-$input, $m-$output operators. With $\overline{\mathscr{BD}}(m_1\times n_1,\ldots,m_N\times n_N)$ we indicate the set of non-square block-diagonal matrices with block sizes $m_i \times n_i$, $i = 1, \ldots, N$.

II. PRELIMINARIES

The approach to stability analysis and virtual coupling design presented in this paper exploits a generalization of the framework in [5]-[8], which is based upon several passivity-related concepts [2], [17] that are recalled below.

Definition 1: (continuous-time passivity). Let Σ be a continuous-time dynamical system with input vector $u(t) \in \mathbb{R}^m$, output vector $y(t) \in \mathbb{R}^m$, and state vector $\psi(t) \in \mathbb{R}^n$. If there exists a continuously differentiable positive definite function $V(\psi) : \mathbb{R}^n \to \mathbb{R}$ (called the *storage function*) and $m \times m$ symmetric matrices Δ and Φ such that along all system trajectories $(\psi(t), u(t), y(t))$, $t \in \mathbb{R}$, the following inequality holds

$$\dot{V}(\psi(t)) < y(t)^T u(t) - y(t)^T \Delta y(t) - u(t)^T \Phi u(t),$$

then, system Σ is *passive* if $\Delta = \Phi = 0$, *output strictly passive with level* Δ ($\Delta-$OSP) if $\Delta > 0$, $\Phi = 0$, *input strictly passive with level* Φ ($\Phi-$ISP) if $\Delta = 0$, $\Phi > 0$, respectively.

Definition 2: (discrete-time passivity). Let Σ_d be a discrete-time dynamical system with input vector $u(k) \in \mathbb{R}^m$, output vector $y(k) \in \mathbb{R}^m$, and state vector $\psi(k) \in \mathbb{R}^n$. If there exists a positive definite function $V(\psi) : \mathbb{R}^n \to \mathbb{R}$ and $m \times m$ symmetric matrices Δ and Φ such that along all system trajectories $(\psi(k), u(k), y(k))$, $k \in \mathbb{N}$, the following inequality holds

$$\begin{array}{l} V(\psi(k+1)) - V(\psi(k)) \\ < y(k)^T u(k) - y(k)^T \Delta y(k) - u(k)^T \Phi u(k), \end{array} \tag{1}$$

then the system is passive if $\Delta = \Phi = 0$, output strictly passive ($\Delta-$OSP) if $\Delta > 0$, $\Phi = 0$, input strictly passive ($\Phi-$ISP) if $\Delta = 0$, $\Phi > 0$, respectively.

Note that Δ and Φ need not necessarily be positive definite: indeed, a dynamical system will be said to lack OSP (ISP) when the above definitions hold for non-positive definite Δ (Φ).

Let Σ_d be a discrete-time time-invariant linear system defined by the state space representation (A, B, C, D), where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$. A straightforward extension of the standard Kalman-Yacubović-Popov lemma [17] applies.

Lemma 1: System Σ_d is passive ($\Delta-$OSP, $\Phi-$ISP) if and only if there exists a symmetric matrix $P \in \mathbb{R}^n$ such that the following two matrix inequalities hold:

$$P > 0$$

$$\left[\begin{array}{cc} A^T PA - P + C^T \Delta C & A^T PB - \frac{C^T}{2} + C^T \Delta D \\ B^T PA - \frac{C}{2} + D^T \Delta C & B^T PB - \frac{D+D^T}{2} + D^T \Delta D + \Phi \end{array} \right] < 0. \tag{2}$$

In order to address the design problems presented in this paper, we find it convenient to use an alternative formulation of the above result in which matrices A, B, C, D appear linearly in the matrix inequalities that define the passivity condition (2). We have the following result whose proof is based on a simple Schur complement argument (see [15]).

Lemma 2: Let $\Delta > 0$. System Σ_d is passive ($\Delta-$OSP, $\Phi-$ISP) if and only if there exists a symmetric matrix $Q \in \mathbb{R}^n$ satisfying the constraints

$$Q > 0, \quad \left[\begin{array}{cc} Q^{-1} & R^T \\ R & S \end{array} \right] > 0,$$

$$R = \left[\begin{array}{c} \frac{C}{2} \\ A \\ C \end{array} \right], \quad S = \left[\begin{array}{ccc} \frac{D^T+D}{2} + \Phi & B^T & D^T \\ B & Q & 0 \\ D & 0 & \Delta^{-1} \end{array} \right].$$

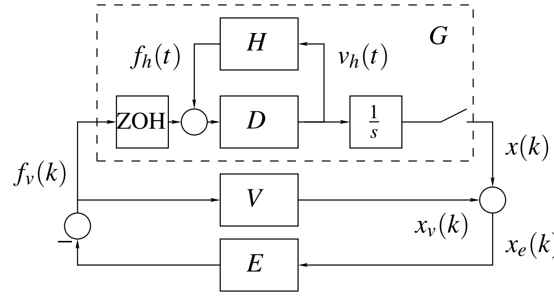


Fig. 1. Haptic loop L

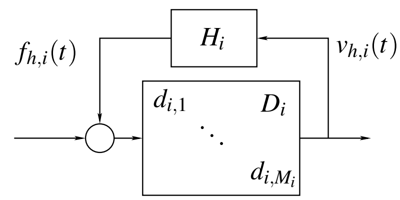


Fig. 2. Interconnection of human operator H_i with haptic devices $d_{i,1}, \ldots, d_{i,M_i}$

III. PROBLEM FORMULATION AND BASIC RESULTS

A haptic system is typically modeled as a sampled-data system (with sampling period T) resulting from the interconnection of four main components described by suitable I/O mappings (see Fig. 1): a human operator block H, a haptic device block D, a computer-simulated virtual environment E, and a virtual coupling V, whose role is to act as a controller [3], [5]-[8]. The mappings H and D are continuous-time, while E and V are described by discrete-time dynamical systems. Let L denote the overall loop.

In this paper, we fit the above model to the case of N human operators H_i, $i = 1, \ldots, N$, each assumed to have m_i degrees of freedom (DOF). Each operator H_i is assumed to interact with M_i different devices denoted by $d_{i,j}$, $j = 1, \ldots, M_i$, where $d_{i,j}$ has $m_{i,j}$ DOF. Clearly, $m_i = \sum_{j=1}^{M_i} m_{i,j}$. All devices are coupled through a computer-simulated $m-$DOF virtual environment E (with $m = \sum_{i=1}^{N} m_i$) and through a virtual coupling V, which are both represented by $m-$input, $m-$output discrete-time dynamical systems. In order to simplify our exposition, we assume the absence of delay in the computations and consider only the impedance causality representation of the haptic system (see [6]), although the proposed results are believed to be easily extendable to cover both the delayed case and admittance causality.

The interaction of each operator H_i with the respective set of devices $d_{i,j}$, $j = 1, \ldots, M_i$ can be described by the feedback loop in Fig. 2, in which $f_{h,i}(t) \in \mathbb{R}^{m_i}$ represents the generalized force vector, $v_{h,i}(t) \in \mathbb{R}^{m_i}$ is the generalized velocity vector presented to the operator by the devices operated by H_i, and

$$D_i = \text{blockdiag}(d_{i,1}, \ldots, d_{i,M_i}). \tag{3}$$

It turns out that the overall system is described by the interconnection L in Fig. 1, where

$$\begin{array}{l} H = \text{blockdiag}(H_1, \ldots, H_N) \in \mathscr{BD}(m; m_1, \ldots, m_N) \\ D = \text{blockdiag}(D_1, \ldots, D_N) \in \mathscr{BD}(m; m_1, \ldots, m_N) \end{array} \tag{4}$$

$$\begin{array}{l} f_h(t) = [f_{h,1}^T(t) \; \ldots \; f_{h,N}^T(t)]^T \\ v_h(t) = [v_{h,1}^T(t) \; \ldots \; v_{h,N}^T(t)]^T \end{array} \tag{5}$$

and where $x(k) \in \mathbb{R}^m$ and $f_v(k) \in \mathbb{R}^m$ are the sampled generalized device displacement vector and the generalized force feedback vector, respectively.

Remark 1: Note that no peculiar structure is enforced a-priori on V. However, it is often the case that each haptic device (or group of devices) has its own controller. This requirement can be taken into account by assuming that V has a suitable block-diagonal structure. In particular, if the block diagonal structure of V matches that of D, i.e., V is of the form

$$V = \mathrm{blockdiag}(V_1, \ldots, V_N) \in \mathscr{BD}(m; m_1, \ldots, m_N), \qquad (6)$$

then the controller is completely decentralized, i.e., each controller block V_i acts (i.e., senses and provides feedback) only on the m_i DOFs pertaining to device D_i.
Clearly, additional requirements arising from decentralized computation and communication restrictions may enforce different contraints on V. For the sake of simplicity, in the sequel we will assume that V may only be constrained to be block-diagonal, which is a structure general enough to take several practical implementation requirements into account.

Passivity-based stability analysis of haptic systems typically relies on the assumption that both the human and the device can be seen as passive operators; in particular, when an impedance causality model is employed, the device dynamics is assumed to be OSP [7]. The OSP level pertaining to a given device can be related to the amount of damping introduced into the system by the device itself. The problem of its computation has been addressed in [5] for linear and in [8] nonlinear devices. In this respect, we would like to point out that the analysis performed in this paper is linear in nature, but it allows for the presence of nonlinear devices, since only the dissipation levels and not the explicit device dynamics are involved.
Motivated by the above observations, the following assumption is made in the context of this paper.

Assumption 1: (a) Each device $d_{i,j}$ is a $\Delta_{d_{i,j}}-$OSP continuous-time dynamical system, and (b) each human block H_i is a passive continuous-time m_i-input, m_i-output operator.
In view of Assumption 1, it is easily seen that the device block D defined by (3),(4) is Δ_D-OSP, where

$$\Delta_D = \mathrm{blockdiag}(\Delta_{D_1}, \ldots, \Delta_{D_N}) \qquad (7)$$

being

$$\Delta_{D_i} = \mathrm{blockdiag}(\Delta_{d_{i,1}}, \ldots, \Delta_{d_{i,M_i}}). \qquad (8)$$

In the context of this paper, the following notion of loop stability is considered [7].

Definition 3: The haptic loop L is stable if the generalized velocity vector $v_h(t)$ goes to zero in steady state.

We look for a loop stability criterion which allows for taking into account the structure of the human-device block. In this respect, we follow an approach that is closely related to the computation of structured Lyapunov functions [18]. To proceed, we find it convenient to introduce a loop transformation parameterized by a diagonal matrix Γ whose form depends on the structure itself.
Let us introduce the set of matrices

$$\begin{aligned}&\mathscr{G}(m; m_1, \ldots, m_N) = \{\Gamma \in \mathbb{R}^{m\times m}\\ &: \ \Gamma = \mathrm{blockdiag}(\gamma_1 I_{m_1}, \ldots, \gamma_N I_{m_N}),\ \gamma_i \geq 1,\ i = 1, \ldots, N\}.\end{aligned} \qquad (9)$$

For any $\Gamma \in \mathscr{G}(m; m_1, \ldots, m_N)$ corresponding to a set of scalars $\gamma_1, \ldots \gamma_N \geq 1$, consider the interconnected system L^Γ in Fig. 3 and denote

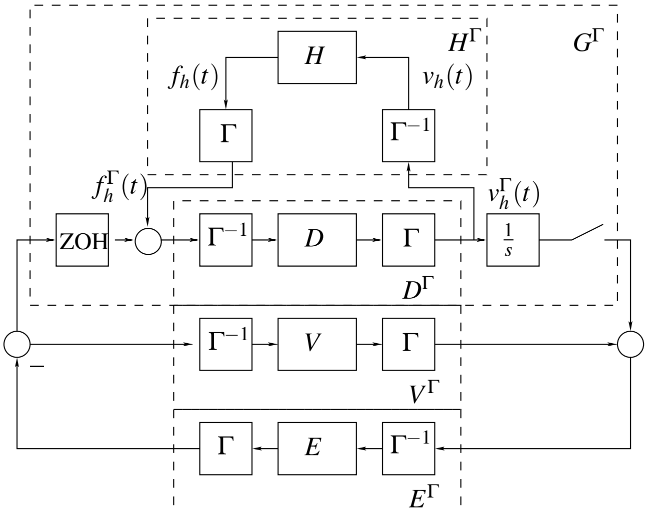


Fig. 3. Equivalent haptic loop L^Γ

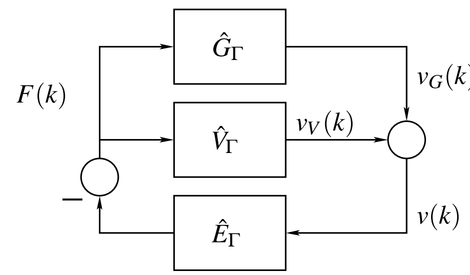


Fig. 4. Transformed haptic loop $\hat{L}^\Gamma$

$$\begin{aligned} f_h^\Gamma(t) &= \Gamma f_h(t) = [\gamma_1 f_{h,1}^T(t) \ \ldots \ \gamma_N f_{h,N}^T(t)]^T \\ v_h^\Gamma(t) &= \Gamma v_h(t) = [\gamma_1 v_{h,1}^T(t) \ \ldots \ \gamma_N v_{h,N}^T(t)]^T. \end{aligned} \qquad (10)$$

The following result states some key properties of the loop L^Γ.

Theorem 1: Let Assumption 1 hold. Then, the following properties pertain to L^Γ:

(P1) for any $\Gamma \in \mathscr{G}(m; m_1, \ldots, m_N)$, the block $H^\Gamma = \Gamma H \Gamma^{-1}$ is passive,

(P2) for any $\Gamma \in \mathscr{G}(m; m_1, \ldots, m_N)$, the block $D^\Gamma = \Gamma D \Gamma^{-1}$ is Δ_D-OSP, where Δ_D is as in (7),(8),

(P3) if there exists $\Gamma \in \mathscr{G}(m; m_1, \ldots, m_N)$ such that the loop L^Γ is stable, i.e., the signal $v_h^\Gamma(t)$ goes to zero in steady state, then so is the original loop L, i.e., the velocity vector $v_h(t)$ goes to zero in steady state.

Proof: See [19]. ■

The loop L^Γ introduced above and the related properties stated by Theorem 1 allow for analyzing stability of the haptic system by exploiting a generalization of the approach in [5]-[8]. To this purpose, consider the further transformed loop $\hat{L}^\Gamma$ in Fig. 4, where

$$\begin{aligned} \hat{G}^\Gamma &= \frac{z-1}{Tz}[G^\Gamma + K], \quad \hat{V}^\Gamma = \frac{z-1}{Tz}[V^\Gamma - K], \\ \hat{E}^\Gamma &= \frac{Tz}{z-1} E^\Gamma \end{aligned} \qquad (11)$$

being K a constant $m \times m$ matrix. The loop $\hat{L}^\Gamma$ is a purely discrete-time system in which the blocks $\hat{G}^\Gamma$, $\hat{V}^\Gamma$ and $\hat{E}^\Gamma$

can be characterized in terms of their OSP or ISP levels. The following result concerns the passivity properties of $\hat{G}^\Gamma$.

Theorem 2: Let $\hat{G}^\Gamma$ be as in (11) and suppose Assumption 1 holds. Assume Δ_D as in (7),(8) and let

$$K = \frac{T}{2}\Delta_D^{-1}. \tag{12}$$

Then, for any $\Gamma \in \mathscr{G}(m; m_1, \ldots, m_N)$, $\hat{G}^\Gamma$ in (11) is (discrete-time) $\Delta_{\hat{G}}-$OSP with

$$\Delta_{\hat{G}} = \Delta_D. \tag{13}$$

Proof: Since D^Γ is Δ_D-OSP and H^Γ is passive for any Γ by Theorem 1, the result follows from a quite straightforward extension of Lemma 2 in [8] applied to G^Γ. ■

The characterization introduced above allows for deriving a sufficient stability condition for the haptic loop L via a generalization of the results in [7]. Indeed, since the loop transformation in (11) is the same as the one performed in [7], it is readily checked that if the discrete-time transformed loop $\hat{L}_\Gamma$ is asymptotically stable, then so is L^Γ, i.e., the signal $v_h^\Gamma(t)$ goes to zero in steady state. Hence, taking property (P3) of Theorem 1 into account, we have the sought stability condition.

Theorem 3: If there exists $\Gamma \in \mathscr{G}(m; m_1, \ldots, m_N)$ such that the transformed loop $\hat{L}^\Gamma$ is asymptotically stable, then the haptic loop L is stable, i.e., the generalized velocity vector $v_h(t)$ presented to the human operator(s) goes to zero in steady state.

Remark 2: It is worth noting that enforcing $\gamma_i \geq 1$, $i = 1, \ldots, N$ in the definition of $\Gamma \in \mathscr{G}(m; m_1, \ldots, m_N)$ in (9) does not affect the generality of Theorem 3 (see Remark 2 in [19]).

The problem now becomes that of assessing stability of the transformed loop $\hat{L}^\Gamma$. This can be done by exploiting the passivity levels of $\hat{G}^\Gamma$, $\hat{V}^\Gamma$ and $\hat{E}^\Gamma$. The following condition generalizes standard results concerning stability of feedback and parallel interconnections of passive systems and can be expressed in matrix inequality form.

Theorem 4: If there exist $\Gamma \in \mathscr{G}(m; m_1, \ldots, m_N)$ and symmetric matrices $\Delta_{\hat{V}}$ $\Phi_{\hat{E}}$ such that

1) $\hat{V}^\Gamma$ is $\Delta_{\hat{V}}-$OSP,
2) $\hat{E}^\Gamma + \Phi_{\hat{E}}$ is passive, i.e., $\hat{E}^\Gamma$ is $(-\Phi_{\hat{E}})-$ISP, and
3) the following matrix inequalities hold:

$$\Delta_{\hat{V}} > 0, \qquad \begin{bmatrix} \Delta_D - \Phi_{\hat{E}} & -\Phi_{\hat{E}} \\ -\Phi_{\hat{E}} & \Delta_{\hat{V}} - \Phi_{\hat{E}} \end{bmatrix} > 0 \tag{14}$$

then, the haptic loop L is stable.

Proof: See [19]. ■

IV. LMI STABILITY CONDITION

In this section, we exploit Theorem 4 to derive a sufficient stability criterion when the virtual environment is described as a linear time-invariant system. This criterion is shown to boil down to an LMI condition.

Let the virtual environment E be given as a linear time-invariant system $E(z)$ with state-space realization (A_E, B_E, C_E, D_E). It is easily seen that a state-space realization of $E(z)(\frac{Tz}{z-1})$ is given by $(A_{\hat{E}}, B_{\hat{E}}, C_{\hat{E}}, D_{\hat{E}})$ where

$$\begin{aligned} A_{\hat{E}} &= \begin{bmatrix} A_E & B_E \\ 0 & I_m \end{bmatrix}, \quad B_{\hat{E}} = \begin{bmatrix} B_E T \\ I_m T \end{bmatrix}, \\ C_{\hat{E}} &= \begin{bmatrix} C_E & D_E \end{bmatrix}, \quad D_{\hat{E}} = [D_E T]. \end{aligned} \tag{15}$$

In turn, for fixed Γ, $\hat{E}^\Gamma$ has the state-space representation $(A_{\hat{E}}, B_{\hat{E}}\Gamma^{-1}, \Gamma C_{\hat{E}}, \Gamma D_{\hat{E}}\Gamma^{-1})$.

The following is the first main result of the paper. It shows that checking the existence of Γ, $\Delta_{\hat{V}}$ and $\Phi_{\hat{E}}$ satisfying the conditions of Thorem 4 can be cast as an LMI problem, and therefore provides a computationally appealing stability criterion.

Theorem 5: Suppose there exist symmetric matrices P, $Q_{\hat{V}}$, $Q_{\hat{E}}$ and X satisfying the LMIs

$$\begin{aligned} & P > 0, \quad X \in \mathscr{G}(m; m_1, \ldots, m_N), \quad Q_{\hat{V}} > 0 \\ & \begin{bmatrix} A_{\hat{E}}^T P A_{\hat{E}} - P & A_{\hat{E}}^T P B_{\hat{E}} - \frac{C_{\hat{E}}^T X}{2} \\ B_{\hat{E}}^T P A_{\hat{E}} - \frac{X C_{\hat{E}}}{2} & B_{\hat{E}}^T P B_{\hat{E}} - \frac{X D_{\hat{E}}}{2} - \frac{D_{\hat{E}}^T X}{2} - Q_{\hat{E}} \end{bmatrix} < 0 \\ & \begin{bmatrix} \Delta_D X - Q_{\hat{E}} & -Q_{\hat{E}} \\ -Q_{\hat{E}} & Q_{\hat{V}} - Q_{\hat{E}} \end{bmatrix} > 0 \end{aligned} \tag{16}$$

and let $\Gamma = \sqrt{X}$, where X solves (16). Then, the haptic loop L is stable for all virtual couplings V such that the corresponding $\hat{V}^\Gamma$ is $\Delta_{\hat{V}}-$OSP with $\Delta_{\hat{V}}$ satisfying

$$\Delta_{\hat{V}} \geq \Gamma^{-1} Q_{\hat{V}} \Gamma^{-1}.$$

Proof: See [19]. ■

Remark 3: We observe that if $Q_{\hat{V}}$ in the LMI problem (16) is constrained to belong to $\mathscr{BD}(m; m_1, \ldots, m_N)$, then for any $\Gamma \in \mathscr{G}(m; m_1, \ldots, m_N)$, it holds that $Q_{\hat{V}} \geq \Gamma^{-1} Q_{\hat{V}} \Gamma^{-1}$.

V. STABILIZING STRUCTURED VIRTUAL COUPLING PARAMETERIZATION

We are now interested in providing a computationally viable parameterization of a class of stabilizing virtual coupling systems with a given structure. More specifically, we seek the parameterization of a set $\mathscr{V}$ of linear virtual coupling systems V which share the following properties:

- (R1) V stabilizes the haptic loop,
- (R2) V has an arbitrarily assigned block-diagonal structure, i.e., $V = \text{blockdiag}(V_1, \ldots, V_{\bar{N}}) \in \mathscr{BD}(m; \bar{m}_1, \ldots, \bar{m}_{\bar{N}})$, where each block V_i is a linear system with state space dimension $\bar{n}_i$, for given $\bar{N}, \bar{m}_i, \bar{n}_i$, $i = 1, \ldots, \bar{N}$.

Let (A_V, B_V, C_V, D_V) denote a state space representation of V. Requirement (R2) is equivalent to the condition that $A_V \in \mathscr{BD}(n; \bar{n}_1, \ldots, \bar{n}_{\bar{N}})$, $B_V \in \mathscr{BD}(\bar{n}_1 \times \bar{m}_1, \ldots \bar{n}_{\bar{N}} \times \bar{m}_{\bar{N}})$, $C_V \in \mathscr{BD}(\bar{m}_1 \times \bar{n}_1, \ldots \bar{m}_{\bar{N}} \times \bar{n}_{\bar{N}})$ and $D_V \in \mathscr{BD}(m; \bar{m}_1, \ldots, \bar{m}_{\bar{N}})$.

For any $\Gamma \in \mathscr{G}(m; m_1, \ldots, m_N)$, let $(A_V^\Gamma, B_V^\Gamma, C_V^\Gamma, D_V^\Gamma)$ be a state space representation of $V^\Gamma = \Gamma V \Gamma^{-1}$. Clearly, a state space representation of V and of V^Γ can be computed uniquely one from the other since $(A_V, B_V, C_V, D_V) = (A_V^\Gamma, B_V^\Gamma \Gamma, \Gamma^{-1} C_V^\Gamma, \Gamma^{-1} D_V^\Gamma \Gamma)$ and moreover the corresponding matrices have the same block diagonal structure.

A straightforward computation yields the following state space representation of the transformed virtual coupling $\hat{V}_\Gamma$:

$$\begin{aligned} A_{\hat{V}_\Gamma} &= \begin{bmatrix} A_V^\Gamma & -\frac{1}{T} B_V^\Gamma \\ 0 & 0 \end{bmatrix}, \quad B_{\hat{V}_\Gamma} = \begin{bmatrix} \frac{1}{T} B_V^\Gamma \\ I_m \end{bmatrix}, \\ C_{\hat{V}_\Gamma} &= \begin{bmatrix} C_V^\Gamma & -\frac{1}{T} D_V^\Gamma + \frac{1}{T} K \end{bmatrix}, \quad D_{\hat{V}_\Gamma} = \frac{1}{T} D_V^\Gamma - \frac{1}{T} K. \end{aligned} \tag{17}$$

We are now ready to state the main design result, which provides the parameterization of a set $\mathscr{V}$ of controllers that satisfy requirements (R1) and (R2).

Theorem 6: Consider the haptic loop L, let Δ_D be the device OSP level as in (7),(8) and $(A_{\hat{E}}, B_{\hat{E}}, C_{\hat{E}}, D_{\hat{E}})$ be a state space realization of $E(z)\frac{Tz}{z-1}$ as in (15). Let

$P, X, Q_{\hat{V}}, Q_{\hat{E}}, \Sigma_{\hat{V}}, Q, Y, A_V^{\Gamma}, B_V^{\Gamma}, C_V^{\Gamma}, D_V^{\Gamma}$ be any solution of the LMI problem

$$
\begin{aligned}
& X \in \mathscr{G}(m; m_1, \ldots, m_N) \\
& A_V^{\Gamma} \in \mathscr{BD}(n; \bar{n}_1, \ldots, \bar{n}_{\bar{N}}) \\
& C_V^{\Gamma} \in \overline{\mathscr{BD}}(\bar{m}_1 \times \bar{n}_1, \ldots \bar{m}_{\bar{N}} \times \bar{n}_{\bar{N}}) \\
& B_V^{\Gamma} \in \overline{\mathscr{BD}}(\bar{n}_1 \times \bar{m}_1, \ldots \bar{n}_{\bar{N}} \times \bar{m}_{\bar{N}}) \\
& \Sigma_{\hat{V}},\ Q_{\hat{V}},\ D_V^{\Gamma} \in \mathscr{BD}(m; \bar{m}_1, \ldots, \bar{m}_{\bar{N}})
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
& P > 0, \quad Q_{\hat{V}} > 0 \\
& \begin{bmatrix} A_{\hat{E}}^T P A_{\hat{E}} - P & A_{\hat{E}}^T P B_{\hat{E}} - \frac{C_{\hat{E}}^T X}{2} \\ B_{\hat{E}}^T P A_{\hat{E}} - \frac{X C_{\hat{E}}}{2} & B_{\hat{E}}^T P B_{\hat{E}} - \frac{X D_{\hat{E}}}{2} - \frac{D_{\hat{E}}^T X}{2} - Q_{\hat{E}} \end{bmatrix} < 0 \\
& \begin{bmatrix} \Delta_D X - Q_{\hat{E}} & -Q_{\hat{E}} \\ -Q_{\hat{E}} & Q_{\hat{V}} - Q_{\hat{E}} \end{bmatrix} > 0
\end{aligned} \tag{19}
$$

$$
Q > 0, \quad \begin{bmatrix} Y & R^T \\ R & S \end{bmatrix} > 0 \tag{20}
$$

where

$$
R = \begin{bmatrix} \frac{C_{\hat{V}_\Gamma}}{2} \\ A_{\hat{V}_\Gamma} \\ C_{\hat{V}_\Gamma} \end{bmatrix}, \quad S = \begin{bmatrix} \frac{D_{\hat{V}_\Gamma} + D_{\hat{V}_\Gamma}^T}{2} & B_{\hat{V}_\Gamma}^T & D_{\hat{V}_\Gamma}^T \\ B_{\hat{V}_\Gamma} & Q & 0 \\ D_{\hat{V}_\Gamma} & 0 & \Sigma_{\hat{V}} \end{bmatrix} \tag{21}
$$

being $A_{\hat{V}}^{\Gamma}, B_{\hat{V}}^{\Gamma}, C_{\hat{V}}^{\Gamma}, D_{\hat{V}}^{\Gamma}$ as in (17), with the further non-convex constraint

$$
\Sigma_{\hat{V}} = Q_{\hat{V}}^{-1}, \quad Y = Q^{-1}. \tag{22}
$$

Then, the virtual coupling V defined by the state space representation

$$
(A_V, B_V, C_V, D_V) = (A_V^{\Gamma}, B_V^{\Gamma}\Gamma, \Gamma^{-1}C_V^{\Gamma}, \Gamma^{-1}D_V^{\Gamma}\Gamma) \tag{23}
$$

where

$$
\Gamma = \sqrt{X} \tag{24}
$$

has the structure $\mathscr{BD}(m; \bar{m}_1, \ldots, \bar{m}_{\bar{N}})$ and stabilizes the haptic loop L.

Proof: Constraints (18) enforce that the virtual coupling computed according to (23),(24) has the required structure. Moreover, (19) and (20),(21),(22) ensure, respectively, that the OSP level $Q_{\hat{V}}$ satisfies the loop stability condition of Theorem 4 for some $Q_{\hat{E}}$ and that it can be achieved by a virtual coupling V of the given structure; the latter property simply follows from Lemma 2. Note that $Q_{\hat{V}}^{-1}$ can be used in place of $\Delta_{\hat{V}}^{-1}$ in (21),(22) by virtue of Remark 3, since $Q_{\hat{V}}$ is constrained to be in $\mathscr{BD}(m; m_1, \ldots, m_N)$. ∎

Remark 4: The LMI problem (18)-(21) with the additional non-convex condition (22) can be reformulated as a sequence of LMI optimization problems via the cone complementarity linearization algorithm in [20]. The details can be found in [19].

VI. Application Example

We consider a haptic system in which the interaction with a $2-$DOF linear virtual environment E is performed through a pair of $1-$DOF haptic devices d_a and d_b. We are interested in stability analysis and virtual coupling design for the haptic loop in the following two cases:

i) both devices are operated by the same (2-DOF) human H_1, i.e., by a single human using both hands,

ii) the two devices are operated by two (1-DOF) humans H_1 and H_2.

According to the characterization in Section III, each block of the haptic loop L is described by a $m-$input, $m-$output system with $m = 2$. Moreover, assumptions i) and ii) enforce the following problem structures, respectively.

$$
\text{i)} \quad \begin{cases} N = 1,\ m_1 = 2,\ M_1 = 2,\ d_{1,1} = d_a,\ d_{1,2} = d_b \\ D_1 = \text{blockdiag}(d_{1,1} = d_a, d_{1,2} = d_b) \\ D = D_1 \\ H = H_1 \\ \Gamma \in \mathscr{G}(2;2) \text{ i.e. } \Gamma = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_1 \end{bmatrix}, \quad \gamma_1 \geq 1 \end{cases} \tag{25}
$$

$$
\text{ii)} \quad \begin{cases} N = 2,\ m_1 = m_2 = 1,\ M_1 = M_2 = 1,\ d_{1,1} = d_a,\ d_{2,1} = d_b \\ D_1 = d_{1,1} = d_a,\ D_2 = d_{2,1} = d_b \\ D = \text{blockdiag}(D_1, D_2) \\ H = \text{blockdiag}(H_1, H_2) \\ \Gamma \in \mathscr{G}(2;1,1) \text{ i.e. } \Gamma = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \quad \gamma_1, \gamma_2 \geq 1 \end{cases} \tag{26}
$$

Let the virtual environment E be the backward Euler discretized version with sample period $T = 0.01$ secs of the mechanical system in Fig. 5, where $x_e = [x_{e,1}^T \ x_{e_2}^T]^T$ is the virtual environment displacement vector and $f_v = [f_{v,1}^T \ f_{v,2}^T]^T$ is the force feedback vector as depicted Fig. 1. Assume

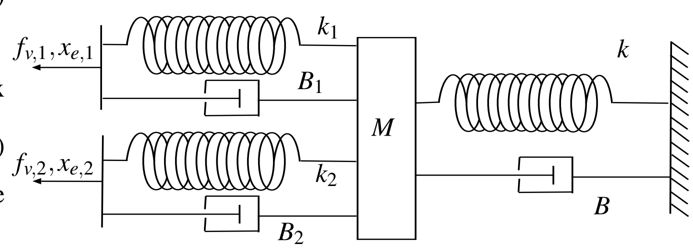


Fig. 5. Example: Virtual environment model

the parameter values $B_1 = 1,\ B_2 = 2,\ k_1 = 800,\ k_2 = 500,\ B = 5,\ k = 1000,\ M = 0.3$, in standard measurement units. The matrices (A_E, B_E, C_E, D_E) of a state space realization of E can be easily obtained and are not reported.

We assume that the two haptic devices are characterized by the OSP levels $\Delta_{d_a} = \Delta_{d_b} = 1.37$. These values are computed according to the results in [8] from the identified dynamics along one axis of Force Dimension's Omega device.

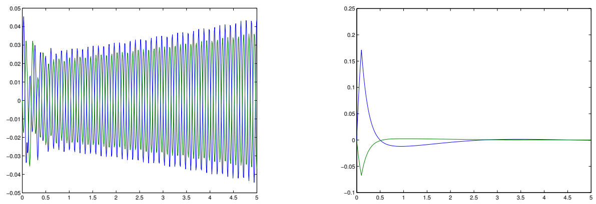
Let us consider the problem structure corresponding to case i) (single operator). In the absence of virtual coupling, the haptic loop may be unstable, as it is clear from the simulation in figure 6(a), which depicts the system response (velocity vector) to an impulsive force perturbation on the devices. In this simulation, the human operator is modeled by the passive continuous-time operator

$$
H = H(s) = \begin{bmatrix} \frac{0.1(s^2+5s+10)}{s^2+s+1} & 0.05 \\ 0.05 & \frac{0.05(s^2+5s+10)}{s^2+s+1} \end{bmatrix}
$$

and the devices are simulated by a first-order (mass-damper) identified model of the Omega.

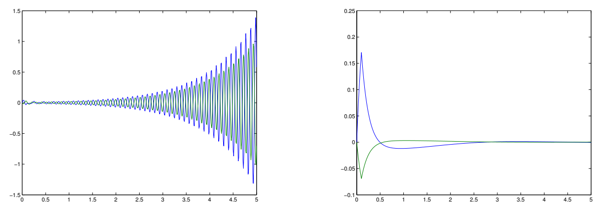
We look for a decentralized stabilizing virtual coupling composed of two first-order SISO controllers each attached to one device. Therefore, we seek a solution of the problem (18)-(22) with the parameter values in (25) and $\bar{N} = 2,\ \bar{n}_1 = \bar{n}_2 = \bar{m}_1 = \bar{m}_2 = 1$.

It turns out that the problem has a non-empty set of solutions.



(a) (b)

Fig. 6. Example, case i): Simulation of haptic loop dynamics (velocity vector) without (a) and with (b) virtual coupling (27)



(a) (b)

Fig. 7. Example, case ii): Simulation of haptic loop dynamics (velocity vector) without (a) and with (b) virtual coupling.

A feasible V is given by

$$A_V = \begin{bmatrix} -0.03732 & 0 \\ 0 & -0.01054 \end{bmatrix}, \quad B_V = \begin{bmatrix} -3.762 & 0 \\ 0 & -6.29 \end{bmatrix},$$
$$C_V = \begin{bmatrix} 0.01477 & 0 \\ 0 & -0.01639 \end{bmatrix}, \quad D_V = \begin{bmatrix} 9.494 & 0 \\ 0 & 9.749 \end{bmatrix} \tag{27}$$

and is a stabilizing one, as it is clear from the simulation in Fig. 6(b). It is apparent that this solution is valid also for case ii) (two operators), since the structure of Γ pertaining to case i) is more restrictive than that of case ii).

We now introduce a perturbation on the virtual environment model E. In particular, we consider the system obtained by pre-multiplying the matrix C_E of the state space representation by the constant matrix $\begin{bmatrix} 1 & 0.3 \\ 0 & 1 \end{bmatrix}$. For the perturbed environment model, it turns out that the above problem no longer has a solution, thus meaning that the proposed method fails to provide a stabilizing virtual coupling of the given class. We look for a solution for case ii) (two operators), corresponding to the parameters in (26) and with the same controller structure.

It turns out that a feasible solution exists. A simulation of the corresponding closed loop system is reported in figure 7, where the two operators are modeled as the diagonal passive transfer function

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} = H(s) = \begin{bmatrix} \frac{0.1(s^2+5s+10)}{s^2+s+1} & 0 \\ 0 & \frac{0.05(s^2+5s+10)}{s^2+s+1} \end{bmatrix}.$$

VII. Conclusion

In this paper, the problem of stability assessment and virtual coupling design for haptic interaction systems involving multiple devices and human operators has been addressed, and a convenient framework based on passivity and Linear Matrix Inequalities has been proposed. The main advantage of the approach is that it easily allows for taking into account the problem structures that may arise in a multi-dimensional scenario, in particular the different configurations in which one or several operators may interact with several devices. As far as control design is concerned, a class of stabilizing virtual coupling controllers which can be parameterized via a sequence of LMI problems has been introduced. Such a class is quite flexible, since it allows for taking into account decentralization constraints imposed on the control system. The study of performance problems such as device and controller transparency within the proposed framework is the subject of current research.

References

[1] M. Minsky, M. Ouh-Young, and O. Steele, "Feeling and seeing: issue in force display," *ACM Trans. Comput.-Human Interactions*, vol. 24, no. 2, pp. 235–243, 1990.

[2] R. Lozano, B. Brogliato, O. Egeland, and B. Maschke, *Dissipative systems analysis and control*. Springer Verlag, 2000.

[3] J. Colgate and G. Schenkel, "Passivity of a class of sampled-data systems: application to haptic interfaces," *Journal of robotic systems*, vol. 14, no. 1, pp. 37–47, 1997.

[4] S. Stramigioli, C. Secchi, A. J. van der Schaft, and C. Fantuzzi, "Sampled data systems passivity and discrete port-hamiltonian systems," *IEEE Transactions on Robotics*, vol. 21, no. 4, pp. 574–586, 2005.

[5] B. E. Miller, J. E. Colgate, and R. Freeman, "Computational delay and free mode environment design for haptic display," in *ASME Dyn. Syst. Cont. Div.*, 1999.

[6] B. E. Miller, *Stability of haptic systems with nonpassive virtual environments*. Ph.D. Thesis, Northwestern University, 2000.

[7] B. E. Miller, J. E. Colgate, and R. Freeman, "Guaranteed stability of haptic systems with nonlinear virtual environments," *IEEE Transactions on Robotics and Automation*, vol. 16, no. 6, pp. 712–719, 2000.

[8] ——, "On the role of dissipation in haptic systems," *IEEE Transactions on Robotics*, vol. 20, no. 4, pp. 768–771, 2004.

[9] P. Naghshtabrizi and J. P. Hespanha, "Designing transparent stabilizing haptic controllers," in *IEEE American Control Conference*, Minneapolis, MI, USA, 2006.

[10] F. Barbagli, D. Prattichizzo, and J. Salisbury, *Multi-Point Physical Interaction with Real and Virtual Objects*, ser. STAR, Springer Tracks in Advanced Robotics. Berlin Heidelberg: Springer Verlag, 2005, vol. 18.

[11] N. Melder and W. S. Harwin, "Improved rendering for multi-finger manipulation using friction cone based god-objects," in *Proc. Eurohaptics Conference*, 2003, pp. 82–85.

[12] F. Barbagli, A. Frisoli, K. Salisbury, and M. Bergamasco, "Simulating human fingers: a soft finger proxy model and algorithm," in *Proc. Haptics Symposium*, Chicago, Illinois, March 2004.

[13] F. Barbagli, D. Prattichizzo, and K. Salisbury, "A multirate approach to haptic interaction with deformable objects single and multipoint contacts," *International Journal of Robotics Research*, vol. 4, no. 29, pp. 703–715, September 2005.

[14] M. Fotoohi, S. Sirouspour, and D. Capson, "Stability and Performance Analysis of Centralized and Distributed Multi-rate Control Architectures for Multi-user Haptic Interaction," *The International Journal of Robotics Research*, vol. 26, no. 9, p. 977, 2007.

[15] G. Bianchini, M. Orlandesi, and D. Prattichizzo, "Analysis and design of multi-contact haptic systems: an LMI approach," in *Proc. of 46th IEEE Conference on Decision and Control*, New Orleans, LA, USA, 2007, pp. 5761–5766.

[16] ——, "Stability analysis and design of multi-dimensional haptic systems," in *Proc. of IEEE Haptics Symposium 2008*, Reno, NV, USA, 2008.

[17] C. Byrnes and W. Lin, "Losslessness, feedback equivalence, and the global stabilization of discrete-time nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 39, no. 1, pp. 83–98, 1994.

[18] S. Boyd and Q. Yang, "Structured and simultaneous Lyapunov functions for system stability problems," *International Journal of Control*, vol. 49, no. 6, pp. 2215–2240, 1989.

[19] G. Bianchini, M. Orlandesi, and D. Prattichizzo, "An LMI framework for analysis and design of multi-dimensional haptic systems," 2008, technical report, available online: http://www.dii.unisi.it/~giannibi/research/papers/cdc08mod.pdf.

[20] L. E. Ghaoui, F. Oustry, and M. AitRami, "A cone complementarity linearization algorithm for static output feedback and related problems," *IEEE Transactions on Automatic Control*, vol. 42, no. 8, pp. 1171–1175, 1997.